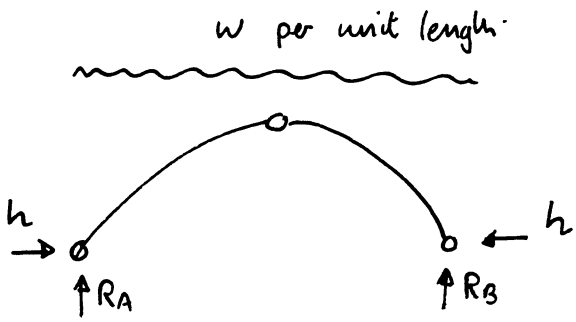
Engineering Tripos Part IA

Paper 2

Structures and Materials

Solutions

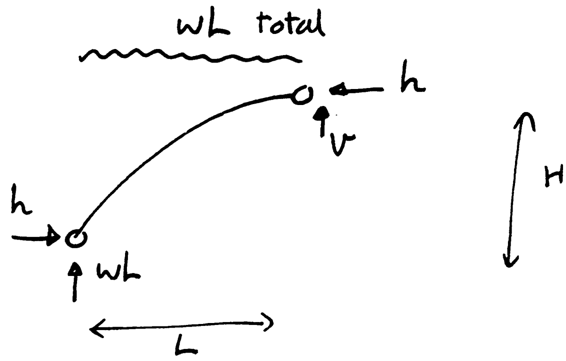
1(a)(i) Consider whole arch.



$R_A = R_B$ (symmetric)

Vertical equilibrium, $R_A = R_B = wL$

Consider half arch.

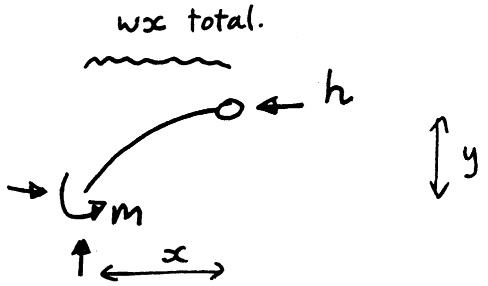


$\updownarrow$ equil, $\underline{\underline{v = 0}}$

Moments about left pin, $h \cdot H = wL \cdot \frac{L}{2}$

$$\underline{\underline{h = \frac{wL^2}{2H}}}$$

(ii) Cut arch at a distance x from centre



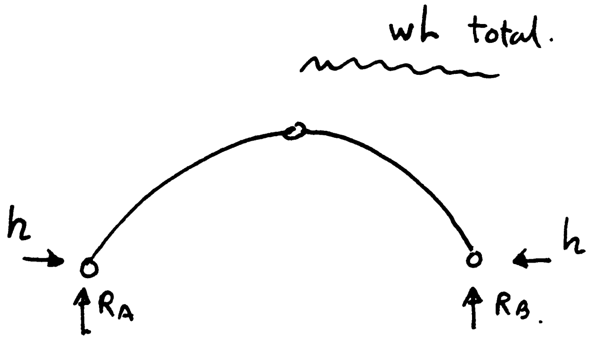
Moments about cut

$$M = wx.\frac{x}{2} - hy$$

substitute $h = \frac{wL^2}{2H}$, $y = H\frac{x^2}{L^2}$

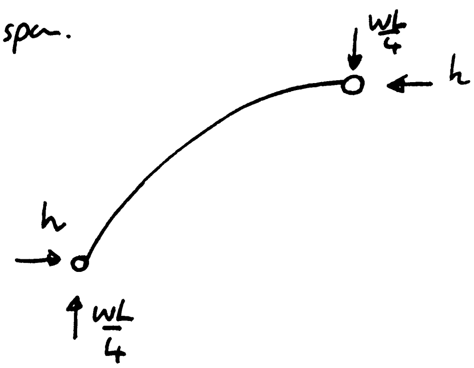
$$M = \frac{wx^2}{2} - \frac{wL^2}{2H}.\frac{Hx^2}{L^2} = 0 \quad \text{for any } x.$$

(b i) Consider whole arch.



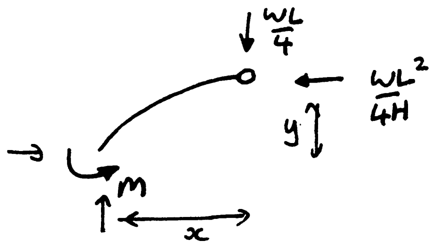
Moments about supports, $R_A = \frac{wL}{4}$ $\quad R_B = \frac{3wL}{4}$

Consider unloaded half-span.



