EGT1
ENGINEERING TRIPOS PART IB

Thursday 7 June 2018 2 to 4.10

Paper 6

INFORMATION ENGINEERING: SOLUTIONS

*Answer not more than **four** questions.*

*Answer not more than **two** questions from each section.*

All questions carry the same number of marks.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

*Write your candidate number **not** your name on the cover sheet.*

STATIONERY REQUIREMENTS
Single-sided script paper, graph paper, semilog graph paper

SPECIAL REQUIREMENTS TO BE SUPPLIED FOR THIS EXAM
CUED approved calculator allowed
Engineering Data Book

10 minutes reading time is allowed for this paper at the start of the exam.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed to do so.

SECTION A

*Answer not more than **two** questions from this section.*

1 (a) Taking LT of both sides, using the formula for integration on the LT page in databook, we see that $\frac{\bar{y}}{\bar{u}} = G(s) = \frac{g}{s}$. Need to assume that $u(t) = 0$ for $t < 0$. [3]

(b) A first order lag with time constant $\tau = 0.2$s has a pole at $1/\tau$, and is therefore of the form $a/(1+\tau s)$. The steady-state gain is obtained by setting s to zero and equals a; so we must have $a = 1$. Conclusion: the transfer function of this sensor is

$$T(s) = \frac{1}{1+0.2s}$$

[3]

(c) A block diagram of the proposed control scheme is shown in Fig. 1 [2]

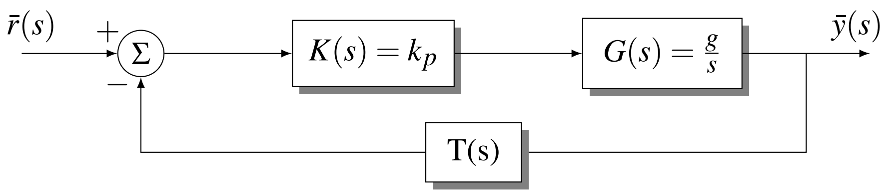


Fig. 1

(d) Due to the presence of an integral action in the loop, if the system stabilizes, then the steady-state error must be zero (indeed, if there were some residual steady-state error left, it would grow linearly through the integrator, contradicting (internal) stability of the loop). [3]

(e) Bode diagrams for $g = 1$ and $g = 100$ are given in Figure 2. For $g = 1$, the phase margin is very large, so the step response is expected to be non-oscillatory. For $g = 100$, the phase margin is small, indicative of a fairly poor damping of the step response, which will be transiently oscillatory. [8]

(f) As g increases, the gain curve shifts upwards, while the phase remains unaffected. Thus, increasing g decreases the phase margin, such that the worst case is obtained for $g = 100$. We can therefore look at the Bode diagram for $g = 100$, and check the gain (call it α) at the frequency where the phase is equal to $-180+20 = -160°$. k_p will need to be no larger than $1/\alpha$ to make sure the phase margin, in this worst case, remains above

