Solutions: IB Paper 6, 2014

ENGINEERING TRIPOS PART IB

Thursday 5 June 2014 2 to 4

Paper 6

INFORMATION ENGINEERING

Version: jl/gv

1 (a) A system is *linear* if, whenever an input $u_1(t)$ results in an output $y_1(t)$, and an input $u_2(t)$ results in an output $y_2(t)$, then an input $u_1(t)+u_2(t)$ results in an output $y_1(t)+y_2(t)$. [or: an input $u_1(t)$ results in an output $y_1(t)$, and an input $u_2(t)$ results in an output $y_2(t)$, then an input $\lambda_1 u_1(t)+\lambda_2 u_2(t)$ results in an output $\lambda_1 y_1(t)+\lambda_2 y_2(t)$, where λ_1, λ_2 are constants.] [5]

(b) We can write down the following for the given closed loop system:

$$\bar{y}(s) = \bar{d}(s) + G(s)\bar{u}(s)$$

$$\bar{u}(s) = K(s)\bar{e}(s)$$

$$\bar{e}(s) = \bar{r}(s) - \bar{y}(s)$$

therefore we can eliminate $\bar{u}$ and $\bar{e}$ to give

$$\bar{y}(s) = \bar{d}(s) + G(s)K(s)(\bar{r}(s) - \bar{y}(s))$$

$$\implies \quad \bar{y}(s) = \frac{1}{1+G(s)K(s)}\bar{d}(s) + \frac{G(s)K(s)}{1+G(s)K(s)}\bar{r}(s)$$

We can also express $\bar{u}$ in terms of $\bar{r}$ and $\bar{d}$:

$$\bar{u}(s) = K(s)[\bar{r}(s) - (\bar{d}(s) + G(s)\bar{u}(s))]$$

$$\implies \quad \bar{u}(s) = \frac{K(s)}{1+G(s)K(s)}\bar{r}(s) - \frac{K(s)}{1+G(s)K(s)}\bar{d}(s)$$

This therefore enables us to answer (i),(ii) and (iii).

(i)

$$\bar{y}(s) = \frac{K(s)G(s)}{1+K(s)G(s)}\bar{r}(s)$$

this should be close to 1 at low frequencies in order to obtain good tracking.

(ii)

$$\bar{y}(s) = \frac{1}{1+K(s)G(s)}\bar{d}(s)$$

this should be small in order to obtain low *sensitivity* to disturbances and modelling errors.

(iii)

$$\bar{u}(s) = \frac{K(s)}{1+K(s)G(s)}\bar{d}(s)$$

this should not be too large, as it determines the control effort in response to disturbances.

[7]

(c) We know that the steady state response to a step input is the transfer function evaluated at $s = 0$. Therefore:

(i) We saw above that the transfer function relating y and r (and assuming $d = 0$), was $\frac{K(s)G(s)}{1+K(s)G(s)}$, so that

$$\left.\frac{KG}{1+KG}\right|_{s=0} = \frac{3/5}{1+3/5} \implies y(t) \to 3/8$$

(ii) We also saw above that the transfer function relating y and d (and assuming $r = 0$), was $\frac{1}{1+K(s)G(s)}$. When applied to a sinusoidal input we know that steady state implies the gain factor, ie the quantity which changes amplitude and phase. Since output for input $e^{j\omega_0 t}$ is the convolution with the impulse response, which then gives $e^{j\omega_0 t}G(j\omega_0)$, we therefore evaluate $G(s)$ at $s = j\omega_0$:

$$\left.\frac{1}{1+KG}\right|_{s=j\omega_0} = \frac{(2j+1)(2j+5)}{(2j+1)(2j+5)+3} = 0.952\angle 0.239(rad)$$

$$\implies y(t) \to 0.952\cos(\omega_0 t + 0.239)$$

(iii) using linearity (d input is now $e^{j(\omega_0 t+\pi/3)}$, so output due to d is also $e^{j(\omega_0 t+\pi/3)}$) we have

$$y \to 2 \times 3/8 + 0.952\cos(\omega_0 t + 0.239 + \pi/3)$$

[8]

(d) In order to achieve zero steady state error we need $K(0) \to \infty$, so the simplest way of achieving this is to require an integral action, e.g. $K(s) = k_p + k_i/s$. One option is therefore to use both $k_i = k_p = 3$, so that $K(s) = 3 + 3/s = 3\frac{(s+1)}{s}$.

