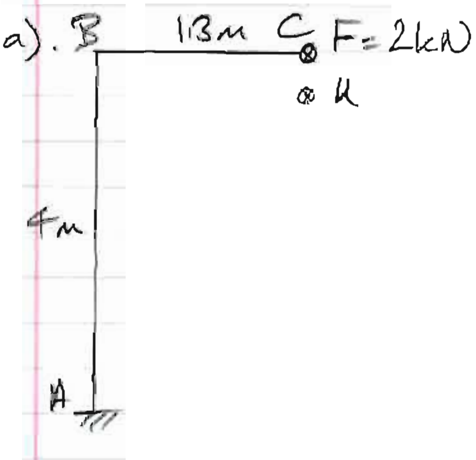
PART IB PAPER 2 : STRUCTURES

(1). Thin-walled section: $d = 200\,mm$, $t = 8\,mm$
$E = 210\,GPa$, $\nu = 0.3$.

$I = \pi r^3 t = \pi \times 100^3 \times 8 = 25.13 \times 10^6\,mm^4 = 25.13 \times 10^{-6}\,m^4$

$J = 2\pi r^3 t = 2I = 50.27 \times 10^{-6}\,m^4$ Alternatively, use radius of 96mm to mid-thickness.

a).



Deflection u (into page) comprises:

i) bending of AB;

ii). bending of BC;

iii) torsion of AB.

Take care over units!

Bending

$$\delta_i = \frac{"WL^3"}{3EI} = \frac{2 \times 10^3 \times 4^3}{3 \times 210 \times 10^9 \times 25.13 \times 10^{-6}} = \underline{8.08\,mm}$$

$$\delta_{ii} = \left(\frac{1.3}{4}\right)^3 \times 8.08 = \underline{0.28\,mm}.$$

Torsion

$$\phi = "\frac{T}{GJ}", \quad G = \frac{E}{2(1+\nu)} = \frac{210 \times 10^9}{2 \times 1.6} = 81\,GPa.$$

$$\therefore \phi = \frac{1.3 \times 2 \times 10^3}{81 \times 10^9 \times 50.27 \times 10^{-6}} = 6.39 \times 10^{-4}\,rad/m$$ N.B. rotation per unit length.

$$\therefore \delta_{iii} = \underbrace{4 \times 6.39 \times 10^{-4}}_{\text{rotation at B}} \times \underbrace{1.3}_{BC} = \underline{3.32\,mm}.$$

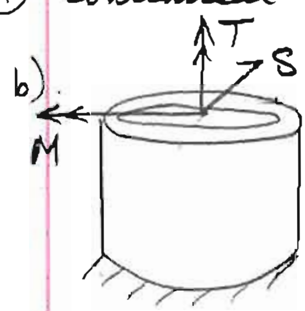
$\therefore$ Total deflection, $u = \delta_i + \delta_{ii} + \delta_{iii}$

$= 8.08 + 0.28 + 3.32$

$= \underline{11.68\,mm}.$

(1) Continued

b)



Helpful to define moment/torque in a vector sense (right-hand rule).

$T = 1.3 \times 2 \times 10^3 = \underline{2.6 \text{ kNm}}$; $S = \underline{2 \text{ kN}}$

$M = 4 \times 2 \times 10^3 = \underline{8 \text{ kNm}}$

Direct stress - bending

$$\sigma_M)_{max} = \text{"}\frac{My}{I}\text{"} = \frac{8 \times 10^3 \times 100 \times 10^{-3}}{25.13 \times 10^{-6}} = 31.83 \text{ MPa.}$$

Shear stress - direct

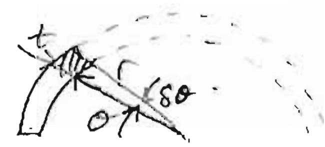
$$\tau_S)_{max} = \text{"}\frac{SA_c\bar{y}}{Ia}\text{"} \quad , \quad a = 2t$$

Mechanics databoo

$$\bar{y} = \text{"}\frac{r\sin\alpha}{\alpha}\text{"} = \frac{2r}{\pi}$$

$$\Rightarrow A_c\bar{y} = \pi r t . \frac{2r}{\pi} = 2r^2 t$$

OR:

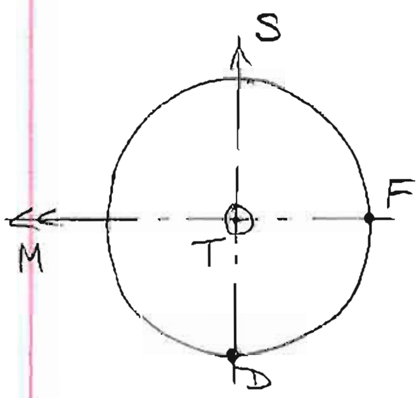


$$A_c\bar{y} = 2\int_0^{\pi/2} t . r\, d\theta . r\sin\theta = 2r^2 t \int_0^{\pi/2} \sin\theta\, d\theta$$

$$\therefore \tau_S)_{max} = \frac{S . 2r^2 t}{\pi r^3 t . 2t} = \frac{S}{\pi r t} = \frac{2 \times 10^3}{\pi \times 100 \times 10^{-3} \times 8 \times 10^{-3}} = \underline{0.80 \text{ MPa.}}$$