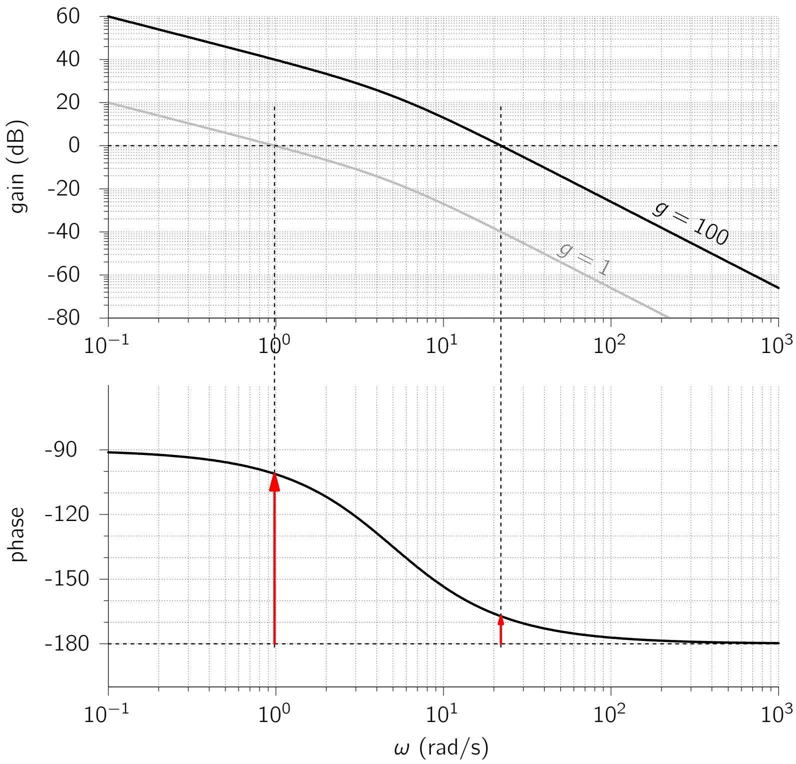


Fig. 2

20°. Reading α off the previous Bode diagram, we have $\alpha \approx 2.4$, and so we must have $k_p < 0.42$. [6]

2 (a) First, it cannot be $G_3(s)$. Indeed, the phase of $G_3(s)$ for large ω goes to $-\pi$, not $-3\pi/2$, whereas the Nyquist diagram provided approaches 0 for large ω from the top. While $G_1(s)$ has the right asymptotic phase, it has real poles only and thus cannot show any "bump" in modulus as a function of ω, whereas the Nyquist diagram provided clearly "moves away from the origin" past $\omega = 1.8$ rad/s. So it has to be $G_2(s)$. Further sanity check: the natural frequency of the second order term in the denominator of $G_2(s)$ is $\sqrt{9} = 3$ rad/s, so we expect the phase to have dropped by a further 90° at $\omega = 3$ rad/s below the initial $-90°$ phase lag imposed by the $1/s$ factor; this is indeed what we see, as the point labelled $\omega = 3$ rad/s lies on the real axis at a phase of $-180°$. [5]

(b) The exact answer is 1.35; students will probably have approximated it as $|G(j\omega_c)| = 1/0.75 = 1.333$, which is acceptable. [4]

(c) The relevant closed loop transfer function is

$$\frac{k_p G(s)}{1 + k_p G(s)}.$$

For $\omega = 3$ rad/s, we get the closed-loop gain as the ratio of: the distance of the Nyquist curve to the origin ($|k_p G(3j)|$ term) to the distance of the Nyquist curve to the -1 point (the $|1 + k_p G(3j)| = |k_p G(3j) - (-1)|$ term). The exact value is $(1.2/1.35)/(1 - 1.2/1.35) = 1.2/0.15 = 8$; students will probably have said

$$\frac{1.2 \times 0.75}{1 - 1.2 \times 0.75} = 9$$

which is acceptable. The phase lag can be computed similarly (easy here because both the numerator and denominator happen to be real for this particular $\omega = 3$ rad/s), and is $-180°$. Conclusion: $y(t) = 8\cos(3t - \pi) = -8\cos(3t)$. Thus, there is quite a bit of resonance at this frequency. (Students will probably have said $-9\cos(3t)$ if their estimate of the gain margin was $1/0.75 = 4/3$.) [6]

(d) Somewhat unusually, such resonance could not have been predicted by considering the phase margin. Indeed, the phase margin is almost 90° here (even after multiplying by $k_p = 1.2$, the only point that intersects the circle of unit radius centered at the origin is on the "vertical lower branch" for small ω), and this point would need to rotated by almost 90° clockwise to hit the -1 point on the real axis. This looked very safe! [4]

(e) We use the principle of superposition, as it applies to this linear system, to assemble the output from:

- the output for $r(t) = 1$ and $d(t) = 0$; this is $y(t) = 1$ too, because the presence of an integral term $1/s$ ensures a unit steady-state gain for constant inputs;
- the output due to $r(t) = 0$ and $d(t) = 0.2\cos(1.8t)$. Using the Nyquist diagram to estimate the gain and phase of $1 + k_p G$, we get a closed loop frequency response of $0.98\angle + 0.45$.

