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ENGINEERING TRIPOS PART IA 1997

PAPER 1: MECHANICAL ENGINEERING

SOLUTIONS

Q1 (a) $T\,ds = dh - v\,dp$

$= c_p dT - v\,dp$ as $dh = c_p dT$

$\therefore ds = c_p \frac{dT}{T} - \frac{v}{T} dp$

$pv = RT \Rightarrow \frac{v}{T} = \frac{R}{p}$

if in quasi-equilibrium

$\therefore ds = c_p \frac{dT}{T} - R\frac{dp}{p}$

$$\therefore s_2 - s_1 = c_p \ln\frac{T_2}{T_1} - R\ln\frac{p_2}{p_1}$$

This is also valid for an irreversible process because it is an equation in properties. The change in properties is independent of the path taken between given states.

(b) (i) Continuity: $\rho_A V_A A_A = \rho_B V_B A_B \Rightarrow \frac{A_A}{A_B} = \frac{\rho_B}{\rho_A}\cdot\frac{V_B}{V_A}$

As T is constant $p = \rho RT \Rightarrow \frac{\rho_B}{\rho_A} = \frac{p_B}{p_A}$

$$\therefore \frac{A_A}{A_B} = \frac{p_B}{p_A}\cdot\frac{V_B}{V_A} = \frac{1.8}{1.5}\times\frac{300}{400} = 0.90$$

(ii) SFEE: $Q = \cancel{W_x} + \cancel{(h_B - h_A)} + \frac{1}{2}(V_B^2 - V_A)^2 + \cancel{g(z_B - z_A)}$

no shafts T const. horizontal pipe

$$\therefore Q = \frac{1}{2}(V_B^2 - V_A^2) = \frac{1}{2}(300^2 - 400^2) = -35\ \mathrm{kJ/kg}$$

Q1 (cont)

(iii) $\Delta s = s_B - s_A = \cancel{C_p \ln \frac{T_B}{T_A}} - R \ln \frac{P_B}{P_A} = -287 \ln \frac{1.8}{1.5}$

$T_B = T_A$

$= \underline{\underline{-52.3 \ J/kgK}}$

To establish direction of flow use the 2nd Law $\Delta s \geq \int \frac{dQ}{T}$

As flow is isothermal $\int \frac{dQ}{T} = \frac{Q}{T}$

If flow is from A → B $\Delta s > \frac{Q}{T}$

$$\frac{Q}{T} = \frac{-35 \times 10^3}{400} = -87.5 \ J/kg\,K$$

From (iii) $\Delta s = -52.3 \ J/kg\,K$

Hence the inequality is satisfied ⇒ <u>flow is from A to B</u>

Completely spurious arguments:

Flow is from B to A as entropy increases that way

Flow is from B to A as pressure decreases that way

Q2 (a) Pressure and temperature are <u>not</u> independent in the saturation region, so in this region measurements of p and T are not sufficient to establish the thermodynamic state.

(i) For the throttle the SFEE gives

$$\cancel{Q} = \cancel{W_x} + (h_2 - h_1) + \cancel{\tfrac{1}{2}(V_2^2 - V_1^2)} + \cancel{g(z_2 - z_1)}$$

adiabatic no shafts negligible horizontal

$\therefore \quad h_2 = h_1$

From Table 9 $\quad h_2 = 2683$ kJ/kg

$\therefore \quad h_1 = 2683$ kJ/kg

(ii) $$h_1 = x h_g + (1-x) h_f$$

$$\therefore \quad x = \frac{h_1 - h_f}{h_g - h_f} = \frac{2683 - 762.6}{2776.2 - 762.6}$$

$$= 0.954$$

The use of the h-s chart is also acceptable.

(b) (i) The basic four stroke cycle is the same for both petrol and diesel engines, the four parts being:

Induction ("suck")

Compression ("squeeze") ⟵ Ignition occurs

Power ("bang")

Exhaust ("blow")

(ii) In a petrol engine the air and fuel are premixed before being drawn into the cylinder during the induction stroke. Combustion is initiated by a spark at or near the end of the compression stroke, followed by rapid flame propagation. Hence the heat addition is approximately "at constant volume" — the Otto cycle.

Q2 (cont)

In a diesel engine fuel is sprayed into the cylinder towards the end of the compression stroke. Combustion occurs spontaneously throughout the cylinder when the pressure / temperature is high enough. Hence heat addition is approximately "at constant pressure" - the diesel cycle.

(iii) In a petrol engine the air-fuel ratio is kept constant and the mass of the charge drawn into the cylinder is regulated to control the power.

In a diesel engine the mass of air drawn into the cylinder is kept constant and the mass of fuel sprayed in is varied to control the power.

(iv) In a petrol engine the compression ratio is limited to a maximum of ~10 by the need to avoid "knock" (spontaneous detonation). This limits the size of cylinders and the thermal efficiency of the engine (η increases with CR).

In a diesel engine the compression ratio is limited only by mechanical considerations. CR has to be higher than for petrol engines to achieve spontaneous combustion and to give comparable (or better) thermal efficiencies.

