

Q1 Vectors $(1\ 1\ 2)^t$ $(0\ 3\ 0)^t$ & $(3\ 1\ 1)^t$

So vectors in plane $(1\ -2\ 2)^t$ and $(3\ -2\ 1)^t$

So eq^n (1) for plane:

$$\underline{r} = (1\ 1\ 2)^t + \lambda(1\ -2\ 2) + \mu(3\ -2\ 1)^t.$$

(+ various others).

Normal to plane:

$$\begin{vmatrix} i & j & k \\ 1 & -2 & 2 \\ 3 & -2 & 1 \end{vmatrix} = (2\ 5\ 4)$$

$\therefore$ unit normal is $\pm \dfrac{(2\ 5\ 4)}{\sqrt{4+25+16}} = \dfrac{\pm 1}{\sqrt{45}}(2\ 5\ 4)$.

Eq^n (2) for plane: of form $\bar{r}.\bar{n} = s$

eg. $(1\ 1\ 2).(254) = 15$

or eq^n: $2x + \underline{5y} + 4z = 15.$

Angle between normal to plane & y axis $= \alpha$.

$\therefore$ " " plane & y axis $= 90 - \alpha$.

But $(0\ 1\ 0).\frac{1}{\sqrt{45}}(2\ 5\ 4) = \cos\alpha$.

$\therefore$ $\cos\alpha = \underline{\frac{5}{\sqrt{45}}}$ and requ^d $\angle$ is $90 - \cos^{-1}\frac{5}{\sqrt{45}}$.

Closest distance: eg. $(1\ 1\ 2).\frac{1}{\sqrt{45}}(2\ 5\ 4)$

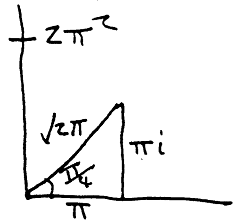
$= \frac{15}{\sqrt{45}}$

So point on plane $= \frac{15}{\sqrt{45}} \times \bar{n} = \frac{15}{\sqrt{45}}.\frac{1}{\sqrt{45}}(2\ 5\ 4)$

or $\frac{1}{3}(2\ 5\ 4)$

2(i) evaluate z where $(\sin^{-1} z)^2 = 2\pi^2 i$

$z = \sin\sqrt{2\pi^2 i}$



or $z = \sin \pm(\pi + \pi i)$

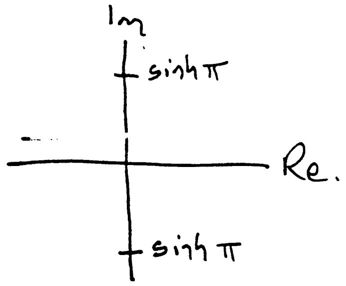
$= \cancel{\sin(\pm\pi)}_{0}\cos \pm\pi i$

$+ \cancel{\cos\pm\pi}_{-1} \sin\pm\pi i$

$= -1 \cdot \sin \pm\pi i$

$= -i\sinh \pm\pi$

$- \pm i \sinh\pi$



2(ii) evaluate $\lim_{x \to \frac{\pi}{2}} \dfrac{1 + e^{2ix}}{\tan^{-1}(x - \frac{\pi}{2})}$

Let $x = \frac{\pi}{2} + \delta \quad \therefore$ as $x \to \frac{\pi}{2}$, $\delta \to 0$.

$$\lim_{\delta \to 0} \frac{1 + e^{2i(\frac{\pi}{2} + \delta)}}{\tan^{-1}\delta} = \lim \frac{1 + e^{i\pi} \cdot e^{i2\delta}}{\tan^{-1}\delta} = \lim \frac{1 - e^{2i\delta}}{\tan^{-1}\delta}.$$

using data book series (or L'Hopital)

$$\lim_{\delta \to 0} \frac{1 - (1 + 2i\delta + O(\delta^2))}{\delta - O(\delta^3)} = -\frac{2i\delta}{\delta} = -2i$$

