EGT1
ENGINEERING TRIPOS PART IB

Friday 5 June 2015 2 to 4

Paper 7

MATHEMATICAL METHODS

Answer not more than ***four*** *questions.*

Answer not more than ***two*** *questions from each section.*

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Write your candidate number ***not*** *your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

1 (a) Sketch the region R in the first quadrant of the x, y plane which lies between the curves: $y = x^2$, $y = 2x^2$, $y = 1/x^2$ and $y = 2/x^2$. [3]

(b) Using the substitutions $u = x^2 y$ and $v = y/x^2$, sketch the equivalent region R in the (u, v) plane, and evaluate the integral

$$I = \int\int_R (y/x)\, dx\, dy$$

[15]

(c) Consider the vector field

$$\mathbf{B} = (y^2/x)\mathbf{i} + y^3\mathbf{j}$$

where $\mathbf{i}$ and $\mathbf{j}$ are the usual unit vectors. Using the result of part (b), evaluate the line integral

$$\Gamma = \oint_C \mathbf{B} \cdot d\mathbf{r}$$

where C is the curve which encloses R in an anticlockwise direction. [7]

2 Let $\mathbf{u} = \nabla g$, where $g = z^3(x^2 + y^2)$.

(a) Calculate $\mathbf{u}$ and $\nabla \cdot \mathbf{u}$, expressing the answers first in Cartesian coordinates, and then in cylindrical polar coordinates. [5]

(b) Evaluate the surface integral

$$\oint_S \mathbf{u} \cdot d\mathbf{A}$$

where S is the surface of the cylinder whose radius is unity, and whose axis runs from $z = -1$ to $z = 1$. [8]

(c) Evaluate the surface integral

$$\oint_S \nabla \times \mathbf{u} \cdot d\mathbf{A}$$

for the same surface. [4]

(d) Evaluate the line integral

$$\Gamma = \oint_C \mathbf{u} \cdot d\mathbf{r}$$

where C is the circle of unit radius defined by $z = 1$ and $x^2 + y^2 = 1$. [4]

(e) What is the scalar potential for the vector field $\mathbf{v} = g\mathbf{u}$? Discuss whether it is possible to find a vector potential for $\mathbf{u}$. [4]

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3 The pressure fluctuation p of a gas in a tube of length a is represented by the wave equation:

$$\frac{\partial^2 p}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} = 0$$

where c is the speed of sound in the tube. The tube is open at $x = a$, so that the pressure fluctuation at this point is zero. There is a loudspeaker at $x = 0$, which modulates the pressure according to $p_0(t) = p_0 \sin\Omega t$, where Ω is a given angular frequency.

(a) Show that a generic solution for the wave equation above is given by $f(x+ct) + g(x-ct)$. [4]

(b) Show by separation of variables that a generic harmonic solution to the problem is given by $p = (A\cos kx + B\sin kx)(C\cos\omega t + D\sin\omega t)$, where $k = \omega/c$. [5]

(c) Show that the solution for the given boundary conditions is given by:

$$p = (C\cos\omega_n t + D\sin\omega_n t)\cos(\omega_n x/c)$$

where $\omega_n = k_n c = \dfrac{c}{a}(2n+1)\dfrac{\pi}{2}$, where n is an integer. [5]

(d) Show that the full solution to the problem can be written as a series: [7]

$$p(x,t) = p_0 \sum_{n=0}^{\infty} (C_n \cos\omega_n t + D_n \sin\omega_n t)\cos(\omega_n x/c)$$

where

$$C_n = \frac{c}{2a}\int_0^{4a/c} \sin\Omega t \cos\omega_n t \, dt$$
$$D_n = \frac{c}{2a}\int_0^{4a/c} \sin\Omega t \, \sin\omega_n t \, dt$$

(e) What would the pressure field be, for a perturbation at the origin equal to a step function at time zero? Sketch the solution, and explain how it could be obtained by the technique of separation of variables. [4]

SECTION B

4 The following equation is to be solved

$$\mathbf{A}\begin{bmatrix} c \\ d \end{bmatrix} = \mathbf{v}$$

where $\mathbf{v}$ is a 3×1 column vector and

$$\mathbf{A} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \\ -2 & 0 \end{bmatrix}$$

(a) Explain how QR decomposition is used for solving least-squares problems. [3]

(b) Find the QR decomposition of $\mathbf{A}$ using Gram-Schmidt. [8]

(c) If

$$\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

what is the least-squares solution for c and d? [6]

(d) Show that for the vector $\mathbf{v}$ below to yield a least squared error of zero, it must lie on a plane. What is the equation of the plane? [8]

$$\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

5 The data received in a system is known to consist of two parts. The first is the signal, X, which has an integer value of zero or greater. The signal is Poisson distributed. In addition there is Gaussian distributed noise, Y, which is added to the signal. The noise has a mean of μ, and variance of σ^2. Thus the received signal, Z, can be written as

$$Z = X + Y$$

The distribution of the signal, X, noise, Y, and received signal Z can be written as $p_x(X)$, $p_y(Y)$ and $p_z(Z)$ respectively. The signal and the noise are independent.

