1st page.

ENGINEERING TRIPOS PART IA 1998

PAPER 1 MECHANICAL ENGINEERING

SECTION A - THERMODYNAMICS

1. (a) $Tds = dh - vdp$

Isentropic $\therefore ds = 0$

$\therefore \int dh = \int v dp$

Incompressible $\therefore v = \text{constant}$

$\therefore h_2 - h_1 = v(p_2 - p_1)$

or $\underline{\underline{\Delta h = v \Delta p}}$

(b) SFEE $\dot{Q} - \dot{W}_x = \dot{m}(h_2 - h_1)$ neglecting KE and PE

$\dot{Q} = 0$ as adiabatic

$\therefore \dot{W}_x = -\dot{m}(h_2 - h_1) = -\dot{m}\Delta h$

As turbine is isentropic and water is effectively incompressible the result from part (a) can be used

$\therefore \dot{W}_x = -\dot{m} v \Delta p$

$\dot{m} v = \dot{V}$ $\therefore$ $\underline{\underline{\dot{W}_x = -\dot{V}\Delta p}}$

(c)

Variables	η	N	D	$\dot{V}$
Dimensions	-	$\frac{1}{T}$	L	$\frac{L^3}{T}$

4 variables 2 dimensions $\Rightarrow$ 4-2 = $\underline{\underline{\text{2 or more}}}$ dimensionless groups

By inspection (or otherwise) $\underline{\underline{\frac{\dot{V}}{ND^3}}}$ is dimensionless

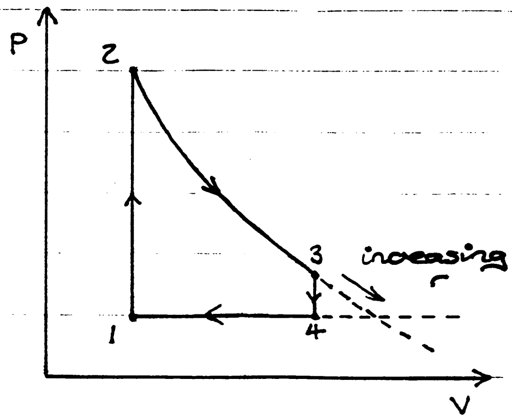
(d) For data given $\phi = \frac{\dot{V}}{ND^3} = \frac{70}{50 \times 3.42^3} = 0.035$

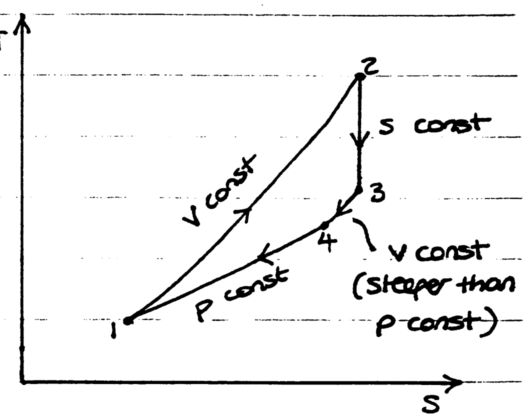
$\Rightarrow \underline{\underline{\eta = 0.78}}$ from graph

$\dot{W}_x = \eta \dot{W}_{xisen} = |\eta \dot{V} \Delta p| = |\eta \dot{V} \rho g \Delta z| = 0.78 \times 70 \times 10^3 \times 9.81 \times 514$

$= \underline{\underline{275 \text{ MW}}}$

2. (a)





(b) Q_{in} only takes place in process A $(1 \rightarrow 2)$

1^{st} Law $\quad q - w = \Delta u$

No volume change $\therefore$ $w = 0$

$$\therefore \; q = \Delta u = c_v(T_2 - T_1) = \underline{\underline{c_v T_1 (\tau - 1)}}$$

(c) For process B $\quad Q_B = 0 \quad$ (reversible + isentropic = adiabatic)

$$\therefore \; W_B = -c_v(T_3 - T_2) = c_v(T_2 - T_3)$$

For an isentropic process $TV^{\gamma-1}$ = constant

$$\therefore \; T_2 V_2^{\gamma-1} = T_3 V_3^{\gamma-1}$$

$$\therefore \; T_3 = T_2 r^{1-\gamma} = T_1 \tau r^{1-\gamma} \quad \text{as } \frac{V_2}{V_3} = \frac{V_1}{V_3} = \frac{1}{r}$$

$$\therefore \; W_B = c_v T_2 (1 - r^{1-\gamma})$$

$$= c_v T_1 \tau (1 - r^{1-\gamma})$$

No work done in process C (constant volume)

For process D $\quad W_D = \int_4^1 p\,dv = p_1(V_1 - V_4) \quad$ as p constant

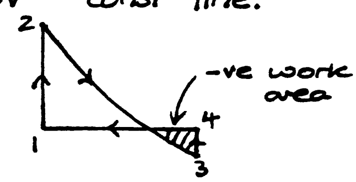
$$= p_1 V_1 (1 - r) \quad \text{as } \frac{V_4}{V_1} = \frac{V_3}{V_1} = r$$

$$= RT_1(1 - r) \quad \text{using } p_1 V_1 = RT_1$$

$$\therefore \; W_{net} = W_B + W_D$$

$$= \underline{\underline{c_v T_1 \tau (1 - r^{1-\gamma}) + RT_1(1 - r)}}$$

(d) The net work is the enclosed area of the p-v diagram. As r is increased point 3 follows the pv^{γ} = const line. For high enough r the cycle looks like



The shaded area represents -ve work and thus the net work reduces.