Conclusion: $y(t) = 1 + 0.2\cos(1.8t + 0.45)$. [6]

3 (a) By taking the Laplace transform of both equations and eliminating $\bar{f}(s)$, we get

$$\bar{r}(s) = \frac{\alpha}{s^2 + (\alpha^2 - 1)} \bar{u}(s)$$

so

$$G(s) = \frac{\alpha}{s^2 + (\alpha^2 - 1)}$$

For $\alpha < 1$, the population dynamics are unstable, as in this case there is a (repeated) real pole at $\sqrt{1-\alpha^2} > 0$. For $\alpha \geq 1$, there are two conjugate poles on the imaginary axis, at $\pm j\alpha$ – hence we have marginal stability. In this case, α determines the resonance frequency. [5]

(b) (i) The closed loop TF is

$$\frac{G(s)K(s)}{1+G(s)K(s)} = \frac{\alpha(k_p + k_d s)}{s^2 + \alpha k_d s + (\alpha^2 - 1 + \alpha k_p)}$$
$$= \frac{2(k_p + k_d s)}{s^2 + 2k_d s + (3 + 2k_p)}$$

[4]

(ii) For $k_d = 0$, the denominator remains of the form $s^2 + (3+2k_p)$, which still has imaginary conjugate poles, and therefore marginal stability. Thus, a proportional controller cannot stabilize the population dynamics. [4]

(iii) For $K(s) = k_p + s$, we have a closed loop TF given by

$$\frac{2(k_p + s)}{s^2 + 2s + (3 + 2k_p)}$$

which has moved the two poles away from the imaginary axis into the stable LHP. The steady state error is given by

$$\mathscr{E} = \frac{3}{3 + 2k_p}$$

which decreases as k_p increases. However, the damping ratio (found by identifying the denominator in the closed loop TF with the canonical form for a second order system) is

$$\zeta = \frac{1}{\sqrt{3 + 2k_p}}$$

For this to not go below $0.5 = 1/\sqrt{4}$, we must have $3 + 2k_p < 4$, resulting in a least achievable steady state error of $\mathscr{E} = 3/4 = 75\%$, which is fairly poor. [6]

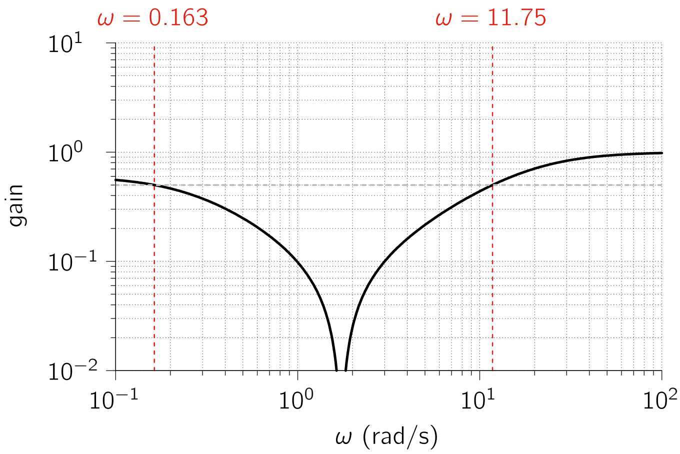


Fig. 3

(iv) The transfer function from the output disturbance to the output is given by the sensitivity function

$$S(s) = \frac{1}{1+K(s)G(s)} = \frac{s^2+3}{s^2+20s+5}$$

We want the range of ω_d for which $|S(j\omega_d)| < 0.5$ (see Fig. 3). Taking the squared modulus instead, and letting $y \equiv \omega_d^2$, we have the condition

$$\frac{(3-y)^2}{(5-y)^2+400y^2} < \frac{1}{4}$$

This is a simple quadratic inequality in y, which yields the following interval (don't forget to transform the obtained range of y into a range for ω_d!): $0.163 \le \omega_d \le 11.75$. [6]

SECTION B

*Answer not more than **two** questions from this section.*

4 (a) The FT and IFT are given by

$$F(\omega) = \int_{-\infty}^{+\infty} f(x)\mathrm{e}^{-j\omega x}dx \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{j\omega x}d\omega$$