Q3 (a) $dq - dw = du$

$dw = p\,dv$

$\therefore dq - p\,dv = du$

$\therefore dq = du + p\,dv$

$dh = du + p\,dv + \cancel{v\,dp}$ 0 for a constant pressure process

Hence $dq = dh$

$\therefore Q = \Delta h$

(b) (i) Process I is reversible and isentropic $\Rightarrow$ adiabatic

$\Rightarrow Q_I = 0$

Process II is reversible and isothermal $\Rightarrow Q = T\Delta S$

From Table 8 $T_2 = 179.9\,°C = 453.1\,K$

$s_2 = 6.583$ kJ/kg K

From h-s chart s_3 (for $T \approx 180°C$, $p = 0.01$ MPa)

$= 8.820$ kJ/kg K

$\therefore Q_{II} = T_2(s_3 - s_2) = 453.1 \times (8.820 - 6.583)$

$= 1.013$ MJ/kg

Process III is constant pressure $\therefore$ from (a) $Q = \Delta H$

From h-s chart h_3 (for $T \approx 180°C$, $p = 0.01$ MPa)

$= 2840$ kJ/kg

h_1 (for $p = 0.01$ MPa, $s = 6.583$ kJ/kg K)

$= 2085$ kJ/kg

$\therefore Q_{III} = h_1 - h_3 = 2085 - 2840 = -755$ kJ/kg

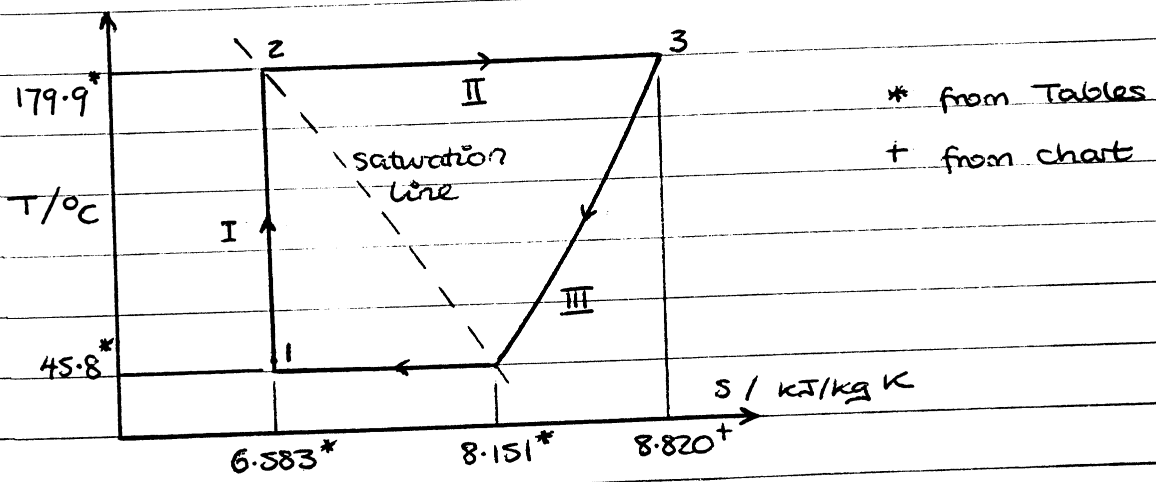
(ii) Thermal efficiency $= \dfrac{\text{net work}}{\text{heat in}} = \dfrac{\text{heat in} - \text{heat out}}{\text{heat in}}$

$= \dfrac{1013 - 755}{1013}$

$= 25.5\%$

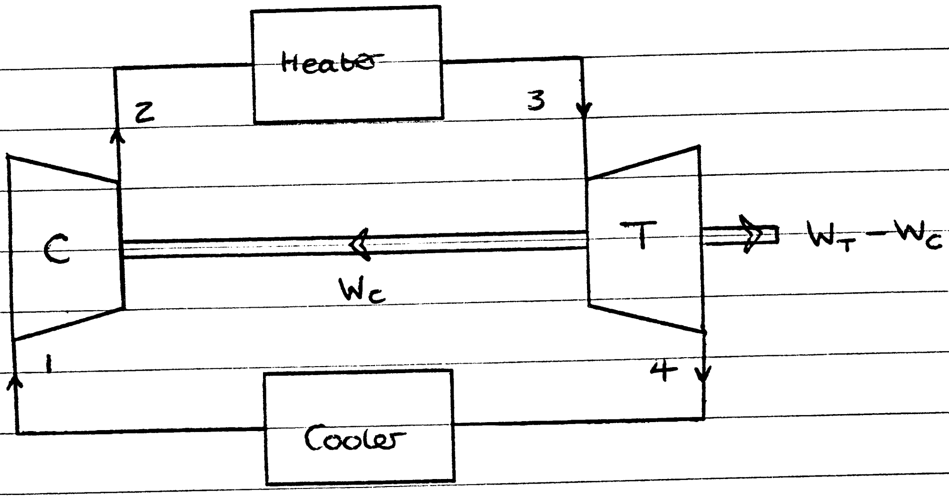
Q 3 (cont)

(iii)



2
3
179·9*
45·8*
T/°C
I
II
III
1
saturation line
s / kJ/kg K
6·583*
8·151*
8·820+
* from Tables
+ from chart

Q4 (a)



