Version SH/5

EGT1
ENGINEERING TRIPOS PART IB

Friday 6 June 2014   2 to 4

Paper 7

MATHEMATICAL METHODS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

Answer not more than **two** questions from this section.

1 A directional point source, shown in Fig. 1, on the surface of a plane through the origin radiates with energy flux $\underline{I} = I_0 \dfrac{\cos\theta}{r^2} \underline{e}_r$, where I_0 is a constant, r is the distance from the source, $\underline{e}_r$ the unit vector in the radial direction, and θ is the angle measured from the normal to the plane. A flat disk of radius R is placed parallel to the plane at a distance H from the source, as shown in Fig. 1.

(a) By considering a hemisphere centred on the point source, find the total power radiated by the source away from the plane. [5]

(b) Find the power emitted over the part of the hemisphere that will be intercepted by the disk in terms of I_0, R and H. [5]

(c) Calculate the total power incident on the surface S of the flat disk, $\int \underline{I} \cdot d\underline{S}$, by direct integration. Show that this agrees with your calculation in Part (b), and justify the agreement using the divergence theorem. [15]

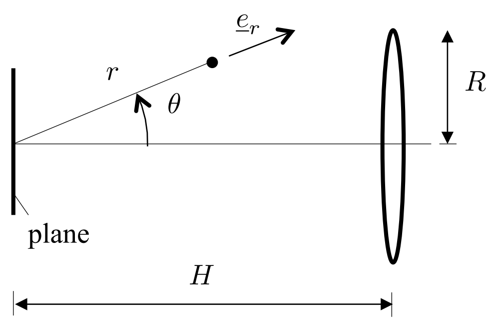

Fig. 1

2 The velocity, $\underline{V}$, of a two-dimensional flow is described in cylindrical coordinates as:

$$V_r = -\frac{Q}{2\pi r} \qquad V_\theta = \frac{\Gamma}{2\pi r}$$

where Q and Γ are positive constants.

(a) Determine the curl of the velocity field by differentiating the expression for $\underline{V}$. [3]

(b) Find $\oint \underline{V} \cdot d\underline{l}$ for the following paths:

(i) the path indicated in Fig. 2, [3]

(ii) a path consisting of a circle of radius $2a$ centred on the origin. [3]

(c) Using the co-ordinate free definition of curl, explain how you would extend the answer obtained in Part (a) to be compatible with your answers for Part (b). [3]

(d) Determine the divergence of the velocity field, ensuring that it is compatible with the application of Gauss' Theorem applied to circles encircling the origin. [5]

(e) Obtain a scalar potential for this velocity field. [4]

(f) Find and sketch the streamlines for this flow. [4]

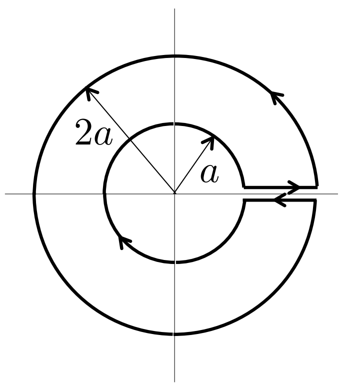

Fig. 2

3 The steady diffusion of heat in a square plate of an anisotropic material is represented by the following equation in two dimensions:

$$\lambda_x \frac{\partial^2 T}{\partial x^2} + \lambda_y \frac{\partial^2 T}{\partial y^2} = 0$$

where T is the temperature, λ_x and λ_y are the constant thermal conductivities for the material in the x and y directions, respectively. The boundary conditions are set as:

$$T(x,0) = T(x,L) = 0$$
$$T(-\frac{L}{2},y) = T(+\frac{L}{2},y) = \sin(\pi y/L)$$

where L is the length of the square.

(a) Using separation of variables, show that a solution to the problem is given by:

$$T(x,y) = A\cosh(\beta\pi x/L)\sin(\pi y/L)$$

where $A = 2\sinh(\frac{\pi\beta}{2})/\sinh(\pi\beta)$ and $\beta^2 = \lambda_y/\lambda_x$. [12]

(b) Determine the heat flux $\underline{q}$, defined as the negative of the product of the gradient of temperature and the conductivity in the respective direction. [3]

(c) Show that the divergence of the heat flux $\underline{q}$ is zero. [3]

(d) Sketch the solution for $\beta = 0$ and $\beta = 1$. [7]

SECTION B

Answer not more than **two** questions from this section.