Therefore:

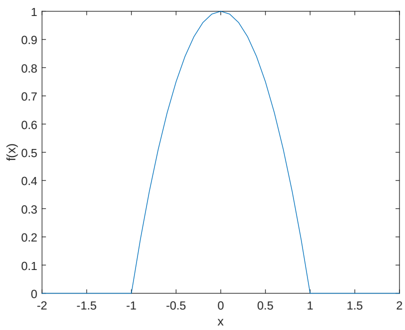
$$\begin{aligned} f(x-a) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\mathrm{e}^{j\omega(x-a)}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{F(\omega)\mathrm{e}^{-j\omega a}\}\mathrm{e}^{j\omega x}d\omega \end{aligned}$$

Showing that $f(x-a)$ has FT $\{F(\omega)\mathrm{e}^{-j\omega a}\}$.
Similarly, differentiating the IFT expression wrt x gives

$$\begin{aligned} \frac{df(x)}{dx} &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} F(\omega)\frac{d\mathrm{e}^{j\omega x}}{dx}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \{j\omega F(\omega)\}\mathrm{e}^{j\omega x}d\omega \end{aligned}$$

so that $\frac{df(x)}{dx}$ has FT $\{j\omega F(\omega)\}$. [4]

(b) $f(x)$ takes the following form:



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FT of this $f(x)$ needs to be found by direct integration:

$$F(\omega) = \int_{-1}^{+1} (1-x^2)\mathrm{e}^{-j\omega x} dx = \int_{-1}^{+1} \mathrm{e}^{-j\omega x} dx - \int_{-1}^{+1} x^2 \mathrm{e}^{-j\omega x} dx$$

Do the first integral:

$$\int_{-1}^{+1} \mathrm{e}^{-j\omega x} dx = \left[\frac{\mathrm{e}^{-j\omega x}}{-j\omega} \right]_{-1}^{+1} = \frac{\mathrm{e}^{j\omega} - \mathrm{e}^{-j\omega}}{j\omega} = 2 \ \mathrm{sinc}\omega$$

Second integral is done via intergration by parts:

$$\begin{aligned} \int_{-1}^{+1} x^2 \mathrm{e}^{-j\omega x} dx &= \left[\frac{x^2 \mathrm{e}^{-j\omega x}}{-j\omega} \right]_{-1}^{+1} + \frac{2}{j\omega} \int_{-1}^{+1} x\mathrm{e}^{-j\omega x} dx \\ &= 2 \ \mathrm{sinc}\omega + \frac{2}{j\omega} \int_{-1}^{+1} x\mathrm{e}^{-j\omega x} dx \end{aligned}$$

and

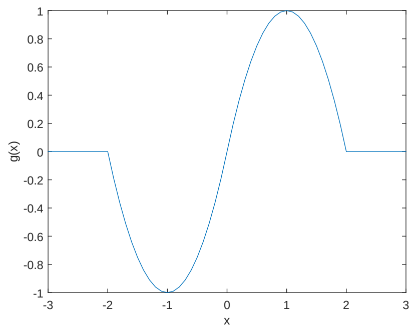
$$\begin{aligned} \int_{-1}^{+1} x\mathrm{e}^{-j\omega x} dx &= \left[\frac{x\mathrm{e}^{-j\omega x}}{-j\omega} \right]_{-1}^{+1} + \frac{1}{j\omega} \int_{-1}^{+1} \mathrm{e}^{-j\omega x} dx \\ &= \frac{\mathrm{e}^{j\omega} + \mathrm{e}^{-j\omega}}{-j\omega} + \frac{2 \ \mathrm{sinc}\omega}{j\omega} \\ &= \frac{2\cos\omega}{-j\omega} + \frac{2 \ \mathrm{sinc}\omega}{j\omega} \end{aligned}$$

Therefore

$$\begin{aligned} \int_{-1}^{+1} (1-x^2)\mathrm{e}^{-j\omega x} dx &= 2 \ \mathrm{sinc}\omega - 2 \ \mathrm{sinc}\omega + \frac{2}{j\omega} \left[\frac{2\cos\omega}{j\omega} - \frac{2 \ \mathrm{sinc}\omega}{j\omega} \right] \\ &= \frac{4}{\omega^2} [\ \mathrm{sinc}\omega - \cos\omega] \end{aligned}$$