(b) $T_{2s} = \left(\frac{P_2}{P_1}\right)^{\frac{\gamma-1}{\gamma}} T_1 = 6^{\frac{0.4}{1.4}} \times 373 = 622.4 \text{ K}$

$$\eta_c = \frac{T_{2s} - T_1}{T_2 - T_1} \Rightarrow T_2 = T_1 + \frac{1}{\eta_c}(T_{2s} - T_1)$$
$$= 373 + \frac{1}{0.9} \times (622.4 - 373)$$
$$= 650.1 \text{ K}$$
$$= \underline{\underline{377\,°C}}$$

$$W_c = c_p(T_2 - T_1) = 1.01 \times (377 - 100) = \underline{\underline{280 \text{ kJ/kg}}}$$

(c) $T_{4s} = \left(\frac{P_4}{P_3}\right)^{\frac{\gamma-1}{\gamma}} T_3 = \left(\frac{1}{6}\right)^{\frac{0.4}{1.4}} \times 1473 = 882.8 \text{ K}$

$$\eta_T = \frac{T_3 - T_4}{T_3 - T_{4s}} \Rightarrow T_4 = T_3 - \eta_T(T_3 - T_{4s})$$
$$= 1473 - 0.92 \times (1473 - 882.8)$$
$$= 930.0 \text{ K}$$
$$= \underline{\underline{657\,°C}}$$

$$W_T = c_p(T_3 - T_4) = 1.01 \times (1200 - 657) = \underline{\underline{548 \text{ kJ/kg}}}$$

(d) $\dot{W}_{net} = \dot{m}(W_T - W_c)$

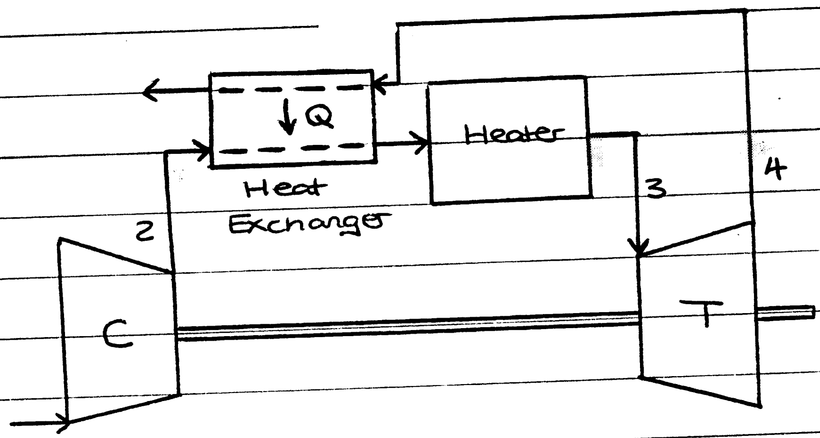
$$\therefore \dot{m} = \frac{\dot{W}_{net}}{W_T - W_c} = \frac{50 \times 10^6}{(548 - 280) \times 10^3} = \underline{\underline{187 \text{ kg/s}}}$$

(e) $Q_{in} = c_p(T_3 - T_2) = 1.01 \times (1200 - 377) = 831 \text{ kJ/kg}$

Q4 (cont)

$$\eta = \frac{W_T - W_C}{Q_{in}} = \frac{548 - 280}{831} = 32.3\%$$

(f) As the exit temperature from the turbine (T_4) is higher than the exit temperature from the compressor (T_2) a heat exchanger can be used to transfer heat from the turbine exhaust to the air before it enters the heater. This reduces the amount of heat which must be added in the heater (Q_{in}), thus raising the thermal efficiency of the cycle.



Q5 (a) Initial K.E. $= \frac{1}{2} \times 10^4 \times 30^2 = 4.5\,MJ$

Momentum conserved during impact, i.e.

$$m_2 V_2 = m_1 V_1$$

$$\therefore V_2 = \frac{10{,}000}{15{,}000} \times 30 = 20\,m/s$$

New K.E. $= \frac{1}{2} \times 15 \times 10^3 \times 20^2 = 3\,MJ$

<u>K.E. lost $= 1.5\,MJ$</u>

(b) K.E. after impact becomes work done against friction + energy in spring

Let distance to 1st stop $= x$

$$3 \times 10^6 = 5000 \times 10 \times 1.5\,x + \tfrac{1}{2} \times 250 \times 10^3\,x^2$$

$$125\,x^2 + 75\,x - 3000 = 0$$

$$x = \frac{-75 \pm \sqrt{5625 + 1.5 \times 10^6}}{250} = 4.608\,m$$

Compressive force in spring $= 250 \times 10^3 \times 4.608\,N$

$= \underline{1.152\,MN}$

Energy now in spring becomes work done vs friction + energy in spring at second stop.