Shear stress - torsion

$$\tau_T = \frac{Tr}{J} = \frac{2.6 \times 10^3 \times 100 \times 10^{-3}}{50.27 \times 10^{-6}} = \underline{5.17 \text{ MPa.}}$$



Hence maximum total shear stress
$= \tau_T + \tau_S)_{max}$

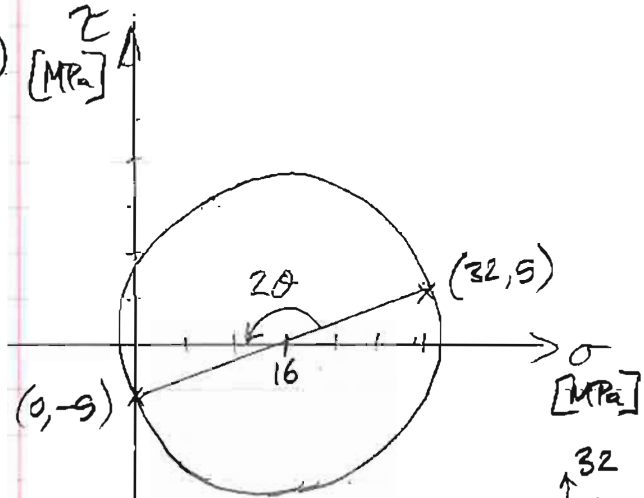
$= \underline{5.97 \text{ MPa}}$

located at F, where torsional and direct shear stress add.

Maximum (tensile) direct stress located at D.

① Continued

c)

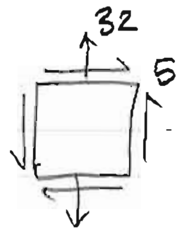


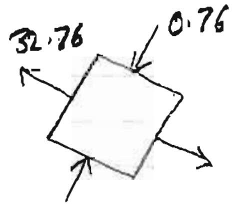
Centre at $\frac{32}{2} = 16$ MPa

Radius $= \sqrt{16^2+5^2} = 16.76$ MPa

$2\theta = 180° - \tan^{-1}\left(\frac{5}{16}\right) = 162.6°$

$\therefore \theta = 81.3°$





Principal stresses: $16 \mp 16.76 = -0.76$ MPa, 32.76 MPa

i.e. $\sigma_1 = 32.76$, $\sigma_2 = 0$, $\sigma_3 = -0.76$ MPa

$\therefore$ von Mises' $\Rightarrow (\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2 = 2Y^2$

$\therefore$ margin against yield $= \sqrt{\dfrac{2\times 245^2}{32.76^2 + 0.76^2 + (-0.76-32.76)^2}}$

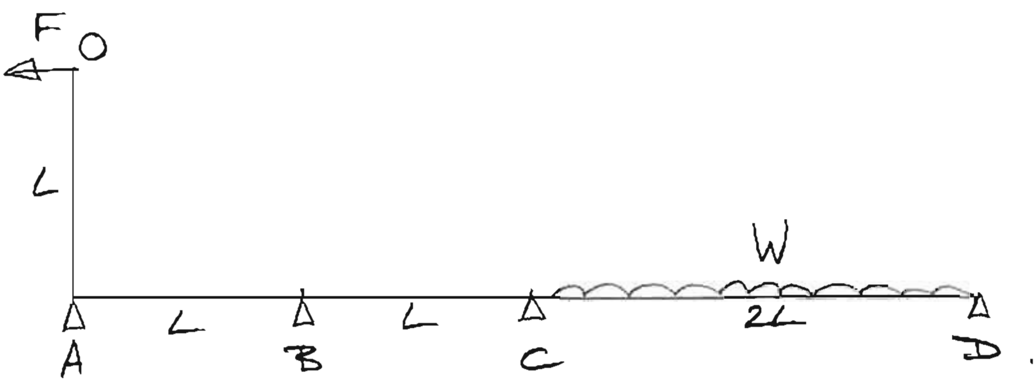
[$Y = \sigma_y = 245$ MPa]

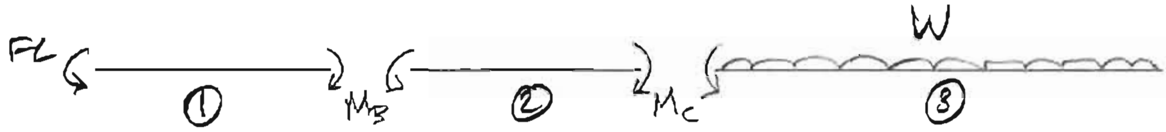
$\underline{7.4}$

Note: full Mohr's circle not required for full marks.

(2).

a).