$\circlearrowleft$ Equil, $\quad h.H = \frac{wL}{4}.L \quad \Rightarrow \quad h = \frac{wL^2}{4H}$

Cut at a distance x from centre



Moments about cut

$$M = \frac{wL}{4}.x - \frac{wL^2}{4H}.y$$

$$M = \frac{wL}{4}.x - \frac{w\cancel{L^2}}{4\cancel{H}}.\frac{\cancel{H}x^2}{\cancel{L^2}} = \frac{w}{4}(xL - x^2) \quad (*)$$

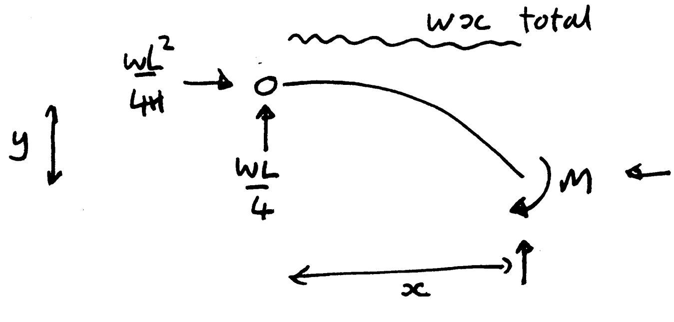
(Check, $M=0$ at $x=0$ and $x=L$ ✓)

$$\frac{dM}{dx} = \frac{w}{4}(L - 2x) = 0 \quad \text{for max/min}$$

$$\Rightarrow x = \frac{L}{2}$$

$\therefore$ $\underline{\underline{M = \frac{wL^2}{16}}}$ at $x = \frac{L}{2}$ is max magnitude.

(bii) Consider loaded half-span, cut at a distance x from centre.



Moments about cut

$$M = -\frac{wL}{4}.x - \frac{wL^2}{4H}.y + wx.\frac{x}{2}$$

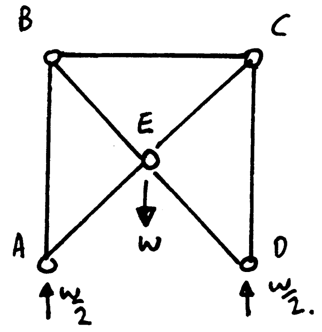
$$M = -\frac{wL}{4}.x - \frac{w\cancel{L^2}}{4\cancel{H}}.\frac{\cancel{H}x^2}{\cancel{L^2}} + \frac{wx^2}{2}$$

$$M = \frac{w}{4}(-xL + x^2)$$

(n.b., $-1 \times$. eqn * for unloaded span)

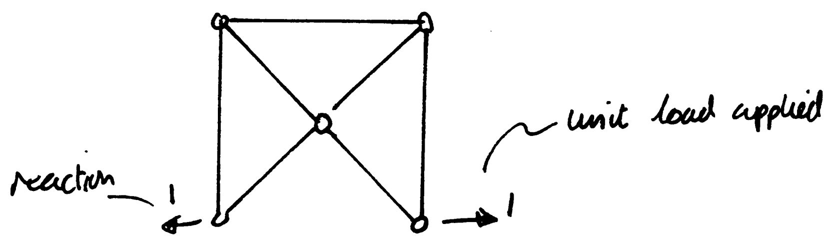
Hence $\underline{\underline{M = -\dfrac{wL^2}{16}}}$ at $x = \dfrac{L}{2}$ is ~~max~~ min magnitude.

2. Real System



For results, see table below.

Virtual System



			REAL			VIRTUAL		
Bar	Length ($\times L$)	Area ($\times A$)	Bar Forces ($\times W$)	Extension due to load, e_1 ($\times \frac{WL}{AE}$)	Extension due to local heat. e_2	Virtual Bar Force T^*	$T^* e_1$ ($\times \frac{WL}{AE}$)	$T^* e_2$
AB	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$	0
BC	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$	0
CD	1	1	$-\frac{1}{2}$	$-\frac{1}{2}$	0	-1	$\frac{1}{2}$	0
AE	$\frac{1}{\sqrt{2}}$	$\frac{1}{4}$	0	0	$\frac{L}{\sqrt{2} \times 100}$	$\sqrt{2}$	0	$\frac{L}{100}$
BE	$\frac{1}{\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{2}}$	2	0	$\sqrt{2}$	$2\sqrt{2}$	0
CE	$\frac{1}{\sqrt{2}}$	$\frac{1}{4}$	$\frac{1}{\sqrt{2}}$	2	0	$\sqrt{2}$	$2\sqrt{2}$	0
DE	$\frac{1}{\sqrt{2}}$	$\frac{1}{4}$	0	0	0	$\sqrt{2}$	0	0

$\Sigma = 7.16 \frac{WL}{AE}$ $\quad \Sigma = \frac{L}{100}$

(a) See table

(b) Virtual Work.