Let extension of spring at 2nd stop $= y$

$$125 \times 10^3\,x^2 = 125 \times 10^3\,y^2 + 500 \times 10 \times 1.5\,(x + y)$$

Q5 (cont)

$$125y^2 + 75y + 75\times4.608 - 125\times4.608^2 = 0$$

$$y = \frac{-75 \pm \sqrt{5625 + 4\times125\times2309}}{250} = 4.008\text{ m}$$

Tensile force in spring $= 250\times10^3 \times 4.008$ N
$= \underline{1.002\text{ MN}}$

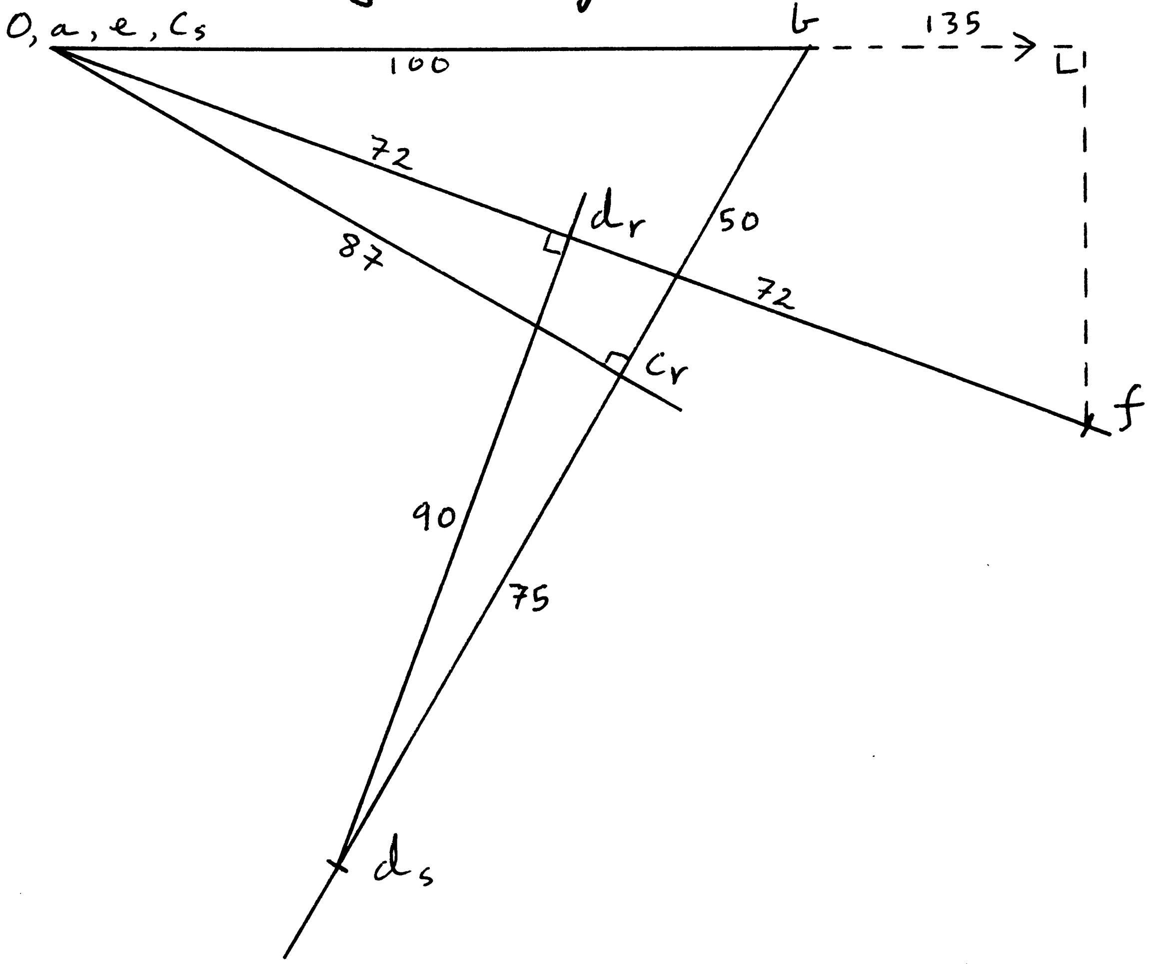
Subsequent tensile + compressive forces will all be smaller.

(c) Max acceleration occurs just <u>before</u> first stop.

Total force on combined mass
$= 1.152\times10^6 + 5000\times10\times1.5$ N
$= 1.227$ MN

$$\text{Acceleration} = \frac{1.227\times10^6}{15\times10^3} = \underline{81.8\text{ m/s}^2}$$

Q6 Velocity diagram:



(a) Angular velocity of BD = $\frac{50+75}{100}$ = $\underline{1.25 \text{ rad/s}}$ ↻

Sliding velocity at C = $\underline{87 \text{ mm/s}}$

(b) Angular velocity of EF = $\frac{144}{60}$ = $\underline{2.40 \text{ rad/s}}$ ↺