2. (d) (cont.)

The net work is maximised when points 3 and 4 coincide,

i.e. $T_3 = T_4$

from (c) $T_3 = T_1 \tau r^{1-\gamma}$

$P_1 = P_4 \quad \therefore \quad \dfrac{T_4}{V_4} = \dfrac{T_1}{V_1} \Rightarrow T_4 = T_1 \dfrac{V_4}{V_1} = T_1 \dfrac{V_3}{V_1} = r T_1$

$$\therefore T_3 = T_4 \Rightarrow T_1 \tau r^{1-\gamma} = r T_1$$

$$\therefore \tau = r^{\gamma}$$

$$\therefore \underline{\underline{r = \sqrt[\gamma]{\tau}}}$$

Alternatively

$$\frac{dW_{net}}{dr} = -c_v T_1 \tau (1-\gamma) r^{-\gamma} - R T_1$$

$\therefore \dfrac{dW_{net}}{dr} = 0$ when $(\gamma - 1) \tau c_v r^{-\gamma} = R$

$\dfrac{R}{c_v} = \gamma - 1 \quad \therefore \quad \tau r^{-\gamma} = 1$

$$\therefore \underline{\underline{r = \sqrt[\gamma]{\tau}}}$$

3\. (a) $dh = c_p\,dT$ at constant pressure

$$= (A + BT)\,dT$$

$$\therefore \int_1^2 dh = \int_1^2 (A + BT)\,dT \quad \text{for a constant pressure process}$$

$$\therefore \quad h_2 - h_1 = \underline{\underline{A(T_2 - T_1) + \tfrac{1}{2}B(T_2^2 - T_1^2)}}$$

Although the pressure is not necessarily constant throughout the device, this equation is valid because enthalpy is a property and the change in a property depends only on the end states and not the path taken.

(b) $T\,ds = dh - v\,dp$

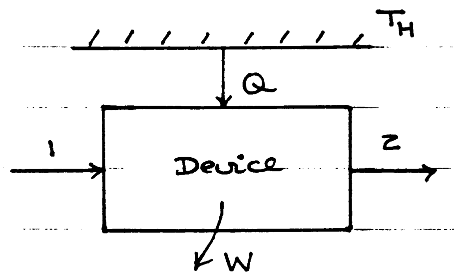
Assume the process is constant pressure $\Rightarrow$ $dp = 0$

$$\therefore \quad T\,ds = c_p\,dT = (A + BT)\,dT$$

$$\therefore \int_1^2 ds = \int_1^2 \left(\frac{A}{T} + B\right) dT$$

$$\therefore \quad s_2 - s_1 = \underline{\underline{A \ln\left(\frac{T_2}{T_1}\right) + B(T_2 - T_1)}}$$

(c) Let Q be the heat flow to the device per kg of gas



Applying the SFEE

$$W = Q - (h_2 - h_1)$$

By the second law

$$s_2 - s_1 \geqslant \frac{Q}{T_H}$$

$$\therefore \quad Q \leqslant T_H(s_2 - s_1)$$

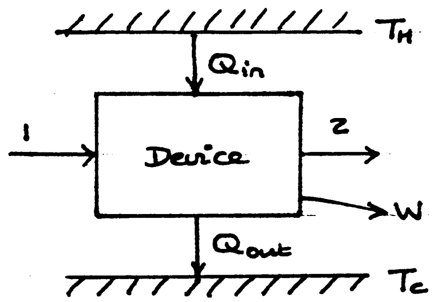
$$\therefore \quad W \leqslant T_H(s_2 - s_1) - (h_2 - h_1)$$

$$\therefore \quad \underline{\underline{W \leqslant AT_H \ln\left(\frac{T_2}{T_1}\right) + BT_H(T_2 - T_1) - A(T_2 - T_1) - \tfrac{1}{2}B(T_2^2 - T_1^2)}}$$

3. (d) If the exit gas is cooled to T_1 then it is in the same state as the entry gas and could therefore be recycled. So the device is now equivalent to a cyclic heat engine. It receives heat at temperature T_H and rejects heat at temperature T_C. The maximum possible thermal efficiency is that for a reversible heat engine operating between these temperatures, i.e. the Carnot efficiency

$$\underline{\underline{\eta_{max} = 1 - \frac{T_C}{T_H}}}$$

Alternatively



State 1 = State 2 (same P and T)