Virtual equil system. Only load is 1 horizontally at D

$$\sum T^* . e_1 = 1 . \delta_1$$

Real extensions and compatible horizontal displacement at D

$$\delta_1 = \sum T^* e_1 = \underline{\underline{7.16 \frac{WL}{AE}}}$$

(c) Virtual Work.

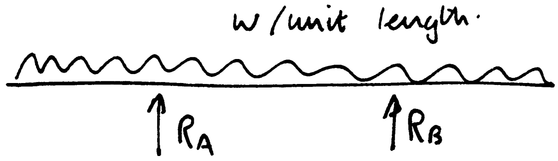
Virtual equil. system as above

$$\sum T^* e_2 = 1 . \delta_2$$

Real extension of AE, and compatible horizontal displacement at D

$$\delta_2 = \sum T^* e_2 = \underline{\underline{\frac{L}{100}}}$$

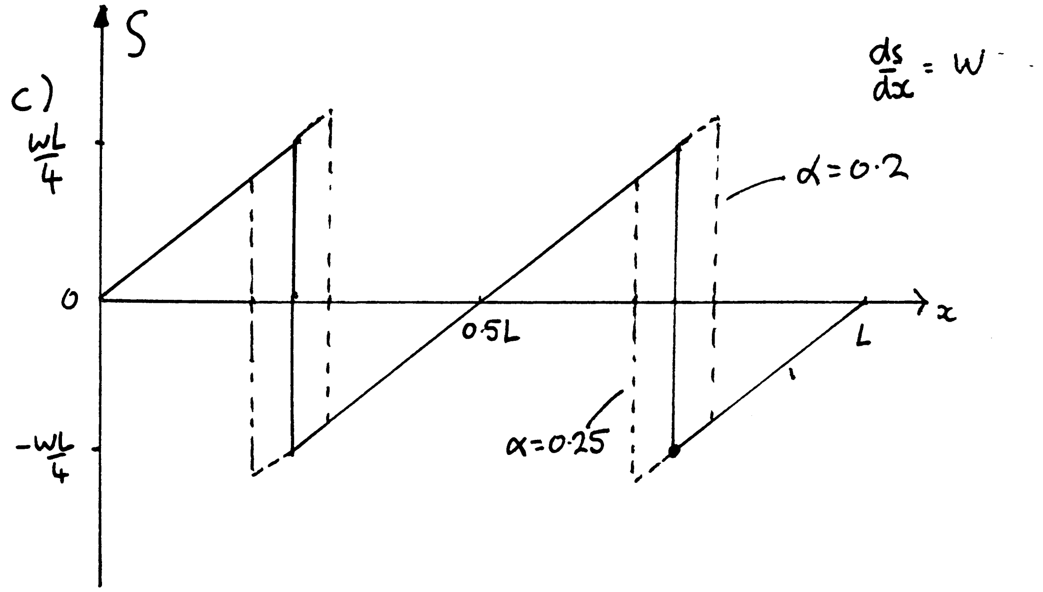
3(a)