Therefore $\alpha = 1$, $\beta = -1$ $p(\omega) = 4/\omega^2$, or other forms where the constants change (e.g. $\alpha = 4$, $\beta = -4$ $p(\omega) = 1/\omega^2$ etc) [10]

(c) The function $g(x)$ is sketched below:



From this sketch we can see that we can write $g(x)$ as

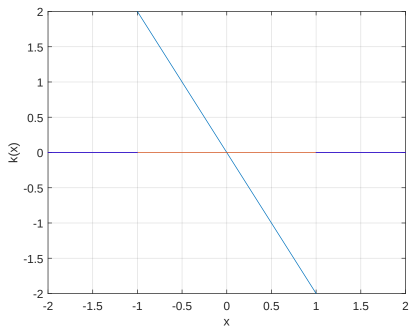
$$g(x) = f(x-1) - f(x+1)$$

Therefore, from the shift rule in part (a) we know that $G(\omega)$ is given by

$$\begin{aligned} G(\omega) &= F(\omega)\mathrm{e}^{-j\omega} - F(\omega)\mathrm{e}^{j\omega} \\ &= -2jF(\omega)\sin\omega \end{aligned}$$

[5]

(d) The function $k(x)$ is sketched below:



Note that $k(x) = \frac{df(x)}{dx}$, so by part (b) we know that

$$K(\omega) = j\omega F(\omega) = \frac{4j}{\omega}[\,\mathrm{sinc}\omega - \cos\omega]$$

Now do this by direct integration (using the results of (b)) :

$$\begin{aligned} K(\omega) &= -2\int_{-1}^{+1} x\mathrm{e}^{-j\omega x}dx \\ &= \frac{4\cos\omega}{j\omega} - \frac{4\ \mathrm{sinc}\omega}{j\omega} \\ &= \frac{4j}{\omega}[\,\mathrm{sinc}\omega - \cos\omega] \end{aligned}$$

Thus we see, as expected, that the two results are equal. [6]

5 (a) We know that if we are to have perfect reconstruction of the signal from its samples, we need to sample at greater than or equal to twice the highest frequency in the signal, ie $2f_B$. We also know that the spectrum of a sampled signal is the original spectrum repeated every interval of the sampling frequency and scaled by $1/T$:

$$F_s(\omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} F(\omega - n\omega_0)$$

Therefore, to extract the original spectrum (in the absence of aliasing) from this sampled spectrum, we need to filter multiply the above with a top hat function:

$$H_r(\omega) = \begin{cases} T, & -2\pi f_B < \omega < +2\pi f_B \\ 0 & \text{otherwise} \end{cases}$$

Multiplying in the frequency domain implies convolving in the time domain – so we therefore convolve our original sampled signal with the spectrum of the top hat function (and we need a factor of 2π). Therefore, our original signal is obtained by convolving/interpolating the sampled signal with a sinc function as given here:

$$f(t) = \sum_{-\infty}^{\infty} f(nT)\text{sinc}\left[\frac{\pi}{T}(t - nT)\right]$$

Therefore, we need to sample at the Nyquist rate ($T \leq 1/(2f_B)$) and our function $g(t)$ is then given by

$$g(t) = \text{sinc}\frac{\pi}{T}t$$

[5]

(b) The formula for the DFT of a set of samples $\{x_n\}$ $n = 0, 1, 2, 3$, is (with $N = 4$)

$$X_k = \sum_{n=0}^{N-1} x_n \mathrm{e}^{-jkn\frac{2\pi}{N}} \quad 0 \leq k \leq N-1$$

Therefore, we compute the DFT coefficients as follows (note that T is not needed for obtaining the samples but will be needed in finding the frequencies that the samples correspond to):

$$k = 0: \quad X_0 = \sum_{n=0}^{3} x_n \mathrm{e}^{-j0n\frac{2\pi}{4}} = \{-1+0+1+0\} = 0$$

$$k = 1: \quad X_1 = \sum_{n=0}^{3} x_n \mathrm{e}^{-j1n\frac{2\pi}{4}} = \{-1+0\mathrm{e}^{-j\pi/2}+1\mathrm{e}^{-j\pi}+0\mathrm{e}^{-j3\pi/2}\} = \{-1+0-1+0\} = -2$$