∴ SFEE gives $W = Q_{in} - Q_{out}$

2nd Law gives $s_2 - s_1 = 0 \geqslant \frac{Q_{in}}{T_H} - \frac{Q_{out}}{T_C}$

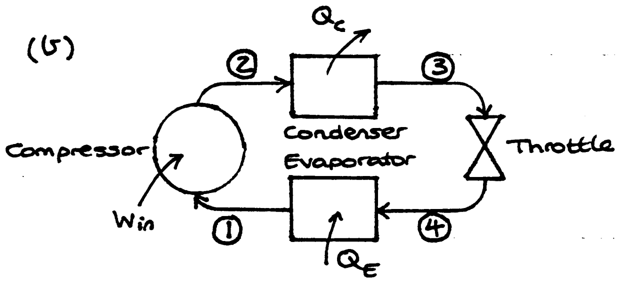
$$\therefore \quad Q_{out} \geqslant \frac{T_C}{T_H} Q_{in}$$

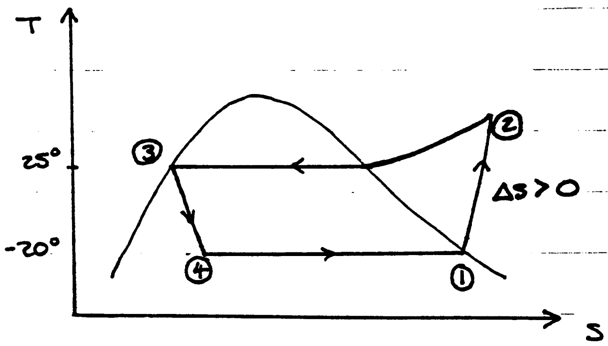
$$\therefore \quad W \leqslant Q_{in} - \frac{T_C}{T_H} Q_{in}$$

$$\therefore \quad \underline{\underline{\frac{W}{Q_{in}} \leqslant 1 - \frac{T_C}{T_H}}}$$

4. (a) Any two properties except pressure and temperature are sufficient to determine the state. Pressure and temperature are not independent in a two phase region.

(b)

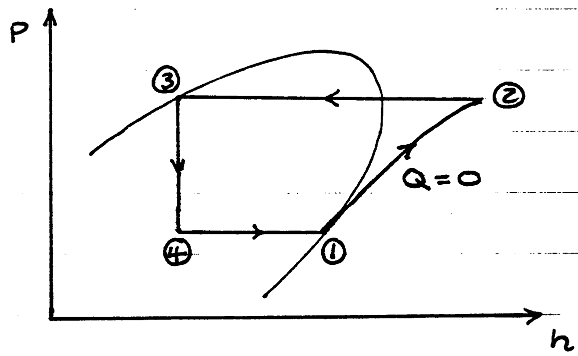




$h_1 = 178.7$ kJ/kg

$h_2 = 212.1$ kJ/kg

$h_3 = 59.7$ kJ/kg $= h_4$



(c) $h_4 = x h_g + (1-x) h_f = x(h_g - h_f) + h_f$

$\therefore \; 59.7 = x(178.7 - 17.8) + 17.8$

$\therefore \; \underline{\underline{x = 0.2604}}$

$v_3 = 0.00076$ m³/kg (Table 13)

$v_4 = x v_g + (1-x) v_f$

$= 0.2604 \times 0.1089 + (1 - 0.2604) \times 0.00069$

$= 0.02887$ m³/kg

Ratio of volumetric flow rates $= \dfrac{v_3}{v_4} = \dfrac{0.00076}{0.02887} = \underline{\underline{0.0263}}$

(d) $W_{in} = h_2 - h_1 = 212.1 - 178.7 = \underline{\underline{33.4 \text{ kJ/kg}}}$

$Q_E = h_1 - h_4 = 178.7 - 59.7 = \underline{\underline{119.0 \text{ kJ/kg}}}$

$COP = \dfrac{Q_E}{W_{in}} = \dfrac{119.0}{33.4} = \underline{\underline{3.563}}$

4.(e) If the throttle is replaced by a turbine the COP is increased:

h_4 is decreased ∴ Q_E is increased

Also work from turbine $(h_3 - h_4)$ can be used to help run the compressor thus reducing W_{in}

However, the turbine is operating with a two-phase fluid and will therefore likely need regular maintenance - a throttle, which has no moving parts, does not have this disadvantage

A throttle is also, of course, cheaper than a turbine.

It may be difficult to match the speeds of the compressor and turbine.

SECTION B

5.