Using the Data Book coefficients (taking care over signs):

$$\theta_{B1} = \frac{M_B L}{3EI} + \frac{(FL)L}{6EI}$$

$$\theta_{B2} = \frac{M_B L}{3EI} + \frac{M_c L}{6EI}$$

$$\theta_{B1} = -\theta_{B2} \quad \therefore \quad \frac{M_B}{3} + \frac{FL}{6} = -\frac{M_B}{3} - \frac{M_c}{6}$$

$$4M_B + M_c + FL = 0 \qquad (A).$$

$$\theta_{C2} = \frac{M_c L}{3EI} + \frac{M_B L}{6EI}$$

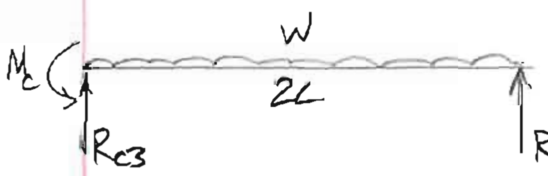
$$\theta_{C3} = \frac{M_c(2L)}{3EI} - \frac{W(2L)^2}{24EI}$$

$$\theta_{C2} = -\theta_{C3} \quad \therefore \quad \frac{M_c}{3} + \frac{M_B}{6} = -\frac{2M_c}{3} + \frac{\cancel{4}WL}{\cancel{24}\,6}$$

$$6M_c + M_B - WL = 0 \qquad (B)$$

$$(A) - 4(B): \quad -23M_c + FL + 4WL = 0 \quad \Rightarrow \quad M_c = \frac{(F+4W)L}{23}$$

$$\therefore \quad M_B = WL - 6M_c = WL - \frac{6(F+4W)L}{23}, \quad M_B = -\frac{(6F+W)L}{23}$$



$$M(c): \quad R_D . 2L = WL - M_c$$

$$\Rightarrow \quad R_D = \frac{W}{2} - \frac{F+4W}{46} = \frac{19W - F}{46}$$

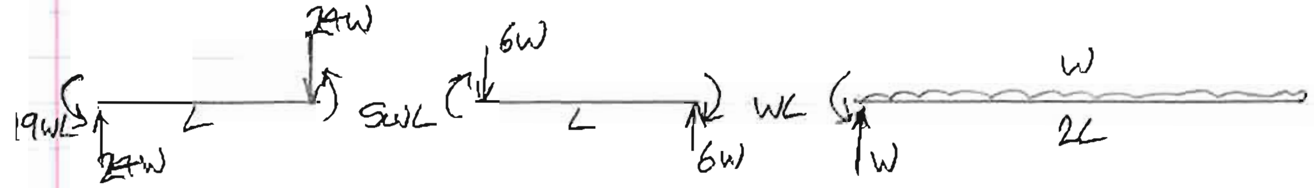
$$\therefore \quad R_D = 0 \quad \Rightarrow \quad F = 19W.$$

b) $F = 19W \Rightarrow M_B = -\frac{(114W + W)L}{23} = -\frac{115WL}{23} = -5WL$

$$M_c = \frac{(19W + 4W)L}{23} = WL$$

$$M_A = FL = 19WL$$

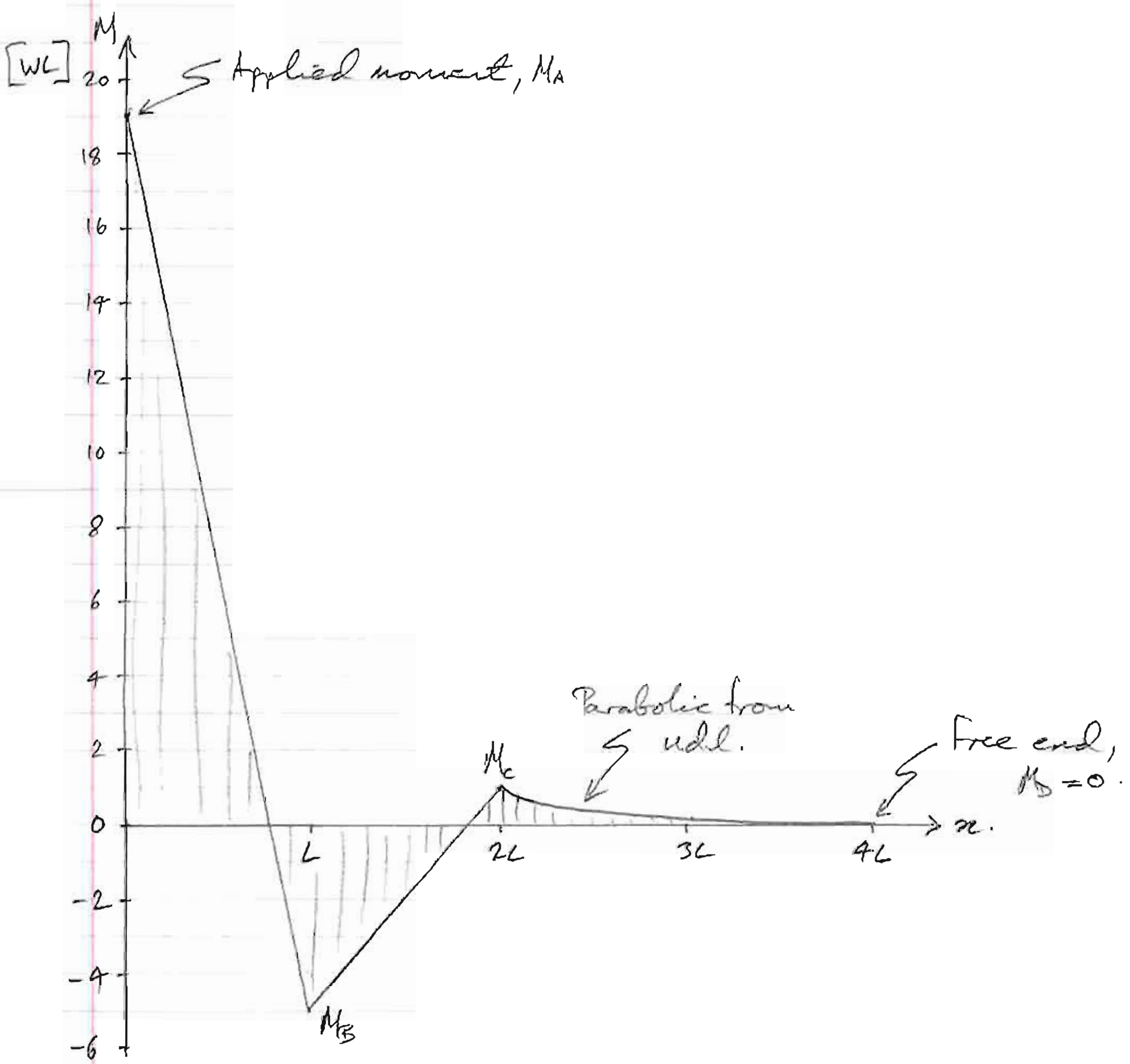
(2) Continued.