 (cont.

$$k=2: \quad X_2=\sum_{n=0}^{3} x_n e^{-j2n\frac{2\pi}{4}}=\{-1+0e^{-j\pi}+1e^{-j2\pi}+0e^{-j3\pi}\}=\{-1+0+1+0\}=0$$

$$k=3: \quad X_3=\sum_{n=0}^{3} x_n e^{-j3n\frac{2\pi}{4}}=\{-1+0e^{-j3\pi/2}+1e^{-j3\pi}+0e^{-j9\pi/2}\}=\{-1+0-1+0\}=-2$$

Therefore we have $\{X_n\} = [0, -2, 0, -2]$. [4]

(c) If $T = 2$ then the frequency of the kth Fourier component is given by $\frac{k}{NT} = \frac{k}{8}$. Since we have a non-zero frequency component at $k = 1$, this is $f = 1/8$ or $\omega = \pi/4$ (other component is just in the conjugate position). This would fit in with our signal taking the form

$$-\cos\frac{\pi}{4}t$$

..and we see that the components of this signal at $T = [0, 2, 4, 6]$ are indeed $[-1, 0, 1, 0]$. [4]

(d) (i) *Amplitude Modulation* (AM) is a form of analogue modulation in which information $x(t)$ modulates the amplitude of the carrier wave. If the carrier wave is $\cos(2\pi f_c t)$ then the transmitted AM signal is

$$s_{AM}(t) = [a_0 + x(t)]\cos(2\pi f_c t)$$

a_0 is a positive constant chosen so that $\max_t |x(t)| < a_0$. The information signal is extracted from the information signal via *envelope detection*.
Frequency Modulation (FM) is a form of analogue modulation in which the information signal, $x(t)$, modulates the *instantaneous frequency*, $f(t)$, of the carrier wave, ie $f(t)$ is varied linearly with $x(t)$:

$$f(t) = f_c + k_f x(t)$$

which gives an instantaneous phase, $\theta(t)$ of

$$\theta(t) = 2\pi f_c t + 2\pi k_f \int_0^t x(u)du$$

The modulated FM signal is therefore

$$s_{FM}(t) = A_c \cos\theta(t) = A_c \cos(2\pi f_c t + 2\pi k_f \int_0^t x(u)du)$$

FM signals are a little harder to demodulate – need a *differentiator and an envelope detector*. [3]

(ii) Spectrum of AM signal is

$$S_{AM(f)} = FT(s_{AM}(t)) = FT\left[[a_0 + x(t)]\frac{e^{j2\pi f_c t} + e^{-j2\pi f_c t}}{2}\right]$$
$$= \frac{a_0}{2}[\delta(f - f_c) + \delta(f + f_c)] + \frac{1}{2}[X(f - f_c) + X(f + f_c)]$$

where X is the FT of x. The bandwidth of an AM signal can be derived from the above form of the spectrum. If $x(t)$ is a baseband signal with (one-sided) bandwidth W, the AM passband signal will have bandwidth $2W$. The power of the AM signal is

$$P_{AM} = \frac{a_0^2}{2} + \frac{P_x}{2}$$

where P_x is the power of $x(t)$. This is derived by forming $\lim_{T\to\infty} \frac{1}{T}\int_0^T s_{AM}^2(t)dt$

Spectrum of FM signal is more complicated (but has been done in the examples sheets!) and is

$$S_{FM}(f) = \frac{A_c}{2}\sum_{n=-\infty}^{+\infty} J_n(\beta)[\delta(f - f_c - nf_x) + \delta(f + f_c + nf_x)]$$

where J_n is an nth order Bessel function of the first kind. The bandwidth of an FM signal is also rather complicated: we usually use *Carson's Rule* to give an effective bandwidth of

$$B_{FM} \approx 2\Delta f + 2W$$

where W is the bandwidth of $x(t)$ and Δf is the frequency deviation around f_c. Since an FM signal has constant carrier amplitude (only phase is changed), it will therefore have constant power dependent on this amplitude. [6]