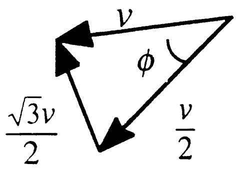
(a) $\dot{\theta} = \frac{\sqrt{3}v}{2r} \Rightarrow \dot{\underline{e}}_r = \omega \underline{e}_\theta = \frac{\sqrt{3}v}{2r}\underline{e}_\theta \text{ , } \dot{\underline{e}}_\theta = -\omega \underline{e}_r = \frac{-\sqrt{3}v}{2r}\underline{e}_r$

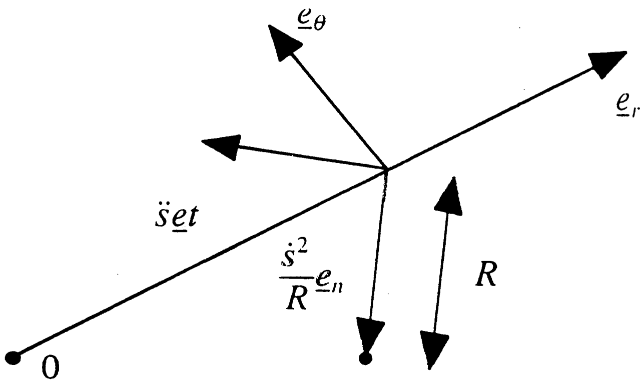
(b) $\underline{r} = r\underline{e}_r$

$$\dot{\underline{r}} = \frac{dr}{dt}\underline{e}_r + r\dot{\underline{e}}_r = -\frac{v}{2}\underline{e}_r + \frac{\sqrt{3}}{2}v\underline{e}_\theta$$

$$\ddot{\underline{r}} = -\underbrace{\cancel{\frac{dv}{dt}}}_{0}\frac{\underline{e}_r}{2} - \frac{v}{2}\dot{\underline{e}}_r + \frac{\sqrt{3}}{2}\underbrace{\cancel{\frac{dv}{dt}}}_{0}\underline{e}_\theta + \frac{\sqrt{3}}{2}v\dot{\underline{e}}_\theta = -\frac{\sqrt{3}}{4}\frac{v^2}{r}\underline{e}_\theta - \frac{3}{4}\frac{v^2}{r}\underline{e}_r$$

(c) Velocity Acceleration





$$\dot{s} = \sqrt{\left(\frac{v}{2}\right)^2 + \left(\frac{\sqrt{3}v}{2}\right)^2} = v \Rightarrow \underset{\sim}{\ddot{s} = 0}$$

Alternatively: $\phi = 60^\circ$

$$\ddot{s} = \ddot{\underline{r}} \cdot \underline{e}_t = -\frac{\sqrt{3}}{4}\frac{v^2}{r}\sin\phi - \frac{3}{4}\frac{v^2}{r}(-\cos\phi) = 0$$

To find the radius of curvature:

Acceleration in $\underline{e}_n$ direction

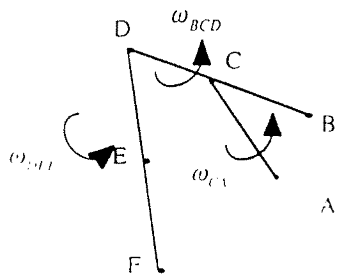
$$\frac{\dot{s}^2}{R} = \ddot{\underline{r}} \cdot \underline{e}_n = -\frac{\sqrt{3}}{4}\frac{v^2}{r}(-\cos\phi) - \frac{3}{4}\frac{v^2}{R}(-\sin\phi)$$

$$= \frac{v^2}{r}\left(\frac{\sqrt{3}}{8} + \frac{3\sqrt{3}}{8}\right) = \frac{\sqrt{3}v^2}{2r}$$