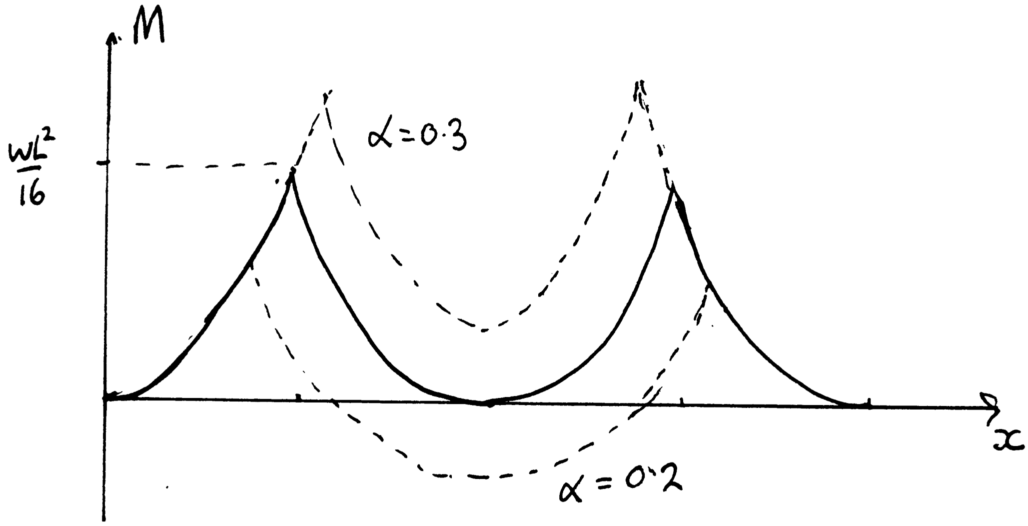


By ~~symmetry~~, $R_A = R_B$ for any α

$\therefore R_A = R_B = \dfrac{wL}{2}$

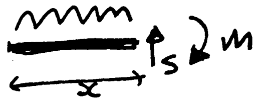
(b, c)





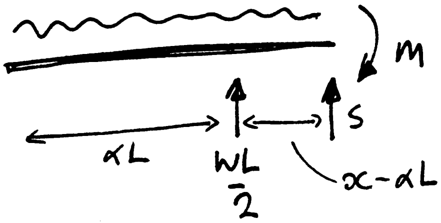
d i) Consider a cut in AB

wx total load



$$M = wx \cdot \frac{x}{2} = \underline{\underline{\frac{wx^2}{2}}}$$

(ii) Consider a cut in BC — wx total load.



$$M = wx \cdot \frac{x}{2} - \frac{wL}{2} \cdot (\cancel{x} - \alpha L)$$

$$\underline{\underline{M = \frac{w}{2}\left(x^2 - Lx + \alpha L^2\right)}}$$

(e) Max hogging at support, $x = \alpha L$

$$M_H = \alpha^2 \frac{wL^2}{2}$$

Max sagging at centre of beam, $x = \frac{L}{2}$

$$M_S = \frac{w}{2}\left(\frac{L^2}{4} - \frac{L^2}{2} + \alpha L^2\right)$$

For these to be equal in magnitude, $M_H = -M_S$

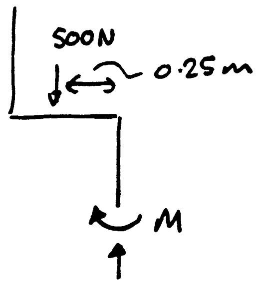
$$\alpha^2 L^2 = -\frac{L^2}{4} + \alpha L^2$$

$\div L^2$

$$\alpha^2 + \alpha - \frac{1}{4} = 0 \quad \Rightarrow \quad \alpha = \frac{-1 \pm \sqrt{1+1}}{2}$$

$\alpha > 0 \quad \Rightarrow \quad \alpha = \underline{\underline{\frac{\sqrt{2}-1}{2}}}$

4\. (a) Free body



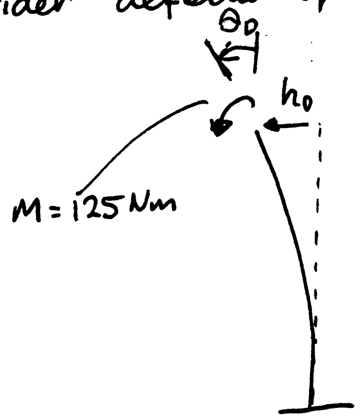
$M = 500\,N \times 0.25\,m$
$= \underline{\underline{125\ Nm}}$

(b) Data Book

$\theta_E = \dfrac{Ml}{3EI}$

$$\theta_E = \frac{125 \times 0.5}{3 \times 2000} = \underline{\underline{0.0104\ \text{rad}}}$$

(c) Consider deflection of DE

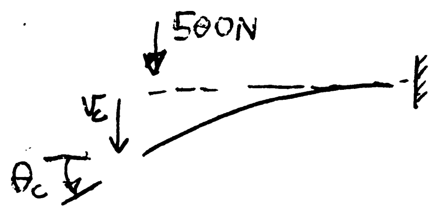


from Data Book

$$\theta_D = \frac{Ml}{EI} = \frac{125 \times 0.5}{2000}$$
$$= 0.03125\ \text{rad}$$

$$h_D = \frac{Ml^2}{2EI} = \frac{125 \times 0.5^2}{2 \times 2000}$$
$$= 0.0078\ m$$

Consider deflection of CD (assuming DEF rigid).