$\implies K(s)G(s) = \frac{3}{s(s+5)}$

There are many other reasonable choices of numbers, although the controller zero should not be at a frequency much greater than 1. [5]

A straightforward and popular question. Well done on the whole, although a large number of candidates had difficulty finding the steady state response to a sinusoidal input.

2 (a) The Bode diagram ($|G|$ on log-log plot [y axis in dB] and $\angle G$ on log-linear plot) is shown in figure 1. Though a correct answer will obtain this curve by sketching the straight line asymptotes and approximations and rounding the corners appropriately.

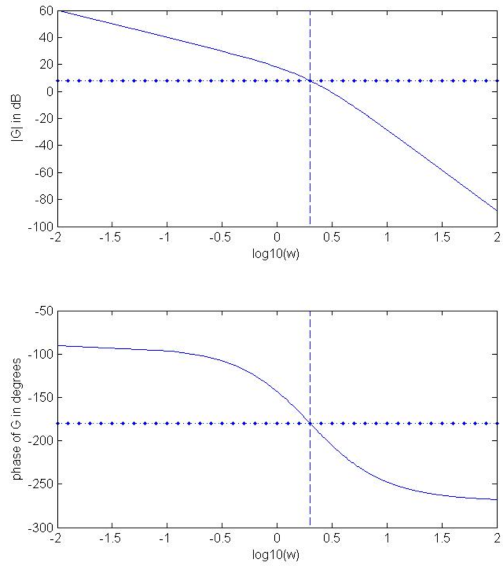


Fig. 1

The way of sketching this is to split the transfer function up into its constituent parts and look at the contribution of each part to the Bode plot.

Note that $\angle G(j\omega) = -\angle(j\omega) - 2\angle(1 + 0.5j\omega) = -\pi/2 - 2 * \tan^{-1}\frac{\omega}{2}$

So that $\angle G = -180^\circ$ at $\omega = 2$ (as second term is then $2 * \pi/4$).

At $\omega = 2$, $|G(j2)| = \frac{10}{2(1^2+1^2)} = 2.5$

$\implies$ need $k_p < \frac{1}{2.5} = 0.4$ [6]

(b) $k_p = 0.4/2 = 0.2, \implies 0.2 * \frac{10}{\omega(\omega^2/4+1)} = 1 \implies \omega \approx 1.4$

The angle when we have a magnitude of 1 (ie at $\omega = 1.4$) is given by
$\angle = -90^\circ - 2\tan^{-1}(1.4 \times 0.5) \approx -160^\circ$
Steady state error $= \frac{1}{1+L(s)}|_{s=0} = 0$ [6]

(c) See plot. Firstly write our new K as $K(s) = 0.2 \times \frac{1+s}{1+s/4}$. Amplitude $= 1$ at $\omega = 2$ and angle $= 37^\circ$ at $\omega = 2$. $\implies$ phase margin $= 37^\circ$ and gain margin $\approx$ 10dB [8]

(d) (c) has greater gain (10dB vs 6dB) and greater phase margins. This implies that the closed loop system is less oscillatory. B/W is greater by a factor of about 2, which implies a faster response. [5]

This question was straightforward, but proved to be time consuming, as they were asked to produce both the uncompensated and compensated diagrams. There were, however, a large number of very good solutions.

3 (a) $4\frac{e^{-0.25j\omega}}{j\omega}$ which we can write as

$$L(j\omega) = \frac{4}{\omega}e^{-j(\omega/4+\pi/2)}$$

So that amplitude is $4/\omega$ and phase is $-(\omega/4+\pi/2)$.

As $\omega \to 0$ we can take just the first two terms in the expansion of the exponential to give

$$4\frac{1-0.25j\omega}{j\omega} = \frac{4}{j\omega} - 1$$

ω	2	5	10
$\|L\|$	2	0.8	0.4
$\angle L$	-0.5 $-\pi/2$	$-1.25-\pi/2$	$-2.5-\pi/2$

The Nyquist diagram is shown (note the asymptote at -1) in figure 2.

[10]

(b) For a phase of $-\pi$ we have $-(\omega/4+\pi/2) = -\pi$, so that $\omega = 2\pi = 6.28$.