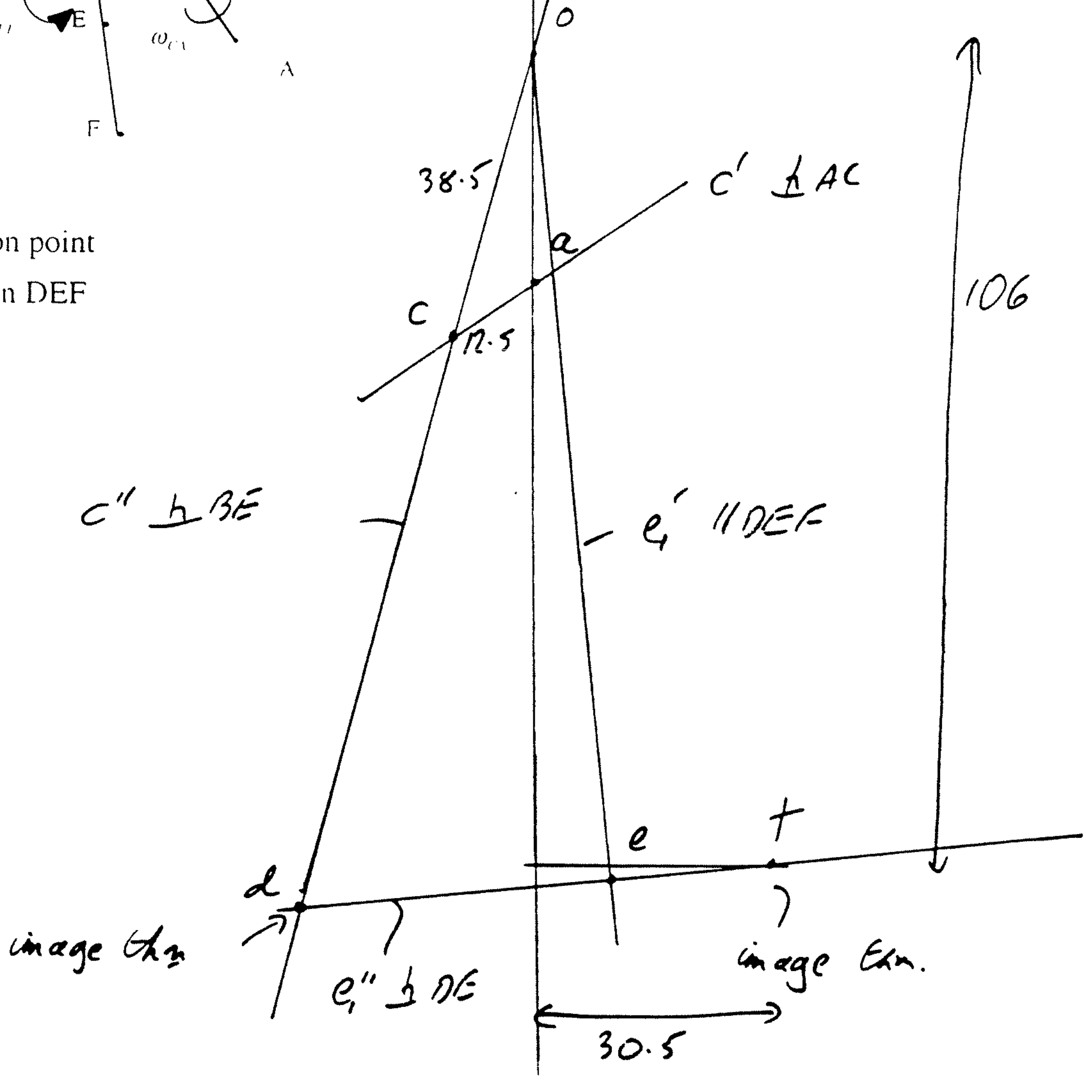
$$\Rightarrow \qquad \Rightarrow R = \frac{2r}{\sqrt{3}}$$

[Fairly well answered. Note that $\ddot{\theta} \neq 0$. In (b) many people derived the databook expressions. This was fine, but explicit expressions with v and r were still needed for (c).]

6.



e_0 on point
e_1 on DEF



(a) $\omega_{BCD} = \dfrac{v}{r} = \dfrac{38.5/10}{30} = 0.31$ rad/s

$\omega_{AC} = \dfrac{12.5}{10 \times 48} = 0.026$ rad/s

(b) $\omega_{DEF} = \dfrac{4.05}{90} = 0.045$ rad/s

$v_F \rightarrow = 3.05$ mm/s

$v_F \downarrow = 10.6$ mm/s

[Necessary to check the signs for ω's to get the friction working right.] All answers subject to drawing errors/inaccuracy.

(c)(i) From work in = work out $10.6 \cdot P = 3 \times R \Rightarrow \quad P = 70.8$N

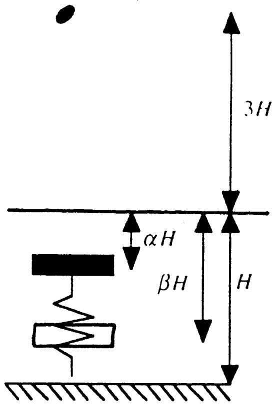
(ii) For all joints ω's are in the same direction - hence difference is needed.

work in = work out + frictional work

$10.6 \cdot P = 3 \times R + 120(|0.13 - 0.026| + |0.13 - 0.045|)$

$\Rightarrow \quad P = 72.9$N

7.



(a) Initial compression $= \alpha H$

$mg = k\alpha H \Rightarrow \alpha H = mg/k = H/30$

(b) loss of PE = gain of KE

$\left(3+\frac{1}{30}\right)H \cdot mg = \frac{1}{2}mv_o^2$

$\Rightarrow \quad v_o = \sqrt{\frac{91gH}{15}}$

(c) Momentum is conserved through the impact.