Sliding velocity at D = $\underline{90 \text{ mm/s}}$

(c) Horizontal velocity of F = 135 mm/s

By Virtual Work, $5Q = 20 \times 0.135$

$\underline{Q = 0.540 \text{ Nm}}$

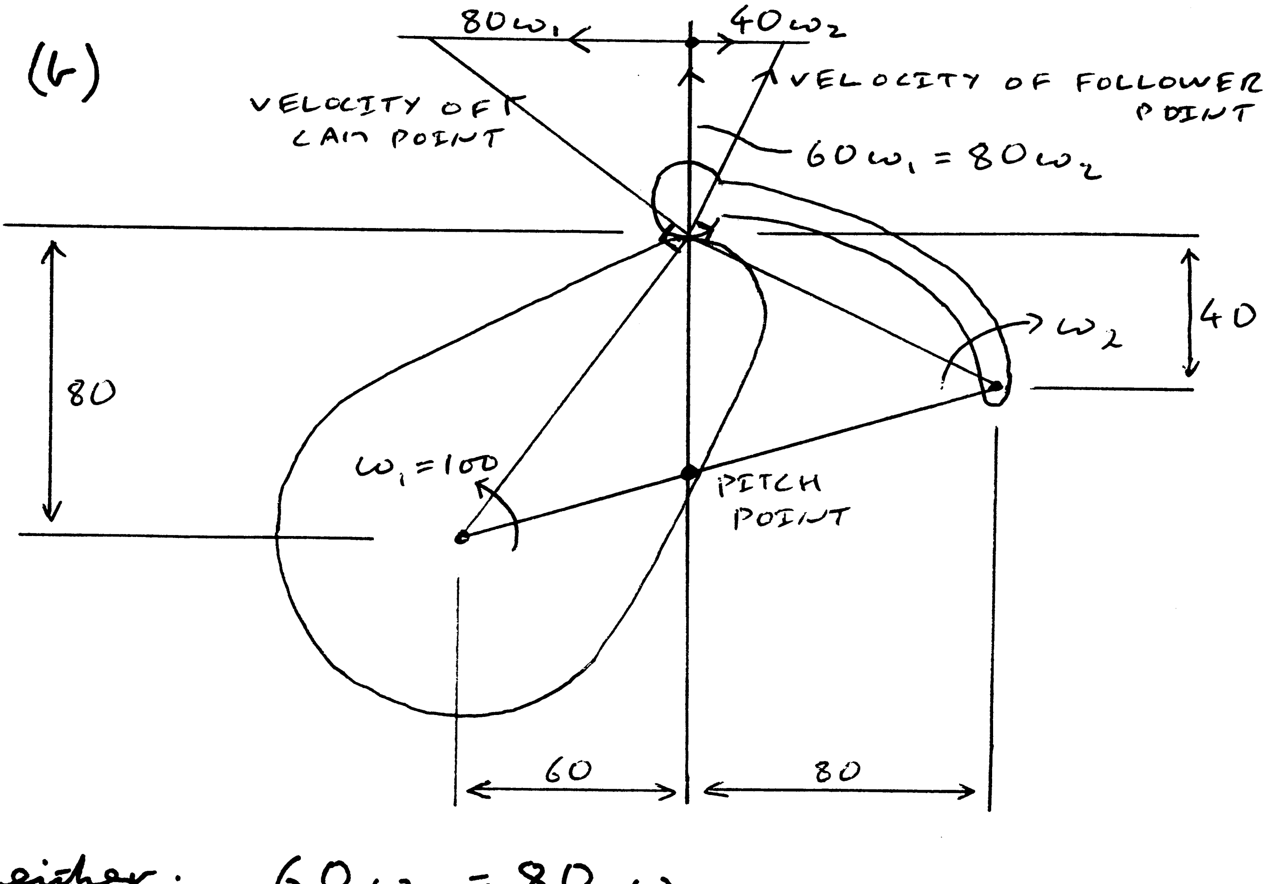
Q 7 (a) The pitch point is where the common normal through the point of contact intersects with the line between the centres of rotation.

There is no sliding if it is coincident with the point of contact.

The ratio of the angular velocities of the laminas stays constant if the pitch point does not move.

The laminas rotate in opposite directions if the pitch point lies between the two centres of rotation.

(b)



<u>either</u>: $60\,\omega_1 = 80\,\omega_2$

$$\therefore \omega_2 = \frac{60}{80}\,\omega_1 = \underline{75 \text{ rad/s}}$$

Q 7 (cont.)

<u>or</u> : Pitch point cuts line between centres in the ratio 3 : 4

$$\therefore \omega_2 = -\tfrac{3}{4} \times 100 = \underline{75 \text{ rad/s}} \circlearrowright$$

Sliding velocity $= 80\,\omega_1 + 40\,\omega_2$ mm/s

$$= \underline{11 \text{ m/s}}$$

(c) <u>Relative</u> angular velocity of wheel is obtained by dividing the sliding velocity by the wheel radius, i.e.

$$\frac{11\,000}{10} = \underline{1100 \text{ rad/s}} \circlearrowright$$

Absolute angular velocity can be found by considering horizontal velocities of wheel centre + contact point :

$50\,\omega_2 = 3750$ mm/s

$80\,\omega_1 = 8000$ mm/s

$$\therefore \omega_3 = \frac{3750 + 8000}{10} = \underline{1175 \text{ rad/s}} \circlearrowright$$

As expected, these two values differ by ω_2.

Q8 (a) At radius r, gravitational force on satellite is:

$$\frac{mMG}{r^2}$$

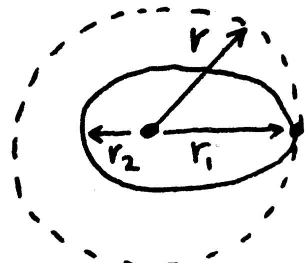
For a circular orbit, this = centrifugal force

i.e. $\dfrac{mV^2}{r} = \dfrac{mMG}{r^2}$

whence $V = \sqrt{MG/r}$

And period $= \dfrac{2\pi r}{V} = 2\pi r\sqrt{\dfrac{r}{MG}}$

(b)



Point of impact = Perigee of new orbit

$\therefore \; r_1 = r$

Also, $V_1 = \dfrac{3V}{4} = \dfrac{3}{4}\sqrt{\dfrac{MG}{r_1}}$ (1)

Method 1:

Conservation of moment of momentum:

$m' V_1 r_1 = m' V_2 r_2 \Rightarrow V_2 = \dfrac{r_1}{r_2} V_1$ (2)

Conservation of energy:

$$\tfrac{1}{2} m'(V_2^2 - V_1^2) = \int_{r_1}^{r_2} \frac{m' MG}{r^2}\, dr = \frac{m' MG}{r_2} - \frac{m' MG}{r_1}$$

i.e. $\tfrac{1}{2}(V_2^2 - V_1^2) = \dfrac{MG}{r_2} - \dfrac{MG}{r_1}$ (3)

Q 8 (cont) Combining equations (1) - (3),

$$\tfrac{1}{2}\left(\left(\frac{r_1}{r_2}\right)^2 - 1\right)\frac{9}{16}\,\frac{MG}{r_1} = \frac{MG}{r_2} - \frac{MG}{r_1}$$

$$\tfrac{9}{32}\left(\left(\frac{r_1}{r_2}\right)^2 - 1\right) = \frac{r_1}{r_2} - 1$$

i.e.

$$9x^2 - 32x + 23 = 0$$

where $x = \dfrac{r_1}{r_2}$

$$x = \frac{32 \pm \sqrt{32^2 - 4\times 9\times 23}}{18} = \frac{32 \pm 14}{18}$$

$$= {}^{23}/_{9} \quad (\text{or } 1)$$

So $\underline{r_1 = r \quad \text{and} \quad r_2 = \dfrac{9r}{23} = 0.391\,r}$

Method 2: using data book formulae

$$h = V_1 r_1 = \frac{3r_1}{4}\sqrt{\frac{MG}{r_1}}$$

$$\therefore\ l = \frac{h^2}{GM} = \frac{9r_1}{16}$$

At Apogee, $r = r_1$ and $\theta = \pi$

$$\therefore\ l/r = 9/16 = 1 - e \implies e = 7/16$$

Then $\dfrac{r_2}{r_1} = \dfrac{1-e}{1+e} = 9/23$ as before.

(c) Orbit time will decrease (despite reduction in velocity) since new orbit radius is always $\leq r$, and orbit time reduces as radius reduces.

Q9 (a) $\quad T\dfrac{d\theta}{dt} + \theta = \phi$

By inspection $\quad$ PI $\quad \theta = \phi$

$\qquad$ CF $\quad \theta = Ae^{-t/T}$

$\qquad$ GS $\quad \theta = \phi + Ae^{-t/T}$

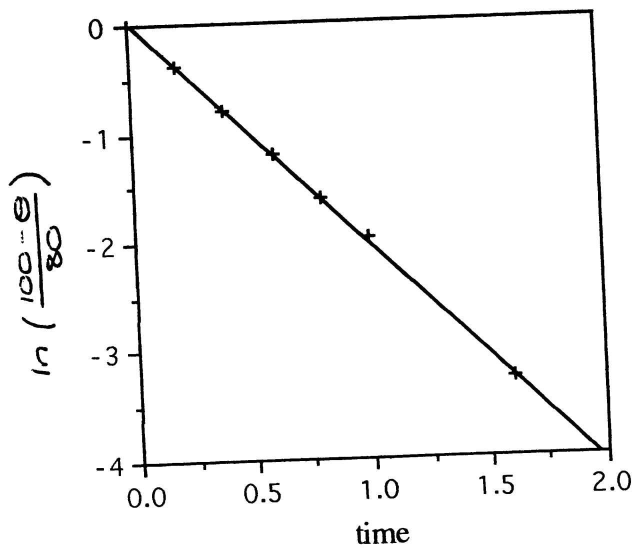
BC $\quad \theta = \theta_0$ at $t = 0 \Rightarrow A = \theta_0 - \phi$

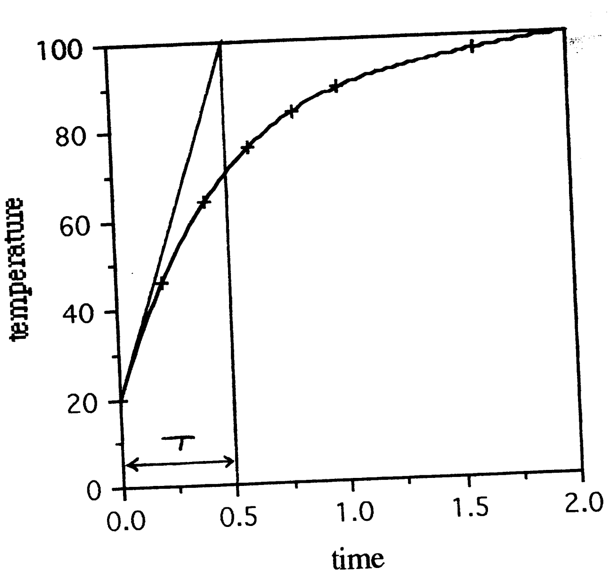
$\therefore$ GS $\quad \theta = \phi + (\theta_0 - \phi)\,e^{-t/T}$

In this case $\quad \theta = 100 - 80\,e^{-t/T}$

$$\therefore \quad \ln\left(\frac{100-\theta}{80}\right) = -t/T$$

Plotting $\ln\left(\dfrac{100-\theta}{80}\right)$ versus t for the data given does yield a straight-line relationship confirming the data is consistent with that for a first-order linear system.