4 A test has been devised for assessing whether the level of a drug in the blood stream is above a particular threshold. The mechanism, associated with the test, results in Gaussian distributed noise being added to the true value. The noise is known to have zero mean and variance σ^2.

(a) If the threshold for the drug is τ, and the true level is x, show that the probability that the measured level is above τ, $\mathscr{F}(x)$, is given by

$$\mathscr{F}(x) = \int_{-\infty}^{(x-\tau)/\sigma} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}z^2)\, dz$$

[5]

(b) The threshold is set at 0.8. If the true value is 0.7 and the value of the variance is 0.1, what is the probability that the test incorrectly classifies the level as being above the threshold? [2]

(c) To improve the accuracy of the result, the test is repeated five times. The noise for each test is independent of the noise for any other test.

(i) If the measured values from the five tests are averaged, what is the probability of an incorrect classification using the same values as Part (b)? [6]

(ii) Instead of averaging the values, the most frequent classification result from the five tests is used as the final classification result. What is the probability of the final classification result being incorrect in this case? Again the same values as Part (b) are to be used. [7]

(iii) Comment on the performance of the two approaches for combining the tests. How do you expect the difference in performance to change as the number of tests is decreased? [5]

5 A set of linear equations have been obtained from a design problem. These may be expressed in terms of target values, $\mathbf{y}$, the design matrix, $\mathbf{A}$ and the values to be found, $\mathbf{x}$, by the expression $\mathbf{y} = \mathbf{A}\mathbf{x}$. For the problem the expression has the following values:

$$\begin{bmatrix} 6 \\ 6 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 4 & a \\ 0 & 3 & 2 \end{bmatrix} \mathbf{x}$$

For any design the cost of the design is k times the magnitude of the vector $\mathbf{x}$. a is a variable that can be tuned.

(a) For a square matrix, what condition must be satisfied for the matrix to be invertible? [2]

(b) If $a = 0$, show that the design matrix is invertible and find the vector $\mathbf{x}$. What is the cost of this design? [5]

(c) The design is to be modified with the aim of decreasing the cost.

(i) What value of a results in the design matrix not being invertible? [4]

(ii) a is set to the value in Part (c)(i). By considering the null space of the design matrix, or otherwise, show that an expression for all possible values of $\mathbf{x}$ that will solve the set of linear equations has the form

$$\mathbf{x} = \mathbf{b} + \lambda \mathbf{c}$$

What are the values of the vectors $\mathbf{b}$ and $\mathbf{c}$? [8]

(iii) By optimising the value of λ, find the value of $\mathbf{x}$ that minimises the design cost, and find the cost of that design. [6]

6 The eigenvalues and vectors of a square matrix $\mathbf{A}$ are to be estimated. The matrix has the following form, where $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ are both square matrices and invertible,

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{bmatrix}$$

(a) What expression must be satisfied by an eigenvalue, λ, and eigenvector, $\mathbf{x}$, of $\mathbf{A}$? [2]

(b) An eigenvector of $\mathbf{A}$, $\mathbf{x}$, is partitioned as:

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{x}_2 \end{bmatrix}$$

where $\mathbf{x}_1$ and $\mathbf{x}_2$ have the same number of rows as $\mathbf{A}_{11}$ and $\mathbf{A}_{22}$ respectively. What can be stated about the eigenvalues of $\mathbf{A}$ when:

(i) $\mathbf{x}_2 \neq \mathbf{0}$; [4]

(ii) $\mathbf{x}_2 = \mathbf{0}$. [4]

Hence show that any eigenvalue of $\mathbf{A}$ must be an eigenvalue of $\mathbf{A}_{11}$ or $\mathbf{A}_{22}$ and state any restrictions on the eigenvectors associated with these eigenvalues. [3]

(c) The matrix $\mathbf{A}$ has the following values

$$\mathbf{A} = \begin{bmatrix} 5 & 1 & 4 & 6 \\ 1 & 5 & 3 & 9 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

(i) What are the eigenvalues of $\mathbf{A}$? [4]

(ii) Find an expression for the vector $\mathbf{y}$ where

$$\mathbf{y} = \mathbf{A}^n \begin{bmatrix} 1 \\ -1 \\ 4 \\ 2 \end{bmatrix}$$

as n tends to large values. The expression should be written in terms of n. [8]

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