[3]

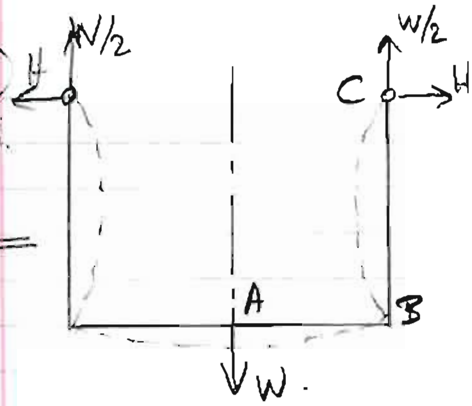
∴ $R_A = 24W$, $R_B = -30W$, $R_C = 7W$, $R_D = 0$

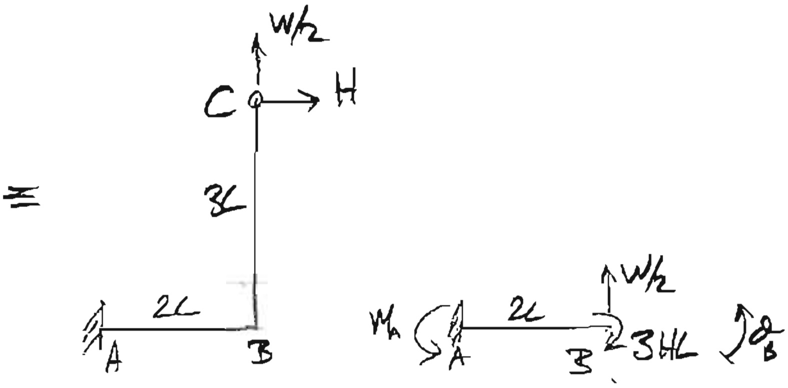
CHECK : $24 - 30 + 7 = 1$ ✓



③ a)

For W





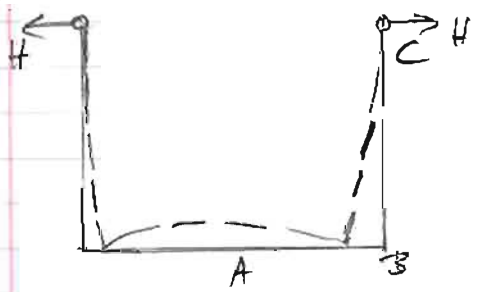
$$AB: \ \theta_B = \frac{W}{2} \cdot \frac{(2L)^2}{2EI} - \frac{3HL \cdot 2L}{EI} = \frac{4WL^2}{4EI} - \frac{6HL^2}{EI} = \frac{WL^2}{EI} - \frac{6HL^2}{EI}$$

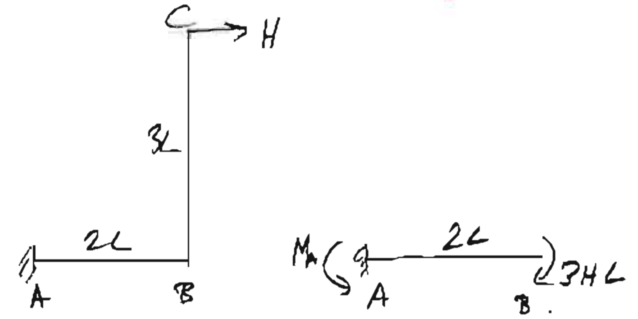
$$BC: \ \delta_C = 0 \ \therefore \left(\frac{WL^2}{\cancel{EI}} - \frac{6HL^2}{\cancel{EI}}\right) \cdot 3L = \frac{H \cdot (3L)^3}{3\cancel{EI}}$$

$$3WL^{\cancel{3}} - 18HL^{\cancel{3}} = 9HL^{\cancel{3}}, \quad 3W = 27H, \quad H = \frac{W}{9}$$

$$AB: \ M_A = M_B - \frac{W}{2} \cdot 2L = 3 \cdot \frac{W}{9} \cdot L - WL = -\frac{2WL}{3}$$ i.e. sagging moment of $\frac{2WL}{3}$