$mv_o = 2mv_1$, where v_1 is velocity after impact

$\Rightarrow v_1 = v_o/2$

KE before impact $= \frac{1}{2}mv_o^2$

KE after impact $= \frac{1}{2}\cdot 2m \cdot v_1^2$

Loss of energy = change in KE = $= \frac{1}{2}mv_o^2 - mv_1^2 = \frac{mv_o^2}{4} = \frac{91}{60}mgH$

(d) Maximum compression $= \beta H$

Equate energy after impact with energy at bottom of bounce.

(1) KE after impact $= mv_1^2 = \frac{91}{60}mgH$

(2) PE after impact (relative to H above ground) $= -2mg\alpha H = -\frac{mgH}{15}$

(3) Elastic stored energy after impact $= \frac{1}{2}k(\alpha H)^2 = \frac{1}{2}\left(\frac{H}{30}\right)^2 \cdot k = \frac{mgH}{60}$

(4) KE at bottom of bounce = 0

(5) PE at bottom of bounce $= -2mg\beta H$

(6) Elastic stored energy at bottom of bounce $= \frac{1}{2}k(\beta H)^2 = 15\, mgH\beta^2$

Equating (1) + (2) + (3) = (4) + (5) + (6)

$\frac{91}{60} - \frac{1}{15} + \frac{1}{60} = -2\beta + 15\beta^2$ (Equilibrium position)

$\beta = \frac{2 \pm \sqrt{4 - 4\cdot 15 \cdot 88/60}}{30} = \frac{1}{15} \pm \frac{\sqrt{\cancel{88}\,92}}{30}$. Take positive root for max. compression.

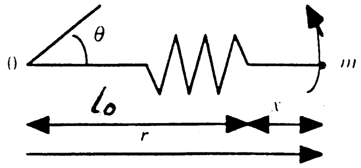
Max. compression $= \beta H = \cancel{0.379H}$ $0.386H$

(e) Assuming that the platform is initially at rest $\frac{29}{30}H$ above the ground (as for (a)) then the same energy is lost on impact, but now some energy goes into frictional dissipation so less goes into elastic stored energy. Hence the spring compresses less.

However, the platform may be at rest at a different position, with friction acting. Now the loss of energy on impact could be reduced (if the platform rests higher than before). Some will go into friction but a full calculation is needed to find the change in compression.

8.

(a)



In steady orbit '$F = ma$'

$$k(r_0 - l_0)^2 = m\dot{\theta}_0^2 r_0$$

$$\dot{\theta}_0 = (r_0 - l_0)\sqrt{\frac{k}{mr_0}}$$

After impact: Tangential velocity increased by Δv

$\Rightarrow$ $v = \dot{\theta}_0 r_0 + \Delta v$

$$\dot{\theta}_1 = (r_0 - l_0)\sqrt{\frac{k}{mr_0}} + \frac{\Delta v}{r_0}$$

(b) Moment of momentum about O is conserved.

$m\dot{\theta}r^2 = m\dot{\theta}_1 r_0^2$, where $\dot{\theta}$ and r vary with time in the subsequent motion.

Radial acceleration $-k(r - l_0^2) = +m(\ddot{r} - r\dot{\theta}^2)$

Eliminating $\dot{\theta}$ and rearranging: $\ddot{r} + \frac{k}{m}(r - l_0)^2 - (\dot{\theta}_1)^2 \frac{r_0^4}{r^3} = 0$

(c) Putting $r = y + r_0 : \ddot{r} = \ddot{y}$

$$\ddot{y} = \frac{k}{m}(y + r_0 - l_0)^2 - (\dot{\theta}_1)^2 \frac{r_0^4}{(y + r_0)^3} = 0$$

Using binomial, and cancelling terms in $(y/r_0)^2$ and higher

$$\ddot{y} + \frac{2ky}{m}(r_0 - l_0) + \frac{k}{m}(r_0 - l_0)^2 - (\dot{\theta}_1)^2 r_0 \left(1 - \frac{3y}{r_0}\right) = 0$$

i.e. $\ddot{y} + ay = b$ with $a = \frac{2k}{m}(r_0 - l_0) + 3\dot{\theta}_1^2$

$$b = -\frac{k}{m}(r_0 - \ell_0)^2 + \dot{\theta}_1^2 r_0$$

Solution to this is $y = A\sin \omega t + B\cos \omega t + b/a$ with $\omega^2 = a$.

Since y varies sinusoidally with time, so does the length of the spring.

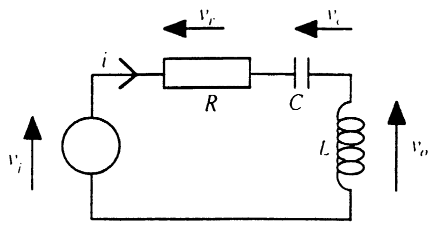
The frequency of oscillation $f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{2k}{m}(r_0 - l_0) + 3\dot{\theta}_1^2}$.

[Relatively poorly answered. Moment of momentum in the radial direction is not conserved through the impact. Some care is needed in the binominal expansion.]

SECTION C

9.