2(iii) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$

Taylor: $f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(x) + \dots$

or for $\sin(\frac{\pi}{4} + x) = \frac{1}{\sqrt{2}} + x\frac{1}{\sqrt{2}} + \frac{x^2}{2!} - \frac{1}{\sqrt{2}} + \frac{x^3}{3!} - \frac{1}{\sqrt{2}} + \dots$

$= \frac{1}{\sqrt{2}}\left[1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots\right]$

multiplying.

$$\frac{1}{\sqrt{2}} \times \Big[1 + x - \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$+ x + x^2 - \frac{x^3}{2} - \frac{x^4}{3!} + \dots$$

$$\frac{x^2}{2!} + \frac{x^3}{2!} - \frac{x^4}{2.2!} + \dots$$

$$+ \frac{x^3}{3!} + O(x^4)$$

$$= \frac{1}{\sqrt{2}}\left[1 + 2x + x^2 + O(x^4)\right].$$

3. $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}, \quad B = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix},$$

If $\theta = \phi = 45^\circ$

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

~~A~~ A followed by B = BA

$$= \frac{1}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{bmatrix}$$

$$\det BA = 1, \quad (BA)^{-1} = (BA^T)$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \\ -1 & -\sqrt{2} & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} + \sqrt{2} \\ -3 + 2\sqrt{2} + 1 \\ -3 - 2\sqrt{2} + 1 \end{bmatrix} = \begin{bmatrix} 4\sqrt{2} \\ 2\sqrt{2} - 2 \\ -2\sqrt{2} - 2 \end{bmatrix}$$

$$AB = \frac{1}{2} \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & -1 \\ 1 & \sqrt{2} & 1 \end{bmatrix}$$

AB is different to BA, but det AB still $= 1$ and $(AB)^{-1} = (AB)^T$ again.

4. Solve: $\frac{1}{2}\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 4e^{-x}\sin x$

c.f. $\lambda = 1 \pm i$, so $y = e^{x}(A\sin x + B\cos x)$

P.I. try $y = e^{-x}(a\sin x + b\cos x)$

$$\frac{dy}{dx} = -e^{-x}(a\sin x + b\cos x) + e^{-x}(a\cos x - b\sin x)$$

$$\frac{d^2y}{dx^2} = \cancel{e^{-x}(a\sin x + b\cos x)} - e^{-x}(a\cos x + b\sin x)$$

$$- e^{-x}(a\cos x - b\sin x) - \cancel{e^{-x}(a\sin x + b\cos x)}$$

$$= -2e^{-x}(a\cos x - b\sin x)$$

$$\text{L.H.S} = -e^{-x}(a\cos x - b\sin x) + e^{-x}(a\sin x + b\cos x)$$

$$- e^{-x}(a\cos x - b\sin x) + e^{-x}(a\sin x + b\cos x)$$

$$= 2e^{-x}(a\sin x + b\cos x) - 2e^{-x}(a\cos x - b\sin x)$$

$\text{R.H.S} = 4e^{-x}\sin x$

Equating coeff's of $e^{-x}\sin x$

$2a + 2b = 4$

Equating coeff's of $e^{-x}\cos x$

$2b - 2a = 0$

$\Rightarrow b = a = 1.$

General Sol^n $y = e^{x}(A\sin x + B\cos x) + e^{-x}(\sin x + \cos x)$

But $y = 1$, $\frac{dy}{dx} = 0$ when $x = 0$

So: $(x=0)$ $1 = 1.B + 1.1 \Rightarrow B = 0$

and $\frac{dy}{dx} = e^{x}A\sin x + e^{x}A\cos x - e^{-x}(\sin x + \cos x)$
$+ e^{-x}(\cos x - \sin x)$