(iii) For Analogue Modulation with information signal of bandwidth W we have:

- AM modulated signal: Bandwidth $2W$, high power, simple receiver using envelope detection.
- FM modulate signal: constant carrier amplitude and therefore constant power Bandwidth depends on both $\beta = \Delta f / f_x$ and W and can be significantly greater than 2W. FM has better robustness to noise than AM – since the information is "hidden" in the phase.

[3]

6 (a) (i) PAM is a digital baseband modulation technique which has two basic components: i) a mapping from bits to real/complex numbers
ii) a unit-energy baseband waveform denoted $p(t)$, called the pulse shape.

The basis of PAM is to modulate a baseband signal using this pulse shape, ie, if X_k are the values our bits are mapped to, then the modulated signal is given by

$$x_b(t) = \sum_k X_k p(t - kT)$$

..ie at each time step we shift the pulse and modulate its amplitude. [4]

(ii) For the 4-ary constellation, $[-3A, -A, A, 3A]$, we have :

$$[00] = -3A,\ \ [01] = -A,\ \ [10] = +A,\ \ [11] = +3A$$

Therefore our 12 digit sequence is split into a 6 element sequence as follows:

$$[-3A, -3A, A, A, -A, A]$$

Similarly, the 8-ary constellation maps via

$$[000] = -7A,\ \ [001] = -5A,\ \ [010] = -3A,\ \ [011] = -A$$

$$[100] = A,\ \ [101] = +3A,\ \ [110] = +5A,\ \ [111] = +7A$$

Therefore our 12 digit sequence is split into a 4 element sequence as follows:

$$[-7A, -3A, A, 5A]$$

[4]

(iii) The pulse shape $p(t)$ should be chosen to satisfy the following important objectives:
a) We want $p(t)$ to decay quickly in time, i.e., the effect of symbol X_k should not start much before $t = kT$ or last much beyond $t = (k+1)T$.
b) We want $p(t)$ to be approximately band-limited. For a fixed sequence of symbols $\{X_k\}$, the spectrum of $x_b(t)$ is

$$X_b(f) = FT\left[\sum_k X_k p(t - kT)\right] = P(f) \sum_k X_k \mathrm{e}^{-j2\pi f kT}$$

by the shift theorem. Therefore the bandwidth of our modulated baseband signal is the same as that of the pulse.

c) The retrieval of the information sequence from the noisy received waveform $x_b(t) + n(t)$ should be simple and relatively reliable. In the absence of noise, the symbols $\{X_k\}$ should be recovered perfectly at the receiver. For this to be the case it is helpful for the pulse to have the *orthonormal shifts* property, ie

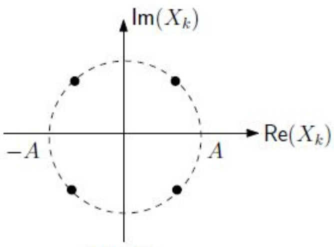
$$\int_{-\infty}^{+\infty} p(t-kT)p(t-mT)dt = 1 \; if \; k=m, \;\; 0 \; if \; k \neq m$$

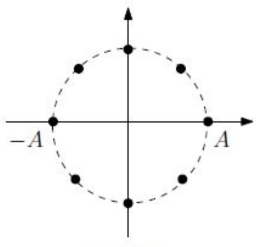
Two common unit energy pulse shapes that satisfy the above properties are the rectangular pulse

$$p(t) = \begin{cases} \frac{1}{\sqrt{T}} & \text{for } t \in (0,T] \\ 0 & \text{otherwise} \end{cases}$$

and the sinc pulse: $p(t) = \frac{1}{\sqrt{T}} \operatorname{sinc}\left(\frac{\pi t}{T}\right)$.
In practice a pulse shape having a raised cosine spectrum can also be used rather than the rectangular pulse. [4]

(b) (i) In PSK, the magnitude of the symbols $\{X_k\}$ is constant, with the information being in the *phase* of the symbol, as shown below