(a)



$$v_R = iR$$
$$v_C = Q/C$$
$$v_o = L\frac{di}{dt}$$

Sum voltages around circuit

$$v_i - iR - \frac{Q}{C} - L\frac{di}{dt} = 0$$

$$i = \frac{1}{L}\int v_o\,dt \quad , \quad Q = \frac{1}{L}\iint v_o\,dt\,dt$$

$$\Rightarrow \quad v_i - \frac{R}{L}\int v_o\,dt - \frac{1}{CL}\iint v_o\,dt\,dt - v_o = 0$$

Differentiate twice with respect to time

$$\Rightarrow \quad \ddot{v}_i = \ddot{v}_o + \frac{R}{L}\dot{v}_o + \frac{1}{CL}v_o$$

Which is case (b) with $\omega_n^2 = \frac{1}{LC}$, $\frac{2C}{\omega_n} = RC$, $x = -v_i$, $y = v_o$

(b) Undamped natural frequency $f_n = \frac{1}{2\pi}\omega_n = \frac{1}{2\pi}\frac{1}{\sqrt{LC}}$

$$\Rightarrow \quad L = \frac{1}{\left(1215\times10^3\times2\pi\right)^2\times0.4\times10^{-9}} = 42.9\ \mu H$$

(c)(i) See page 11 of the databook. The resonance peak is lower and broader, and the frequency at which the maximum response occurs is higher.

(ii) $c = \frac{RC\omega_n}{2} = \frac{1.5\cdot 0.4\times10^{-9}\cdot 1215\times10^3\times2\pi}{2} = 2.29\times10^{-3}$

$$Q = \frac{1}{2c} = 218$$

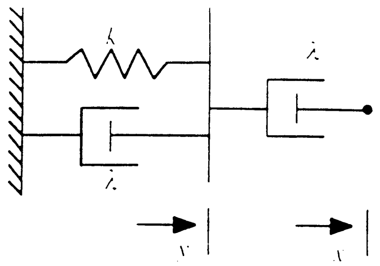
From databook $|Y|_{\max} = \frac{|X|}{2c\sqrt{1-c^2}}$

$$\Rightarrow \quad |v_o|_{\max} = \frac{V_i}{2c\sqrt{1-c^2}} = 218\,V_i$$

Maximum response occurs at $\frac{\omega}{\omega_n} = \frac{1}{\sqrt{1-2c^2}}$ (for small c ✓)

% change $= 100\left(\frac{\omega-\omega_n}{\omega_n}\right) = 100\left(\frac{1}{\sqrt{1-2c^2}} - 1\right) = 5.2\times10^{-4}\%$

10.



(a) Equating forces on massless central platform

$\Rightarrow \quad -ky - \lambda \dot{y} + \lambda(\dot{x} + \dot{y}) = 0$

$\dfrac{2\lambda}{k}\dot{y} + y = \dfrac{\lambda}{k}\dot{x}$ - form as required with

$T = \dfrac{2\lambda}{k}$, $b = \dfrac{\lambda}{k} = \dfrac{T}{2}$

(b) $T\dot{y} + y = \dfrac{AT}{2t_0}$

P.I. $y = \dfrac{AT}{2t_0}$, C.F. $y = Be^{-t/T}$

B.C. $y = 0$ @ $t = 0 \Rightarrow y = \dfrac{AT}{2t_0}\left(1 - e^{-t/T}\right)$

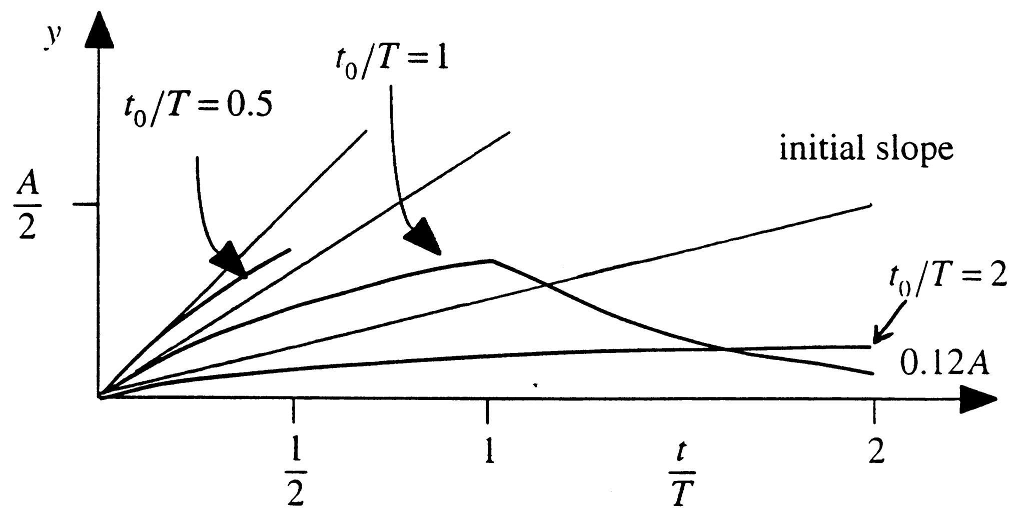
(c) Three different values of $t_0/T \Rightarrow$ 3 different curves.