$(x=0)$ $0 = 0 + A - \cancel{1.1} + \cancel{1.1}$

$\therefore A = 0.$

$\therefore y = e^{-x}(\sin x + \cos x)$

When $x = \pi$, $y = -e^{-\pi}$

Q5 Series $1, 2, 2, 1, 6, -3, \ldots$ is described by a difference equation of the form:

$$a_n = Pa_{n-2} + Qa_{n-4}$$

Find P, Q, gen. sol^n

Two independent series:

odd $1, 2, 6, \ldots$

even $2, 1, -3, \ldots$

Let $m = \frac{n+1}{2}$ (n odd), $m = \frac{n}{2}$ (n even)

$$\therefore a_m = Pa_{m-1} + Qa_{m-2}$$

odd: $6 = 2P + Q$ —— (1)

even: $-3 = P + 2Q$ —— (2)

(1) $- 2\times$(2)

$12 = -3Q$ or $Q = -4$, $P = 5$.

$$\Rightarrow a_m = 5a_{m-1} \cancel{+} -4a_{m-2}$$

Let $a_m = \lambda^m$

$$\therefore \lambda^2 - 5\lambda + 4 = 0$$

or $(\cancel{\ }\lambda - 1)(\lambda - 4) = 0$

Odd series: $a_m = A.1^m + B.4^m$

$m = 1$ $\quad 1 = A + B.4$

$m = 2$ $\quad 2 = A + B.16$ $\Big\}$ $12B = 1$, $B = \frac{1}{12}$

and $1 = A + \frac{4}{12}$ so $A = \frac{2}{3}$.

$$\therefore a_m = \frac{2}{3} + \frac{1}{12}.4^m \text{ (odd terms)}$$

Even series $a_m = C.1^m + D.4^m$

$m = 1$, $\quad 2 = C + 4D$ — (3)

$m = 2$, $\quad 1 = C + 16D$ — (4)

(3) $-$ (4): $\quad 1 = -12D$ or $D = -\frac{1}{12}$

$2 = C - \frac{1}{3}$ or $C = \frac{7}{3}$.

$$\therefore a_m = \frac{7}{3} - \frac{1}{12}.4^m$$

Check, odd series, $m = 3$

$a_m = \frac{2}{3} + \frac{64}{12} = \frac{2}{3} + \frac{16}{3} = 6$ ✓✓

Check, even series, $m = 3$

$a_m = \frac{7}{3} - \frac{1}{12}.64 = \frac{7}{3} - \frac{16}{3} = -3$ ✓✓

6. From $A\bar{U}_1 = \lambda_1 U_1$

Then $A \begin{bmatrix} U_{1a} & U_{2a} & U_{3a} \\ U_{1b} & U_{2b} & U_{3b} \\ U_{1c} & U_{2c} & U_{3c} \end{bmatrix} = \begin{bmatrix} \lambda_1 U_{1a} & \lambda_2 U_{2a} & \lambda_3 U_{3a} \\ \lambda_1 U_{1b} & \lambda_2 U_{2b} & \lambda_3 U_{3b} \\ \lambda_1 U_{1c} & \lambda_3 U_{2c} & \lambda_3 U_{3c} \end{bmatrix}$

or $A\bar{U} = \bar{U}\Lambda$.

Using data book.

$$\bar{U} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$$

So

$$A = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & -1 \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \frac{a}{\sqrt{2}} & \frac{a}{\sqrt{2}} & 0 \\ 0 & 0 & b \\ \frac{c}{\sqrt{2}} & -\frac{c}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{2}+\frac{c}{2} & \frac{a}{2}-\frac{c}{2} & 0 \\ \frac{a}{2}-\frac{c}{2} & \frac{a}{2}+\frac{c}{2} & 0 \\ 0 & 0 & b \end{bmatrix}$$

Inverse, eigenvectors same, eigenvalues $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$

If $a=3, b=2, c=1$

$$\bar{A} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad \bar{A}^{-1} = \begin{bmatrix} \frac{1}{6}+\frac{1}{2} & \frac{1}{6}-\frac{1}{2} & 0 \\ \frac{1}{6}-\frac{1}{2} & \frac{1}{6}+\frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & +\frac{2}{3} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

Multiply $\bar{A}\bar{A}^{-1}$:

$$= \begin{bmatrix} \frac{4}{3}-\frac{1}{3} & -\frac{2}{3}+\frac{2}{3} & 0 \\ \frac{2}{3}-\frac{2}{3} & -\frac{1}{3}+\frac{4}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

7 (soln.)