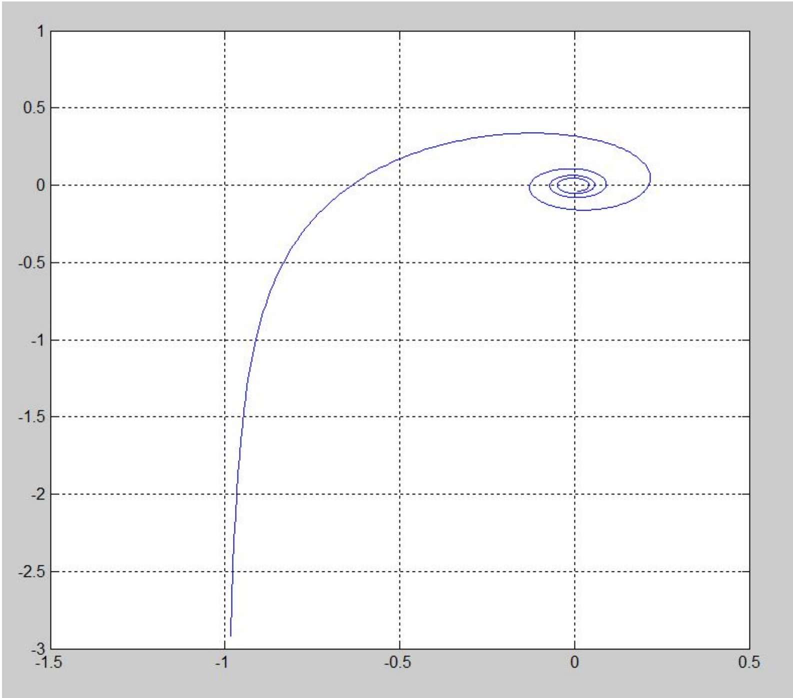


Fig. 2

Magnitude is $4/\omega = 4/(2\pi) = 2/\pi$, so that the Gain Margin $= \pi/2 = 1.57$. [5]

(c) If $k = 4$ our amplitude is $4/\omega$ which should be 1, therefore $\omega = 4$.

The phase is then $-(\omega\tau + \pi/2)$ – equating this to $-\pi$ gives us $\tau = \pi/8 = 0.39$. [5]

(d) From part (b), we know that $\omega = 2\pi$, so that we need $k/\omega = (2)^{-1}$ giving $k = \pi$.

In general we have $\tau\omega = \pi/2$ so amplitude $= k/\omega = k/(\pi/2\tau) = 2\tau k/\pi$. If this quantity is $1/2$ for a GM of 2, we have $k = 0.25\pi/\tau$ so that k varies like $1/\tau$. [5]

Although this was the least popular question of this section, it was well answered by most who attempted it.

4 (a) Since we are told to do this by *direct integration* we need to evaluate the following integral:

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\mathrm{e}^{-j\omega x}\,dx$$

$$= \int_{-a}^{a} \mathrm{e}^{-j\omega x}\,dx = \left[\frac{\mathrm{e}^{-j\omega x}}{-j\omega}\right]_{-a}^{a} = -\frac{1}{j\omega}[\mathrm{e}^{-j\omega a} - \mathrm{e}^{j\omega a}]$$

$$= 2a\mathrm{sinc}\,\omega a$$

For a pulse of height a' and width b centred on the origin, the databook gives the FT as

$$a'b\mathrm{sinc}\frac{\omega b}{2}$$

So, if $a' = 1$ and $b = 2a$ this gives $2a\mathrm{sinc}\,\omega a$, which agrees with our answer above. [5]

(b) The inverse FT of $F(\omega)$ is given by:

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{j\omega x}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} 2a\frac{\sin a\omega}{a\omega}\mathrm{e}^{j\omega x}\,d\omega$$

Thus

$$f(0) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\frac{\sin a\omega}{\omega}\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} 2\frac{\sin a\omega}{\omega}\,d\omega$$

Since the integrand is even. If we then take $a = 2$, this gives

$$1 = \frac{1}{\pi}\int_{0}^{\infty} 2\frac{\sin 2\omega}{\omega}\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} 2\frac{2\sin\omega\cos\omega}{\omega}\,d\omega$$

Which gives us the required result:

$$\int_{0}^{\infty} \frac{\sin\omega\cos\omega}{\omega}\,d\omega = \frac{\pi}{4}$$

[6]