$\dfrac{t_0}{T} = 0.5 \qquad y\big|_{t=t_0} = \dfrac{A}{2}\dfrac{1}{0.5}\left(1 - e^{-0.5}\right) = 0.39A$

$\dfrac{t_0}{T} = 1 \qquad y\big|_{t=t_0} = 0.32A$

$\dfrac{t_0}{T} = 2 \qquad y\big|_{t=t_0} = 0.22A$

Slope $\dfrac{dy}{d(t/T)} = \dfrac{A}{2}\dfrac{T}{t_0}e^{-t/T} = \dfrac{A}{2}\dfrac{T}{t_0}$ @ $t = t_0$



(d) For $t > t_0$, $T\dot{y} + y = 0 \Rightarrow y = Ce^{-t/T}$

B.C. $t = t_0$, $y = 0.32A \Rightarrow y = 0.32A\,e^{-(t-t_0)/T}$

$= 0.86Ae^{-t/T}$

[Many people got mixed up over t, t_0 and T , most sketching three curves, one for each value of t_0/T .]

11.

(a) $$[m]=\begin{bmatrix} m & 0 \\ 0 & m/10 \end{bmatrix}$$ by inspection.

$$[k]=\begin{bmatrix} k+\alpha k & -\alpha k \\ -\alpha k & \alpha k \end{bmatrix}$$ using influence coefficients

(b) $$-[m]\,\omega^2 \underline{y}+[k]\,\underline{y}=\underline{f}$$

$$\underline{y}=\left[[k]-[m]\omega^2\right]^{-1}\underline{f}$$

$$=\begin{bmatrix} k(1+\alpha)-m\omega^2 & -\alpha k \\ -\alpha k & \alpha k-m\omega^2/10 \end{bmatrix}^{-1}\underline{f}$$

$$Y_1=\frac{k\alpha-m\omega^2/10}{\Delta}F$$ where Δ is the determinant of

$$\Delta=\left(k\alpha-\frac{m\omega^2}{10}\right)\left(k(1+\alpha)-\omega^2 m\right)-k^2\alpha^2$$

(c) Resonances occur when $\Delta=0$

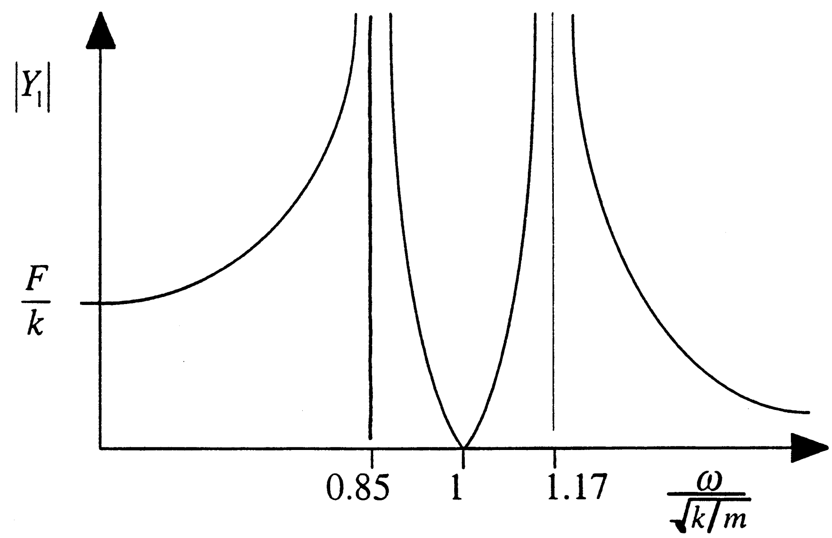
$$\frac{\omega^4 m^2}{10k^2}-\omega^2 m\left(k\alpha+\frac{k(1+\alpha)}{10}\right)-(k\alpha)^2+k^2\alpha(1+\alpha)=0$$

$$\Rightarrow \quad \frac{\omega^4 m^2}{k^2}-\frac{2.1\omega^2 m}{k}+1=0$$

$$\frac{\omega^2 m}{k}=\frac{2.1\pm\sqrt{2.1^2-4}}{2}$$

$$=0.73,\ 1.37$$

$$\omega=(0.85,\ 1.17)\sqrt{k/m}$$



$|Y_1|=0$ when $k\alpha=\omega^2 m/10 \Rightarrow \omega=\sqrt{k/m}$ (resonant frequency for added mass)

$|Y_1|=F/k$ for $\omega\to 0$ (static response)