For ΔT





As before, $\delta_C = 0 \Rightarrow \frac{6HL^2}{EI} \cdot 3L + \frac{H(3L)^3}{3EI} = \alpha \Delta T \cdot 2L$

$$27HL^{\cancel{3}2} = 2\alpha \cancel{L} \Delta T EI, \quad H = \frac{2\alpha \Delta T EI}{27L^2}$$

$$\therefore M_A = M_B = 3 \cdot \frac{2\alpha \Delta T EI}{27L^2} \cdot L = \frac{2\alpha \Delta T EI}{9L}$$ i.e. sagging moment of $-\frac{2\alpha \Delta T EI}{9L}$

So combined moment, $M = -\frac{2WL}{3} + \frac{2\alpha \Delta T EI}{9L}$

i.e. the magnitude of the sagging moment is $\frac{2WL}{3} - \frac{2\alpha \Delta T EI}{9L}$

b) $$\delta_B^W = \frac{W}{2} \cdot \frac{(2L)^3}{3EI} - \frac{(3HL)(2L)^2}{2EI} = \frac{4WL^3}{3EI} - \frac{6L^3}{EI} \cdot \frac{W}{9} = \frac{2WL^3}{3EI}$$

$$\delta_B^{\Delta T} = \frac{(3HL)(2L)^2}{2EI} = \frac{6L^3}{\cancel{EI}} \cdot \frac{2\alpha \Delta T \cancel{EI}}{27\cancel{L^2}} = \frac{12\alpha L \Delta T}{27} = \frac{4\alpha L \Delta T}{9}$$

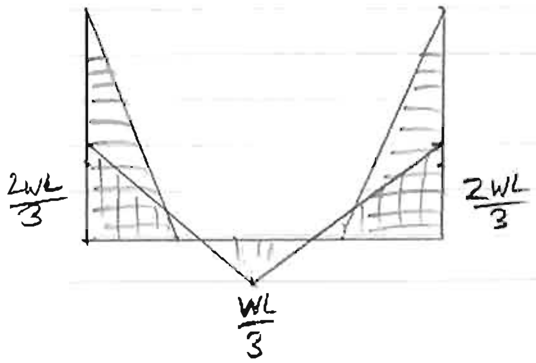
(3) Continued.

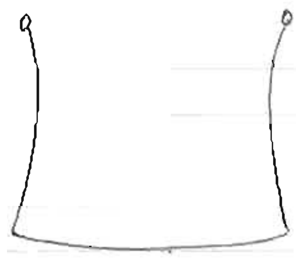
$\therefore$ require $\dfrac{2WL^3}{3EI} = \dfrac{4\alpha L \Delta T}{9} \Rightarrow \Delta T = \underline{\dfrac{3WL^3}{2\alpha EI}}$

$\Rightarrow H = \dfrac{W}{9} + \dfrac{\cancel{2\alpha EI}}{27\cancel{L^3}} \cdot \dfrac{3W\cancel{L^3}}{\cancel{2\alpha EI}} = \dfrac{2W}{9}$

$\therefore M_B = 3HL = \dfrac{2WL}{3}$

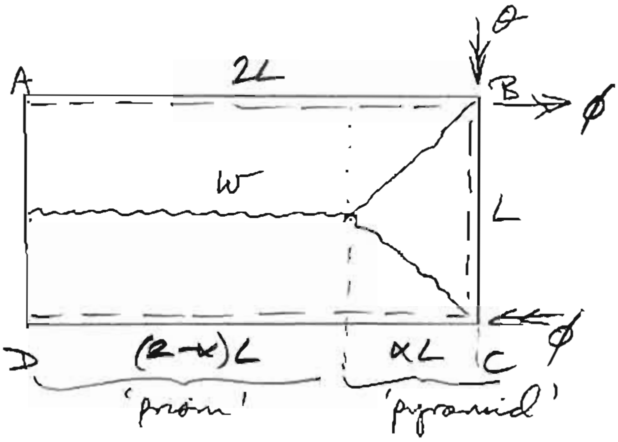
$M = -\dfrac{2WL}{3} + \dfrac{\cancel{2\alpha EI}}{9\cancel{L}} \cdot \dfrac{3WL^{2}}{\cancel{2\alpha EI}} = -\dfrac{WL}{3}$ ie. $\dfrac{WL}{3}$ sagging.

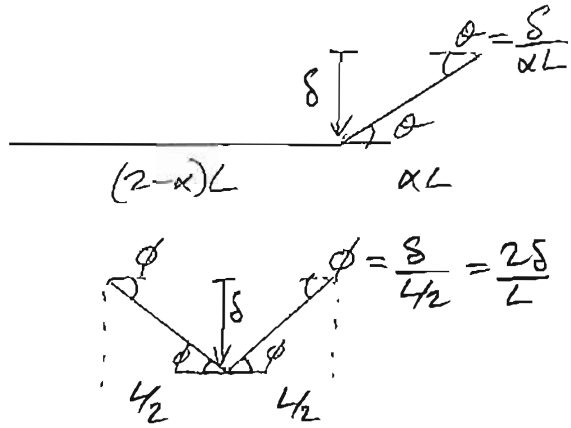




4

a).