(c) The FT of $h(x)$ is given by

$$H(\omega) = \int_{-\infty}^{\infty} f(x)g(x)\mathrm{e}^{-j\omega x}\,dx$$

Substituting for $g(x)$ in terms of G gives

$$H(\omega) = \int_{-\infty}^{\infty} f(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u) \mathrm{e}^{jux} \mathrm{e}^{-j\omega x}\, du dx = \int_{-\infty}^{\infty} f(x) \frac{1}{2\pi} \int_{-\infty}^{\infty} G(u) \mathrm{e}^{-j(\omega-u)x}\, du dx$$

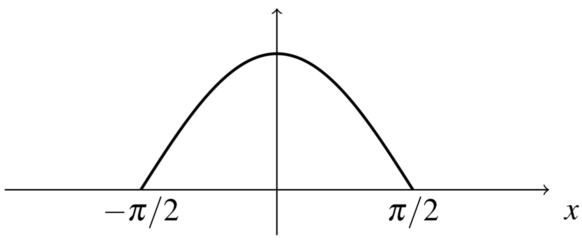
Now let $\omega' = \omega - u$, so that $u = \omega - \omega'$ and $du = -d\omega'$, the above equation now becomes

$$H(\omega) = \frac{1}{2\pi} \int_{x=-\infty}^{\infty} \left[\int_{\omega'=-\infty}^{\infty} f(x) \mathrm{e}^{-j\omega' x} dx \right] G(\omega - \omega') d\omega'$$

$$= \frac{1}{2\pi} \int_{\omega'=-\infty}^{\infty} F(\omega') G(\omega - \omega') d\omega'$$

as required. [6]

(d) The function $h(x)$ is a half cosine pulse between $-\pi/2$ and $\pi/2$, which is sketched below:



The FT of the pulse $g(x)$ is $G(\omega) = \pi \mathrm{sinc} \frac{\omega\pi}{2}$ (from Part (a)) and the FT of $f(x)$ is $F(\omega) = \pi [\delta(\omega - 1) + \delta(\omega + 1)]$

Thus using the result in part (c) we can convolve to give the FT of $h(x)$

$$H(\omega = \frac{1}{2\pi} \int_{\omega'=-\infty}^{\infty} \pi \left[\delta(\omega' - 1) + \delta(\omega' + 1) \right] \pi \mathrm{sinc} \frac{(\omega - \omega')\pi}{2} d\omega'$$

$$= \frac{\pi}{2} \left[\mathrm{sinc} \frac{(\omega - 1)\pi}{2} + \mathrm{sinc} \frac{(\omega + 1)\pi}{2} \right]$$

In the databook we have the following result for a half-cosine pulse of height a and width b centred on the origin:

$$\frac{ab}{2}\left[\text{sinc}\frac{(\omega b-\pi)}{2}+\text{sinc}\frac{(\omega b+\pi)}{2}\right]$$

which corresponds to the result above putting $a = 1$ and $b = \pi$. [8]

This was the most popular question in Section B. Part (a) was straightforward and was done easily by almost all candidates, though many thought that saying ' the form therefore agrees with the Databook' was a sufficient answer for the last part. Part (b) caused the most difficulty, with many students barely attempting it – luckily Part (c) did not depend on (b). Most candidates made a good attempt at Part (c), though there were many answers which made little sense – with a common mistake being lack of care with variables when substituting. Part (d) was generally well done, though the same comment as above re the databook applied.

5 (a) (i) The DFT is given by

$$X_k = \sum_{n=0}^{N-1} x_n e^{\frac{-2\pi ikn}{N}} \quad 0 \leq k \leq N-1$$

Here, N=4, so we evaluate $\{X_0, X_1, X_2, X_3\}$.
For $k = 0$:

$$X_0 = 1e^{\frac{-2\pi i0\times 0}{4}} + 0 - 1e^{\frac{-2\pi i0\times 2}{4}} + 0 = 1 - 1 = 2$$

For $k = 1$:

$$X_0 = 1e^{\frac{-2\pi i1\times 0}{4}} + 0 - 1e^{\frac{-2\pi i1\times 2}{4}} + 0 = 1 - e^{-i\pi} = 2$$

For $k = 2$:

$$X_0 = 1e^{\frac{-2\pi i2\times 0}{4}} + 0 - 1e^{\frac{-2\pi i2\times 2}{4}} + 0 = 1 - e^{-2i\pi} = 0$$