(a) P.I. $y = 1$ $\qquad$ aux. eqn. $\lambda + 1/\alpha = 0$

C.F. $y(t) = A e^{-t/\alpha}$

$\Rightarrow$ gen. soln. $y(t) = 1 + A e^{-t/\alpha}$

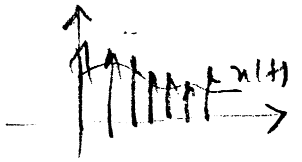
$y(t)$ must be continuous at $t=0$, otherwise $\dot{y}$ has an impulse and equation does not hold. Thus

$y(0^+) = y(0^-) = 0 \Rightarrow A = -1 \Rightarrow$ step response $= 1 - e^{-t/\alpha}$.

(b) Impulse response $= d/dt$ (step response) $= e^{-t/\alpha} \cdot \frac{1}{\alpha}$

(c) $x(t)$ can be approximated by an impulse chain:

$$x(t) = \sum_{k=0}^{\infty} x(kT)\,\delta(t-kT)\,T$$



which gives a response (since system is linear)

$$y(t) = \sum_{k=0}^{\infty} x(kT)\,T\,g(t-kT)$$

Letting $T \to 0$ (and $kT \to \tau$, $T \to d\tau$) gives

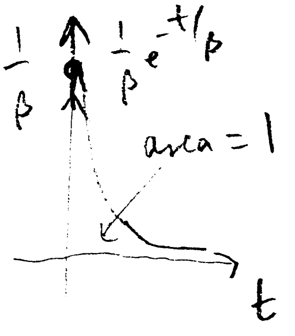
$$y(t) = \int_0^{\infty} x(\tau)\,g(t-\tau)\,d\tau = \int_0^{t} x(\tau)\,g(t-\tau)\,d\tau$$

since $g(t) = 0$ for $t < 0$.

(d)
$$y(t) = \frac{1}{\alpha\beta}\int_0^t e^{-\tau/\beta}\, e^{-(t-\tau)/\alpha}\,d\tau$$

$$= \frac{e^{-t/\alpha}}{\alpha\beta}\int_0^t e^{+\tau(\alpha^{-1}-\beta^{-1})}\,d\tau = \frac{e^{-t/\alpha}}{\alpha\beta}\left[e^{+\tau(\alpha^{-1}-\beta^{-1})}\right]_0^t \frac{1}{\alpha^{-1}-\beta^{-1}}$$

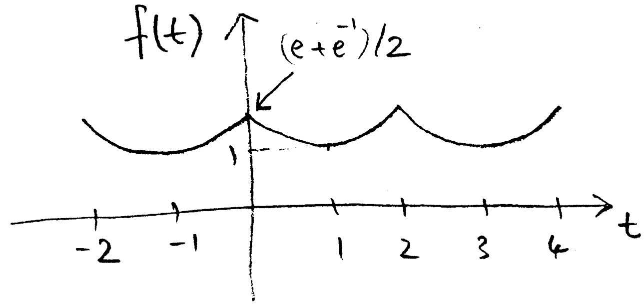
$$= \frac{e^{-t/\alpha}}{\beta-\alpha}\left\{e^{+t(\alpha^{-1}-\beta^{-1})}-1\right\} = \frac{1}{\beta-\alpha}\left\{e^{-t/\beta}-e^{-t/\alpha}\right\}$$



$$= \frac{e^{-t/\alpha} - e^{-t/\beta}}{\alpha-\beta}.$$

Tends to $\frac{e^{-t/\alpha}}{\alpha}$ = impulse response as $\beta \to 0$.

$x(t)$ behaves like an impulse in the limit.



8 (soln.)