We will take $[p_1, p_2, p_3, p_4]$ to have phases of $[\pi/4, -\pi/4, -3\pi/4, 3\pi/4]$ for QSPK. For 8-PSK we take $[p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8]$ to have phases of $[0, -\pi/4, -\pi/2, -3\pi/4, \pi, 3\pi/4, \pi/2, \pi/4]$.
Therefore we have for QPSK:

$$p_1 = [00], p_2 = [01], p_3 = [10], p_4 = [11]$$

and for 8PSK

$$p_1 = [000], p_2 = [001], p_3 = [010], p_4 = [011], [p_5 = [100], p_6 = [101], p_7 = [110], p_8 = [111]$$

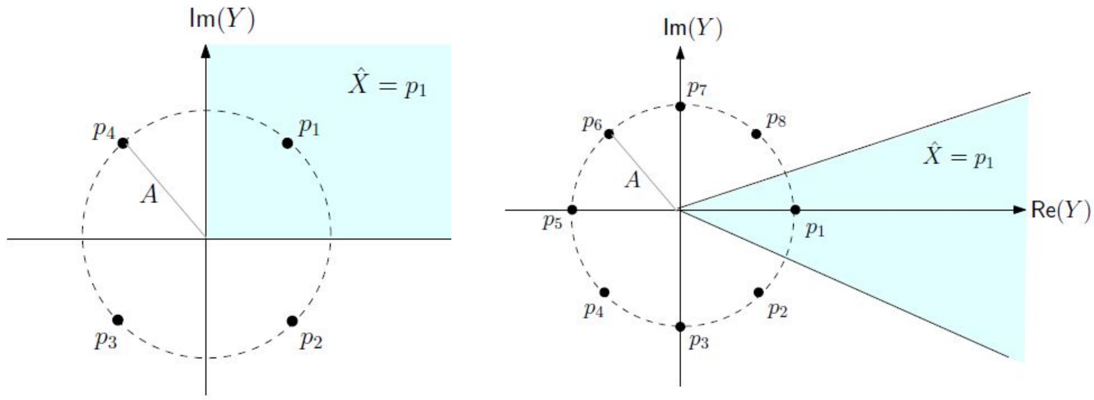
Therefore, for the 12 digit sequence in part(a)(ii) our mapping is:

$$QPSK: \;\; [Ae^{j\pi/4}, Ae^{j\pi/4}, Ae^{-j3\pi/4}, Ae^{-j3\pi/4}, Ae^{-j\pi/4}, Ae^{-j3\pi/4}]$$

$$8-PSK: \;\; [A, Ae^{-j\pi/2}, Ae^{j\pi}, Ae^{j\pi/2}]$$

[5]

(ii) The decision regions for the two schemes are shown below



[3]

(iii) Assume all constellation symbols are equally likely. If our observation is Y and our actual symbol is X, with noise N, then

$$Y = X + N$$

where X,Y,N are all complex numbers. If p_1 is the true mapping, we want $p(\hat{X} = p_1|X = p_1)$, which occurs if Y lies in the region shown in part(b)(ii). If $X = [X^r, X^i]$ etc, then $N = (Y - X)$ and we assume that each of the real and imaginary components are distributed as a gaussian ($\mu = 0, \sigma^2$), and that they are independent.

$$p(\hat{X} = p_1|X = p_1) = p(Y^r > 0 \text{ and } Y^i > 0|X = p_1)$$

since we assign p_1 if we observe Y to be in the first quadrant. Subsituting for $Y = X + N$ and using the fact that p_1 corresponds to $Ae^{j\pi/4} = A/\sqrt{2} + iA/\sqrt{2}$:

$$p(\hat{X} = p_1|X = p_1) = p(N^r > -\frac{A}{\sqrt{2}} \text{ and } N^i > -\frac{A}{\sqrt{2}})$$

Since N^r and N^i are independent we can write the above as

$$p(\hat{X} = p_1|X = p_1) = \left[\Phi\left(\frac{A}{\sigma\sqrt{2}}\right)\right]^2$$

[5]

END OF PAPER

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