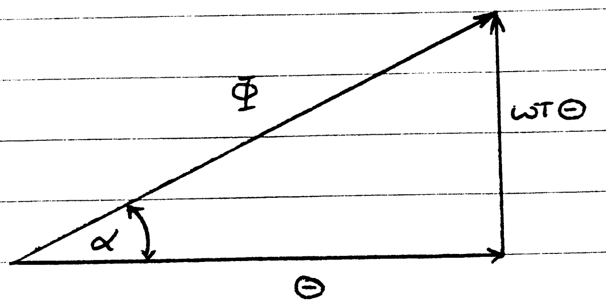


The time constant can be estimated from the gradient of the log plot or from a plot of θ vs t as shown. From either $\underline{\underline{T \approx 0.5\,s}}$

Q9 (G) $\phi = \Phi e^{i\omega t}$

So $\theta = \Theta e^{i\omega t}$

$\therefore (i\omega T + 1)\Theta = \Phi$ substituting in the governing DE



Phasor diagram

$$\Theta = \frac{\Phi}{\sqrt{1+\omega^2T^2}}$$

$$\omega = \frac{2\pi}{10} \text{ rad/s}$$

$$\therefore \Theta = \frac{25}{\sqrt{1 + \frac{4\pi^2}{100} \times 0.5^2}} = 23.85\,°C$$

$$\therefore \text{error} = \Phi - \Theta = \underline{\underline{1.15\,°C}}$$

$$\alpha = \tan^{-1}\omega T = \tan^{-1}\left(\frac{2\pi \times 0.5}{10}\right) = \underline{\underline{17.4°}}$$

Q10 (a) $V_E = iR + L\frac{di}{dt} + V$

$i = C\frac{dV}{dt}$

$\therefore V_E = LC\frac{d^2V}{dt^2} + RC\frac{dV}{dt} + V$

$\therefore \underline{\underline{A = LC}} \qquad \underline{\underline{B = RC}}$

(b) $A = 10 \times 250 \times 10^{-6} = 2.5 \times 10^{-3}$ for values given

$B = 400 \times 250 \times 10^{-6} = 0.1$

Data book form is $\frac{1}{\omega_n^2}\frac{d^2V}{dt^2} + \frac{2c}{\omega_n}\frac{dV}{dt} + V = V_E$

$\therefore \omega_n^2 = \frac{1}{A} = 400 \Rightarrow \omega_n = 20$ rad/s

$\frac{2c}{\omega_n} = B \Rightarrow c = \frac{B\omega_n}{2} = \frac{0.1 \times 20}{2} = \underline{\underline{1}}$

As $c = 1$ system is critically damped

(c) (i) $\underline{\underline{V = 0}}$ — the capacitor is initially uncharged

(ii) Current through an inductor cannot change instantaneously

$\therefore$ initial current $= \frac{V_E}{R}$

But

$i = C\frac{dV}{dt} \quad \therefore \frac{V_E}{R} = C\frac{dV}{dt} \Rightarrow \underline{\underline{\frac{dV}{dt} = \frac{V_E}{RC}}}$ initially

(iii) As $t \to \infty$ $\underline{\underline{V \to V_E}}$

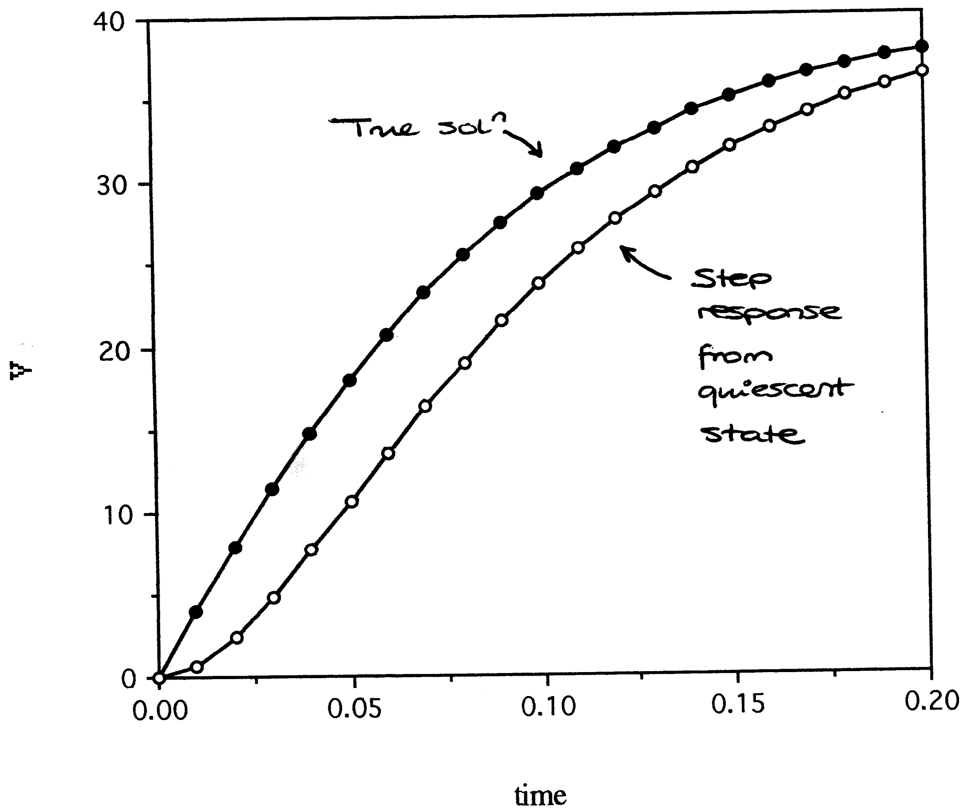
(d) The step response on page 6 can match the initial and final conditions on V but not the initial condition on $\dot{V}$ — the data book response is for a system in an initially quiescent state, i.e.

