ENGINEERING TRIPOS PART IA

Monday 10 June 1996 9 to 12

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **five** *may be taken from Section A and not more than* **three** *from Section B.*

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

Answer not more than **five** *questions from this section.*

1 A plane passes through the points with position vectors $\begin{pmatrix} 1 & 1 & 2 \end{pmatrix}^t$, $\begin{pmatrix} 0 & 3 & 0 \end{pmatrix}^t$ and $\begin{pmatrix} 3 & 1 & 1 \end{pmatrix}^t$. Give the vector equation for this plane in two different forms. Also give the equation of the unit normal to the plane.

What is the angle between the plane and the y axis? What is the closest point on the plane to the origin, and how far is it from the origin?

2 (a) Evaluate z where $\left(\sin^{-1} z\right)^2 = 2\pi^2 i$, and show all solutions on an Argand diagram.

(b) Evaluate

$$\lim_{x \to \frac{\pi}{2}} \frac{1 + e^{2ix}}{\tan^{-1}\left(x - \frac{\pi}{2}\right)} .$$

(c) Express $e^x \sin\left(\frac{\pi}{4} + x\right)$ as a power series in x, neglecting terms of $O\left(x^4\right)$.

3 Given a right handed set of axes x, y, z write down the matrix which represents a transformation **A** of rotation θ about the x axis. Also give the matrix which represents the transformation **B** of rotation $-\phi$ about the y axis. Rotations are taken as positive if they appear clockwise when viewed outwards along the positive axis in question.

If θ and ϕ are both $45°$, calculate the matrix which represents **A** followed by **B**. What are the determinant and inverse of this matrix?

Point $(x,\ y,\ z)$ is subjected to **A** then **B** and finishes at $(3, 2, 1)$. What are x, y, and z ? Calculate the matrix representing **B** followed by **A** and comment on your result.

4 Find the general solution of:

$$\frac{1}{2}\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 4e^{-x}\sin x\ .$$

If $y = 1$ and $\dfrac{dy}{dx} = 0$ when $x = 0$, find y when $x = \pi$.

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5 A difference equation of the form:

$$a_n = Pa_{n-2} + Qa_{n-4}$$

results in the sequence with first six terms, starting with a_0,

$$1,\ 2,\ 2,\ 1,\ 6,\ -3,\ \ldots\ .$$

Use the initial terms of the sequence to find P and Q. Hence find a general solution for a_n in terms n.

6 Show that:

$$\mathbf{AU} = \mathbf{U\Lambda}$$

where $\mathbf{\Lambda}$ is the diagonal matrix whose elements are the eigenvalues of $\mathbf{A}$, and $\mathbf{U}$ is the matrix whose columns are the normalised eigenvectors of $\mathbf{A}$.

Construct the real symmetric 3×3 matrix $\mathbf{A}$ which has eigenvectors $\begin{pmatrix} 1 & 1 & 0 \end{pmatrix}^t$, $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^t$ and $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^t$ and eigenvalues $a,\ b,\ c$.

What is the relationship between the eigenvectors and eigenvalues of $\mathbf{A}$ and $\mathbf{A}^{-1}$? Hence calculate $\mathbf{A}^{-1}$ if $a = 3,\ b = 2$ and $c = 1$. Verify that your result is indeed $\mathbf{A}^{-1}$.

SECTION B

Answer not more than **three** *questions from this section.*

7 A linear system with input $x(t)$ is governed by the equation

$$\alpha \dot{y} + y = x.$$

(a) *Without using Laplace transforms*, find the step response of the system, assuming $y(t) = 0$ for $t < 0$.

(b) Find the impulse response, $g(t)$, of the system.

(c) Explain why the response to a general input $x(t)$ is given by the convolution integral

$$y(t) = \int_0^t x(\tau) g(t - \tau) d\tau.$$

(d) Find the response to an input

$$x(t) = \begin{cases} \frac{1}{\beta} e^{-t/\beta} & t \geq 0, \\ 0 & t < 0, \end{cases}$$

where $\beta < \alpha$. What happens to the response as $\beta \to 0$? Comment briefly on your result.

8 An even function $f(t)$ is periodic with period $T = 2$, and $f(t) = \cosh(t-1)$ for $0 \leq t \leq 1$. Sketch $f(t)$ in the range $-2 \leq t \leq 4$. Find a Fourier series representation for $f(t)$.

Deduce that:

$$\sum_{n=1}^{\infty} \frac{1}{1 + n^2 \pi^2} = \frac{1}{e^2 - 1}.$$

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9 Consider a system described by the simultaneous differential equations

$$\begin{aligned} \dot{x}(t) + 2y(t) &= h(t), \\ \dot{y}(t) - 2x(t) &= h(t-3), \end{aligned}$$

where $h(t)$ is the unit step function, $x(0^-) = 1$ and $y(0^-) = 0$. Using Laplace transforms, *and not otherwise*, determine $x(t)$.

10 Derive a condition for

$$P(x, y)dx + Q(x, y)dy$$

to be a perfect differential.

For

$$\begin{aligned} P(x, y) &= \frac{1}{x^2 + 2} + \frac{\alpha}{y}, \\ Q(x, y) &= xy^\beta + 1, \end{aligned}$$

find values of the constants α and β for which $Pdx + Qdy = df$ is a perfect differential, and find a corresponding function $f(x, y)$.

What is the direction of steepest ascent for $z = f(x, y)$ at the point $(1, 1)$?

END OF PAPER