External work

$$WD = \underset{\text{prism}}{(2-\alpha)L.L.w.\frac{\delta}{2}} + \underset{\text{pyramid}}{\alpha L.L.w.\frac{\delta}{3}} = \frac{(2-\alpha)}{2}wL^2\delta + \frac{\alpha}{3}wL^2\delta = \left(1-\frac{\alpha}{6}\right)wL^2\delta$$

Internal work

$$WD = 2Lm.4\phi + Lm.2\theta = 2\cancel{L}m.\frac{8\delta}{\cancel{L}} + 2\cancel{L}m.\frac{\delta}{\alpha\cancel{L}} = 16m\delta + \frac{2m\delta}{\alpha}$$

$$\therefore \left(1-\frac{\alpha}{6}\right)wL^2\cancel{\delta} = 16m\cancel{\delta} + \frac{2m\cancel{\delta}}{\alpha},$$

$$\frac{6-\alpha}{6}.wL^2 = 16m + \frac{2m}{\alpha}, \quad \alpha(6-\alpha)wL^2 = 96m\alpha + 12m = 12m(1+8\alpha)$$

$$\therefore w = \frac{12m(8\alpha+1)}{L^2\alpha(6-\alpha)}$$

b). For the minimum upper bound, require $\frac{dw}{d\alpha} = 0$

$$\Rightarrow \frac{d}{d\alpha}\left(\frac{8\alpha+1}{\alpha(6-\alpha)}\right) = 0$$

$$\frac{(6\alpha-\alpha^2).8 - (8\alpha+1)(6-2\alpha)}{\alpha^2(6-\alpha)^2} = 0$$

$$\Rightarrow \cancel{48\alpha} - 8\alpha^2 - \cancel{48\alpha} + 16\alpha^2 - 6 + 2\alpha = 0$$

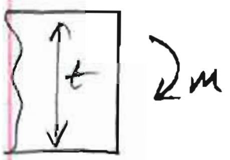
$$8\alpha^2 + 2\alpha - 6 = 0, \quad 4\alpha^2 + \alpha - 3 = 0$$

$$\alpha = \frac{-1 \pm \sqrt{1+16\times 3}}{8} = \frac{-1\pm\sqrt{49}}{8} = 0.75 \quad \text{(+ve root only)}$$

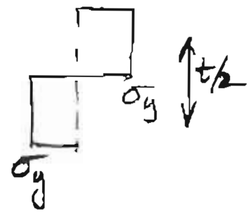
$$\therefore w_{min} = \frac{12m(8\times 0.75+1)}{L^2\, 0.75(6-0.75)} = 21.33\frac{m}{L^2}.$$

(4) Continued.

$L = 300\,mm$, $t = 6\,mm$, $\sigma_y = 250\,MPa = 250\,N/mm^2$



$\equiv$



$$m = \sigma_y \cdot \frac{t}{2} \cdot \frac{t}{2} = \frac{\sigma_y t^2}{4}$$

$$= \frac{250 \times 6^2}{4}$$

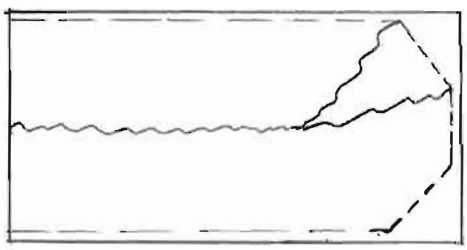
$$= 2250 \ \frac{Nm}{m}$$

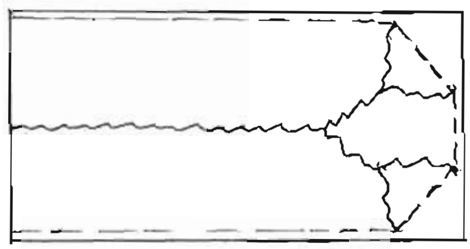
$$\therefore\ w_{min} = 21.33 \times \frac{2250}{0.3^2}$$

$$= 533.25 \times 10^3$$

$$= 533.3\ kN/m^2$$

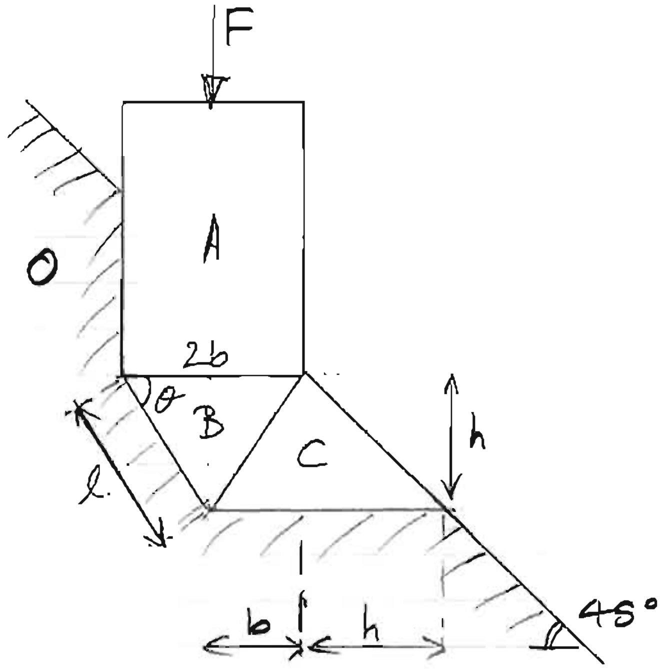
c) Other possible mechanisms include:



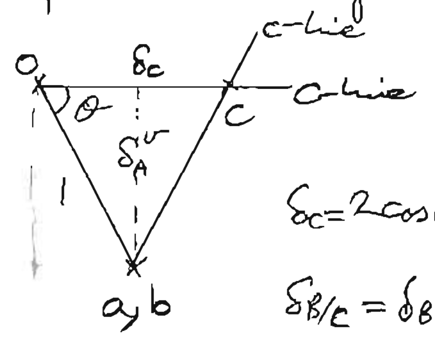


(5)

a).



Displacement diagram



$\delta_c = 2\cos\theta = 2b/l$

$\delta_{B/c} = \delta_B = 1$

$\delta_A^v = \sin\theta = h/l$.

Work done by F = energy dissipated along slip lines

$\Rightarrow F.\delta_A^v = kl\delta_B + k(b+h)\delta_c + kl\delta_{B/c}$

$F\frac{h}{l} = kl.1 + k(b+h)\frac{2b}{l} + kl.1$

$F = \frac{k}{h}\left(l^2 + 2b(b+h) + l^2\right)$ But $l^2 = b^2 + h^2$

$= \frac{k}{h}\left(2b^2 + 2h^2 + 2b^2 + 2bh\right)$

$= 2k\left(\frac{2b^2}{h} + h + b\right)$

Hence minimum upper bound when $\frac{dF}{dh} = 0$

$\Rightarrow -\frac{2b^2}{h^2} + 1 = 0$, $h = \sqrt{2}b$.

$\Rightarrow F_{min} = 2k\left(\frac{2b^{\cancel{2}}}{\sqrt{2}\cancel{b}} + \sqrt{2}b + b\right)$

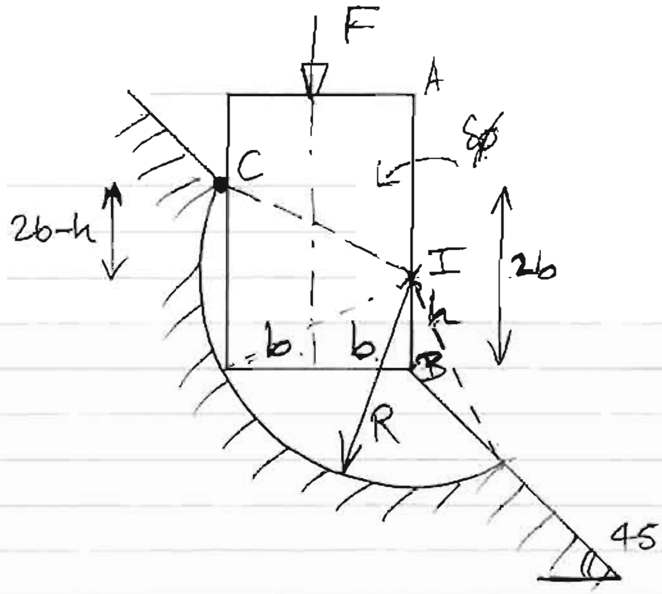
$= 2k\left(\frac{2b + 2b + \sqrt{2}b}{\sqrt{2}}\right)$

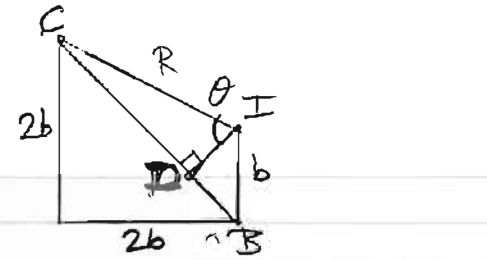
$= \sqrt{2}kb(4+\sqrt{2})$

$= 2kb(1+2\sqrt{2}) = 7.66\,kb$.

(5) Continued.

b)





$CD = 2\sqrt{2}b - b/\sqrt{2} = 3b/\sqrt{2}$

Consider centre of rotation I. This must be positioned such that R is minimised whilst enabling a kinematically viable mechanism

$$\text{ie. } \sqrt{(2b)^2 + h^2} = \sqrt{(2b)^2 + (2b-h)^2}$$

$$\Rightarrow h^2 = (2b-h)^2 = 4b^2 - 4bh + h^2, \quad 4b^2 = 4bh, \quad h = b.$$

$$\Rightarrow R = \sqrt{5}\,b.$$

For this, considering work done:

$$F.b = k.\,R(2\theta)R = 2kR^2\theta = 2k.5b^2\theta = 10kb^2\theta$$

$$\text{But } CD = \frac{3b}{\sqrt{2}} \Rightarrow \sin\theta = \frac{3b/\sqrt{2}}{R} = \frac{3b/\sqrt{2}}{\sqrt{5}\,b} = \frac{3}{\sqrt{10}}$$

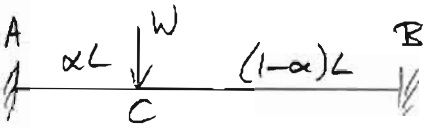
$$\therefore F = 10kb\theta = 10kb.\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$$

$$= 12{\cdot}49\,kb$$

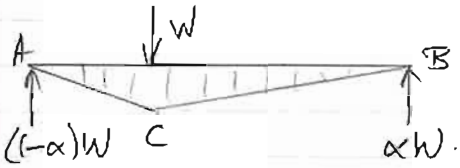
Hence sliding block mechanism remains the optimum mechanism out of the two.