For $k = 3$:

$$X_0 = 1e^{\frac{-2\pi i3\times 0}{4}} + 0 - 1e^{\frac{-2\pi i3\times 2}{4}} + 0 = 1 - e^{-3i\pi} = 2$$

Therefore, our DFT sequence is $\{0, 2, 0, 2\}$ which is given in figure **??**:
The kth DFT component corresponds to a frequency of $\frac{k}{NT} = k/(4 \times 0.25) = k$, therefore the frequencies of X_n are (in Hz) $\{0, 1, 2, 3\}$. [5]

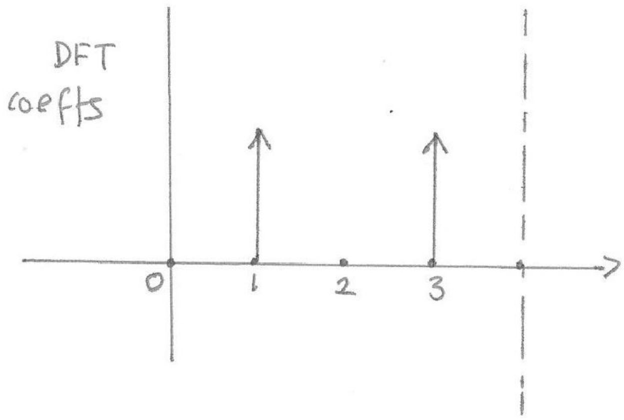


Fig. 3

(ii) We see from above that there are two non-zero terms, at 1Hz and 3Hz, in the DFT. The signal is a pure sinusoid of frequency 1Hz ($\omega = 2\pi$ so that $f = \omega/(2\pi) = 1$Hz). However we know that the DFT is the FT of the periodic repetition of the sample signal and therefore also gives a periodic discrete signal – of which we just take the first N components. The frequencies of ± 1Hz are therefore repeated at every interval of the sampling frequency, so that the -1Hz manifests itself as 3Hx in the 4 DFT components.

We also know that the second half of the DFT space is the complex conjugate of the first half of the space (can see this easily from the equation), so we know the 4th component has to be 2 if the second component is 2. [5]

(b) (i) From the Nyquist sampling theorem, we know that the minimum sampling rate required is twice the bandwidth:

$$2B = 2 \times 22\text{kHz} = 44000\text{samples per second}$$

[3]

(ii) For a sine wave taking values between $+V$ and $-V$ the RMS value is $V/\sqrt{2}$, ie the signal power is $V^2/2$. With an n-bit quantiser, we have 2^n levels and the stepsize is therefore

$$\Delta = \frac{2V}{2^n}$$

The quantisation noise power is

$$\frac{\Delta^2}{12} = \frac{V^2}{3} 2^{-n}$$

Therefore the SNR is given by

$$\frac{V^2/2}{V^2/(3 \times 2^n)} = 3 \times 2^{2n-1} \equiv 1.76 + 6.02n\text{dB}$$

(see lecture notes).

Therefore, we need $1.76 + 6.02n \geq 48$ or

$$n \geq (48 - 1.76)/6.02, \implies n \geq 7.68$$

Since n is an integer, we require a minimum of 8 bits per sample. [5]

(iii) We have 44000 samples/sec and each sample is represented by 8 bits. Therefore the bit rate of the digital stream is 8bit/sec = 352kbits/sec. Thus the minimum capacity of the communication link to support streaming is

352kbit/sec

[5]

Part (a) was reasonably well done but a disappointing number of candidates could not evaluate the DFT even for this very simple sequence. One surprisingly common mistake was to evaluate X_i with i from 1 to 4 rather than from 0 to 3. There were, however, many perfect answers to Part (a).

Part (b) was similarly well done. One common mistake was to use 20 to multiply the log in the dB expression (when dealing with powers). There were also a large number of algebraic errors (though not many marks were lost for this).