$T = 2 \Rightarrow w_0 = \pi$

even function

Elec. and Inf. Data Book $\Rightarrow$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi t)$$

where

$$a_n = \frac{2}{T}\int_{-1}^{1} f(t)\cos(n\pi t)\,dt = \frac{4}{T}\int_0^1 f(t)\cos(n\pi t)\,dt$$

$$= 2\int_0^1 \overset{dv}{\cosh(t-1)}\ \overset{u}{\cos(n\pi t)}\,dt$$

$$= 2\Big[\sinh(t-1)\cos n\pi t\Big]_0^1 + 2n\pi\int_0^1 \overset{dv}{\sinh(t-1)}\ \overset{u}{\sin(n\pi t)}\,dt$$

$$= -2\sinh(-1) + 2n\pi\left\{\cancel{\Big[\cosh(t-1)\sin(n\pi t)\Big]_0^1} - \int_0^1 \cosh(t-1).\,n\pi\cos(n\pi t)\right.$$

$$= 2\sinh(1) - n^2\pi^2 a_n$$

$$\Rightarrow a_n = \frac{2\sinh(1)}{1+n^2\pi^2}$$

Hence $$f(t) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\cos(n\pi t)$$

Setting $t=0 \Rightarrow$ $$\cosh(-1) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}$$

$$\Rightarrow \frac{\cancel{e}+e^{-1}}{2} = \frac{\cancel{e}-e^{-1}}{2} + (e-e^{-1})\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2} = \frac{1}{e^2-1}$$

Taking Laplace transforms:

$$sX - 1 + 2Y = \frac{1}{s}$$

$$sY - 2X = \frac{e^{-3s}}{s}$$

$$\Rightarrow \quad s^2X + 2sY = s + 1$$

$$\Rightarrow \quad s^2X + 2\left(2X + \frac{e^{-3s}}{s}\right) = s + 1$$

$$\Rightarrow \quad (s^2+4)X = s+1 - \frac{2e^{-3s}}{s}$$

$$\Rightarrow \quad X = \frac{s+1}{s^2+4} - \frac{2e^{-3s}}{s(s^2+4)}$$

$$\frac{1}{s(s^2+4)} = \frac{\frac{1}{4}}{s} + \frac{As+B}{s^2+4} \quad : \quad 1 = \frac{1}{4}(s^2+4) + s(As+B)$$

$$\Rightarrow \quad B = 0, \quad A = -\frac{1}{4}$$

$$\Rightarrow \quad X = \frac{s+1}{s^2+4} - 2e^{-3s}\left(\frac{\frac{1}{4}}{s} \quad \frac{-\frac{1}{4}s}{s^2+4}\right)$$

$$= \frac{s+1}{s^2+4} - \frac{1}{2}e^{-3s}\cdot\frac{1}{s} + \frac{1}{2}\frac{se^{-3s}}{s^2+4}$$

$$\Rightarrow \quad x(t) = \frac{1}{2}\sin 2t + \cos 2t - \frac{1}{2}h(t-3) + \frac{1}{2}\cos(2(t-3))\,h(t-3)$$

10 (soln.)

If $df = P\,dx + Q\,dy$ then $P = \frac{\partial f}{\partial x}$ and $Q = \frac{\partial f}{\partial y}$ by the chain rule. Equality of mixed 2nd partial derivatives implies

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

$$\frac{\partial P}{\partial y} = \frac{-\alpha}{y^2}, \quad \frac{\partial Q}{\partial x} = y^\beta$$

Hence we require $\alpha = -1$, $\beta = -2$.

$$\frac{\partial f}{\partial x} = \frac{1}{x^2+2} - \frac{1}{y}$$

$$\Rightarrow f = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + f_1(y) \qquad (1)$$

$$\frac{\partial f}{\partial y} = \frac{x}{y^2} + 1$$

$$\Rightarrow f = -\frac{x}{y} + y + f_2(x) \qquad (2)$$

(1) and (2) together imply that f should take the form

$$f(x,y) = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) - \frac{x}{y} + y + c$$

where c is constant.

Steepest ascent is in direction $\nabla f = (P, Q)$.

At $(1,1)$: $\quad \nabla f = \left(-\frac{2}{3}, 2\right)$.