Q10 (cont)

$V = \dot{V} = 0$ at $t = 0$.

The impulse response on page 7 can match the initial conditions on V and $\dot{V}$ but not the final condition on V.

(e) The exact solution is as plotted here:



The key features required of a sketch are:

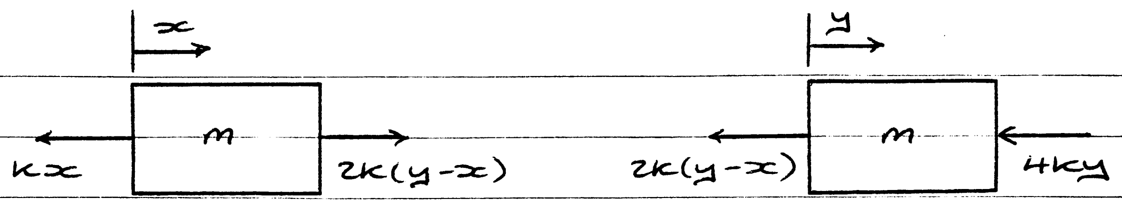
(i) Correct initial value of V (0 V),

(ii) Correct final value of V (40 V),

(iii) Initial gradient of V (non-zero),

(iv) Critically damped response – no overshoot.

Q11 (a)



$m\ddot{x} = 2k(y-x) - kx$ $\qquad$ $m\ddot{y} = -2k(y-x) - 4ky$

$\therefore\ m\ddot{x} + 3kx - 2ky = 0$ $\qquad$ $m\ddot{y} + 6ky - 2kx = 0$

So in matrix form

$$\underbrace{\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}}_{\text{mass matrix } [M]} \begin{bmatrix} \ddot{x} \\ \ddot{y} \end{bmatrix} + \underbrace{\begin{bmatrix} 3k & -2k \\ -2k & 6k \end{bmatrix}}_{\text{stiffness matrix } [K]} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

(b) Natural frequencies are solutions of

$$\left| [K] - \omega^2 [M] \right| = 0$$

$$\therefore \begin{vmatrix} 3k - \omega^2 m & -2k \\ -2k & 6k - \omega^2 m \end{vmatrix} = 0$$

$$\therefore\ (3k - \omega^2 m)(6k - \omega^2 m) - 4k^2 = 0$$

$$\therefore\ 18k^2 - 9k\omega^2 m + \omega^4 m^2 - 4k^2 = 0$$

$$\therefore\ m^2\omega^4 - 9km\omega^2 + 14k^2 = 0$$

$$\therefore\ (m\omega^2 - 7k)(m\omega^2 - 2k) = 0$$

$$\therefore\ \omega = \sqrt{\frac{7k}{m}} \text{ or } \sqrt{\frac{2k}{m}}$$

For $\omega^2 = \frac{7k}{m}$ $\quad [K] - \omega^2 [M] = \begin{bmatrix} -4k & -2k \\ -2k & -k \end{bmatrix}$

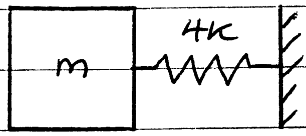
$\Rightarrow \quad 2x = -y \qquad$ i.e. $\{1, -2\}$ mode

For $\omega^2 = \frac{2k}{m}$ $\quad [K] - \omega^2 [M] = \begin{bmatrix} k & -2k \\ -2k & 4k \end{bmatrix}$

$\Rightarrow \quad x = 2y \qquad$ i.e. $\{2, 1\}$ mode

Q11 (cont)

(c) If the two masses move with equal amplitude and phase the middle spring is unstretched and transmits no force. The right hand mass then behaves like a simple single mass-spring system:



$$\omega^2 = \frac{4k}{m}$$

$$\therefore \quad \omega = 2\sqrt{\frac{k}{m}}$$

Otherwise under the specified harmonic excitation

$$[M]\begin{pmatrix}\ddot{x}\\ \ddot{y}\end{pmatrix} + [K]\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}F\cos\omega t\\ 0\end{pmatrix}$$

$\ddot{x} = -\omega^2 x$ & $\ddot{y} = -\omega^2 y$ if $x = X\cos\omega t$ & $y = Y\cos\omega t$

$$\therefore \begin{bmatrix} 3k - m\omega^2 & -2k \\ -2k & 6k - m\omega^2 \end{bmatrix}\begin{pmatrix}x\\ y\end{pmatrix} = \begin{pmatrix}F\\ 0\end{pmatrix}$$

If $x = y$ (equal amplitude and phase), the second equation gives

$$-2k + 6k - m\omega^2 = 0$$

$$\therefore \quad m\omega^2 = 4k$$

$$\therefore \quad \omega = 2\sqrt{\frac{k}{m}}$$