6 (a) (i)

$$s_1(t) = 10\cos(2\pi f_c t) + m(t))\cos(2\pi f_c t) = 10\cos(2\pi f_c t) + \frac{m(t)}{2}\left(\mathrm{e}^{j2\pi f_c t} + \mathrm{e}^{-j2\pi f_c t}\right)$$

so that

$$S_1(f) = 10\pi\left[\delta(f-f_c)+\delta(f+f_c)\right]+\frac{1}{2}\left[M(f-f_c)+M(f+f_c)\right]$$

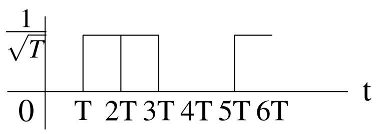
And if $s_2(t) = m(t)\cos(2\pi f_c t)$ we have

$$S_2(f) = \frac{1}{2}\left[M(f-f_c)+M(f+f_c)\right]$$

since if we shift in the frequency domain we multiply by a complex exponential in the time domain. [4]

(ii) $s_1(t)$ is amplitude modulation (AM) and $s_2(t)$ is double side-band suppressed carrier. $s)2(t)$ has smaller power (as we do not transmit the carrier) but needs more complex circuitry at the receiver to demodulate. $s_1(t)$ can be demodulated using a simple envelope detector. [4]

(b) (i) Given the definition of $X(t)$ above, the PAM waveform will look like



[4]

(ii) Know that the FT of a pulse width T, height $1/\sqrt{T}$ centred on the origin is

$$\sqrt{T}\ \mathrm{sinc}\frac{\omega T}{2} \equiv \sqrt{T}\ \mathrm{sinc}(\pi f T)$$

If we shift this pulse to the left by $T/2$ we get $p(t)$, which means that its FT is given by

$$\mathrm{e}^{j\pi f T}\sqrt{T}\ \mathrm{sinc}\pi f T$$

the magnitude of the spectrum(FT), $P(f)$, of $p(t)$ is shown in figure 3:

[5]

(iii) As $X_k \in \{0, A\}$, and $Y_k = X_k + N_k$, the optimal detector is

$$\hat{X}_k = \begin{cases} 0 & \text{if} \quad Y_k \le A/2 \\ A & \text{if} \quad Y_k > A/2 \end{cases}$$

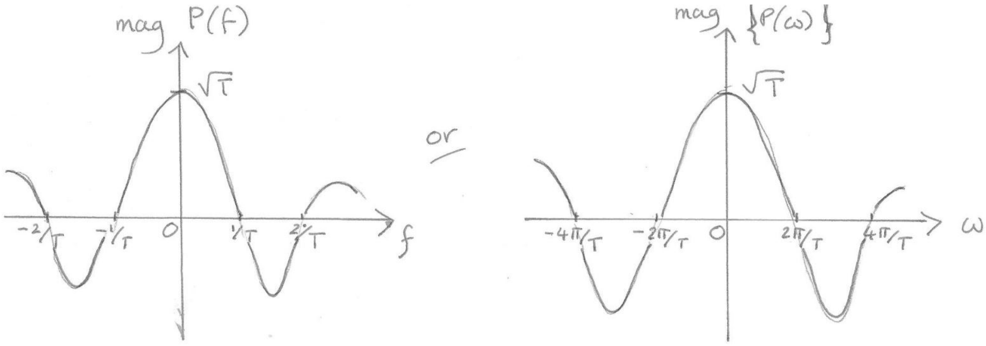


Fig. 4

The probability of detection error, P_e, is therefore

$$P_e = P(X_k = 0)P(\hat{X}_k \neq 0|X_k = 0) + P(X_k = A)P(\hat{X}_k \neq A|X_k = A)$$

$$= \frac{1}{2}P(Y_k > A/2|X_k = 0) + \frac{1}{2}P(Y_k \leq A/2|X_k = A)$$

$$= \frac{1}{2}P(N_k > A/2) + \frac{1}{2}P(N_k \leq -A/2)$$

$$= \frac{1}{2}P(N_k/\sigma > \frac{A}{2\sigma}) + \frac{1}{2}P(N_k/\sigma \leq -\frac{A}{2\sigma})$$

$$= \frac{1}{2}Q\left(\frac{A}{2\sigma}\right) + \frac{1}{2}Q\left(\frac{A}{2\sigma}\right) = Q\left(\frac{A}{2\sigma}\right)$$

This was the least popular question on the whole paper – perhaps because it looked long and was at the end of the paper! Those who did attempt it generally did reasonably well. Part (a) threw up a surprising number of candidates who could not find the spectra of the simple functions s_1 and s_2, though most had learned what these signals were and what were their advantages and disadvantages. Part (b) was done well with (ii) being the part that most marks were lost on. Though most people knew the spectrum was a sinc function, most did not realise that as we had a shifted pulse, the sinc was multipled by a complex exponential.

END OF SOLUTIONS

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