(6)

a)

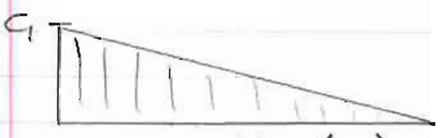


Particular Solution



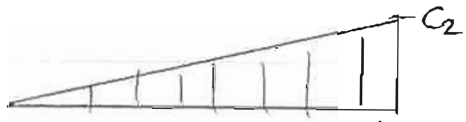
$M_C = -\alpha L(1-\alpha)W$

Self Stress 1



$M_A = C_1$, $M_C = (1-\alpha)C_1$

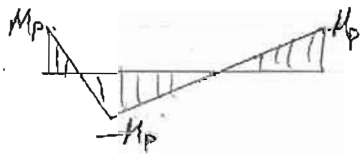
Self Stress 2



$M_B = C_2$, $M_C = \alpha C_2$

Superposing stress states, $M_C = -\alpha L(1-\alpha)W + (1-\alpha)C_1 + \alpha C_2$

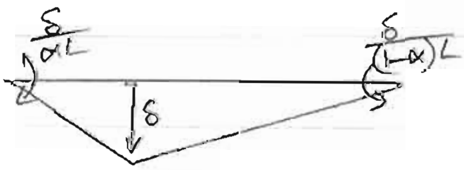
Optimal design requires $|M_A| = |M_B| = |M_C| = M_p$.



$\therefore\ -M_p = -\alpha L(1-\alpha)W + (1\cancel{-\alpha})M_p \cancel{+\alpha} M_p$

$$-2M_p = -\alpha L(1-\alpha)W \Rightarrow W = \frac{2M_p}{\alpha(1-\alpha)L}$$

b).



$$W\delta = M_p\left(\frac{\delta}{\alpha L} + \left(\frac{\delta}{\alpha L} + \frac{\delta}{(1-\alpha)L}\right) + \frac{\delta}{(1-\alpha)L}\right)$$

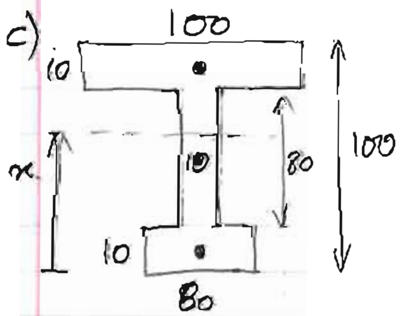
$$W = \frac{M_p}{L}\left(\frac{2}{\alpha} + \frac{2}{1-\alpha}\right)$$

$$= \frac{M_p}{L}\left(\frac{2 \cancel{-2\alpha} \cancel{+2\alpha}}{\alpha(1-\alpha)}\right)$$

$$= \frac{2M_p}{\alpha(1-\alpha)L}$$

Which is the same collapse load calculated as a lower bound estimate. Hence this is the actual collapse load.

c)

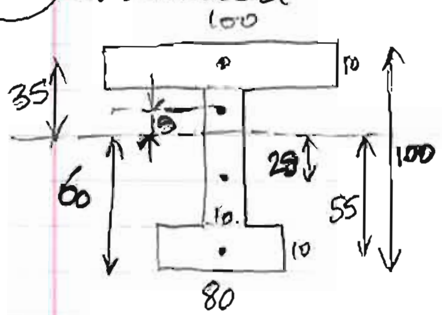


Equal areas axis at x:

$$80\times10 + (x-10)\times10 = \frac{100\times10 + 80\times10 + 80\times10}{2}$$

$\Rightarrow x = 60\,mm.$

6 Continued



$$Z_p = \sum A_i y_i$$

$$= 100 \times 10 \times 35 + 10 \times 30 \times 15 + 50 \times 10 \times 25 + 80 \times 10 \times 55$$

$$= 96{,}000 \text{ mm}^3$$

$$M_p = Z_p \sigma_y = 96000 \times 245 = 23.52 \times 10^6 \text{ Nmm}$$

$$= \underline{23.52 \text{ kNm}}$$

The maximum safe load is for a mid-span load, i.e. $\alpha = 0.5$

$$\Rightarrow W = \frac{2 \times 23.52 \times 10^3}{0.5 \times 0.5 \times 6} = \underline{31.36 \times 10^3 \text{ N}}$$

$\Rightarrow W_{max} = \underline{15.68 \text{ kN}}$, for a load factor of 2.

d). Any initial stresses have no effect on the final collapse load, given the ductility of steel. Hence safe working load remains unchanged.

J.P. Talbot 16/4/17