(a) The moment generating function (MGF) for a Poisson distribution, $g_x(s)$, is

$$g_x(s) = \exp(-\lambda(1-s))$$

where λ is the parameter of the Poisson distribution. Using this MGF, derive the mean and variance of the Poisson distribution in terms of λ. [5]

(b) Derive an expression for the distribution of the received signal, $p_z(Z)$, in terms of $p_x(X)$ and $p_y(Y)$. [4]

(c) The moments of the received signal are required.

(i) Derive the MGF of a Gaussian distribution with mean, μ, and variance, σ^2. [7]

(ii) The MGF from the received signal can be written as

$$g_z(s) = \exp\left(s^2\sigma^2/2 - s\mu + \lambda(\exp(-s) - 1)\right)$$

Find the mean and variance of the received signal from the MGF (and not otherwise). [4]

(d) In order to estimate the original signal, X, the received signal is assigned to the closest integer value of zero or greater. The mean of the noise is known to be zero. Show that the the probability, P_{e} of an error in the estimate of X, can be expressed as

$$P_{\mathrm{e}} = b\,\Phi(a)$$

where $\Phi(v)$ is the cumulative probability distribution

$$\Phi(v) = \int_{-\infty}^{v} \frac{1}{\sqrt{2\pi}} \exp\left(-z^2/2\right) dz$$

Find expressions for a and b in terms of λ and σ. [5]

6 The following under-specified equation is to be solved

$$\mathbf{A}\mathbf{x} = \mathbf{y}$$

where $\mathbf{x}$ is a 4×1 column vector, $\mathbf{y}$ is a 3×1 column vector and

$$\mathbf{A} = \begin{bmatrix} 2 & 4 & 1 & 0 \\ 4 & 3 & z & 5 \\ 2 & 5 & 1 & -1 \end{bmatrix}$$

(a) By performing LU decomposition on the matrix $\mathbf{A}$, find the value of z that makes the rank of $\mathbf{A}$ equal to 2. [8]

(b) For the value of z in part (a), find the general solution for $\mathbf{x}$, when

$$\mathbf{y} = \begin{bmatrix} -4 \\ 2 \\ -6 \end{bmatrix}$$

[8]

(c) What is the *left null space* of the matrix $\mathbf{A}$ for the value of z in part (a)? [5]

(d) How would the solutions to parts (b) and (c) change if the value of z is not the same as the one derived in part (a)? It is not necessary to recompute the sub-spaces for the matrix. [4]

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Answers
Section A

1.
(b) $\frac{1}{4}$
(c) $-\frac{1}{2}$

2.
(a) $\mathbf{u} = 2z^3(x\mathbf{i} + y\mathbf{j}) + 3z^2(x^2 + y^2)\mathbf{k} = 2z^3(x\mathbf{i} + y\mathbf{j}) + 3z^2(x^2 + y^2)\mathbf{k}$
(b) 0
(c) 0
(d) 0
(e) $\pm\frac{1}{2}g^2$

3.

Section B
4.
(b)

$$\mathbf{Q} = \frac{1}{3}\begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & -2 \\ -2 & 2 & 1 \end{bmatrix};\ \mathbf{R} = \begin{bmatrix} 3 & 2 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

(c) $c = -19/9; d = 4/3$
(d) $2x - 2y + z = 0$

5.
(a) λ
(b) $\sum_{X=0}^{\infty} p_x(X)p_y(Z - X)$
(i) $\exp\left(s^2\sigma^2/2 - s\mu)\right)$
(ii) $\mu + \lambda,\ \sigma^2 + \lambda$
(d) $(2 - \exp(-\lambda))\Phi(-0.5/\sigma)$

6.
(a) 2

(b)

$$\mathbf{x} = \begin{bmatrix} 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + \alpha_1 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix} + \alpha_2 \begin{bmatrix} 0 \\ 1 \\ -4 \\ 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 7 \\ -1 \\ -5 \end{bmatrix}$$