$$\theta_C = \frac{wl^2}{6EI} = \frac{500 \times 0.25^2}{6 \times 2000}$$
$$= 0.00\ \text{rad}$$

$$V_C = \frac{wl^3}{3EI} = \frac{500 \times 0.25^3}{3 \times 2000}$$
$$= 0.0013\ m$$

u_c : Horizontal deflection of $C = h_D = \underline{\underline{0.0078\,m}}$

v_c = Vertical deflection of $C = \Theta_D \times 0.25\,m + v_c$

$= \underline{\underline{0.0091\ m}}$

ϕ_c = Rotation of C $= \Theta_D + \Theta_C$

$= \underline{\underline{0.0391\ rad}}$

(d) Vertical deflection of A

$= v_c + \phi_c \times 0.25m + \Theta_E \times 0.5m$

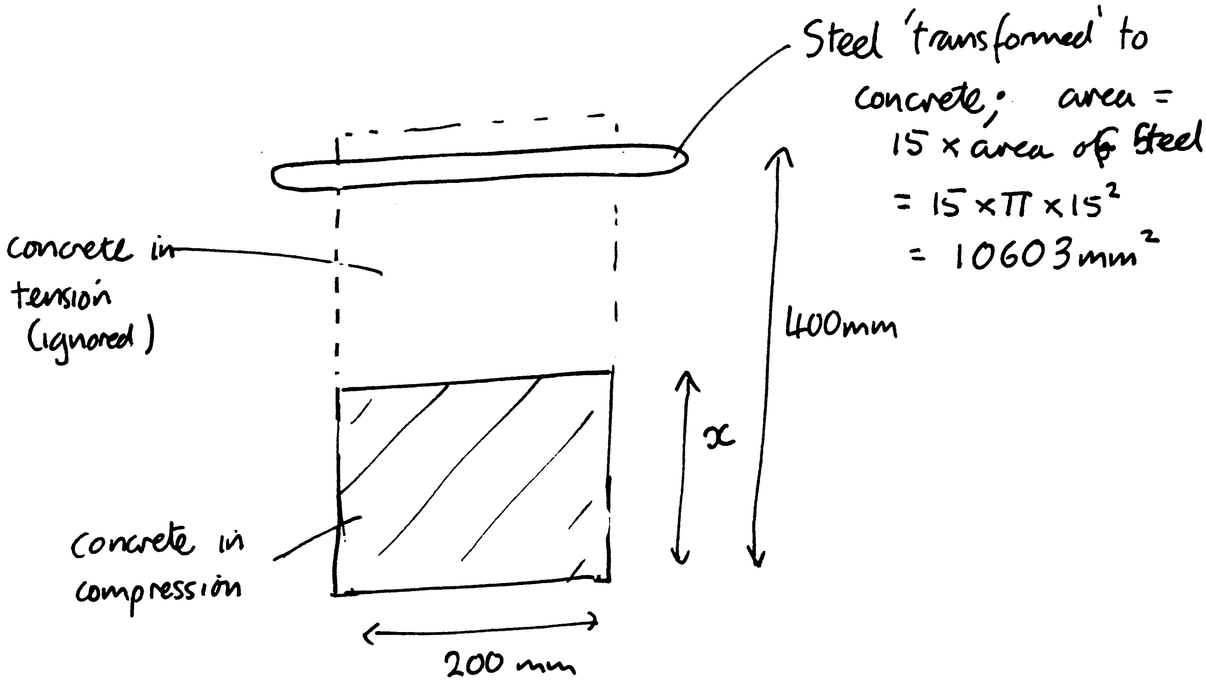
$= \underline{\underline{0.0241\ m}}$

(e) Horizontal deflection of A

$= u_c + \phi_c \times 0.5m + \Theta_E \times 1.0m$

$= \underline{\underline{0.0378\ m}}$

5 (a) $\dfrac{E_{steel}}{E_{concrete}} = \dfrac{210}{14} = 15$



(b) Neutral axis will be distance x from bottom face.
'see-saw' analogy so 1st moment of area = 0 about NA

$$\underbrace{200.x}_{\text{area}} \times \underbrace{\frac{x}{2}}_{\substack{\text{distance}\\\text{to}\\\text{centroid}}} = \underbrace{10603}_{\text{area}} \times \underbrace{(\cancel{4}00 - x)}_{\substack{\text{distance}\\\text{to}\\\text{centroid}}}$$

$$\therefore 100x^2 + 10603x - 4.241 \times 10^6 = 0$$

(x must be +ve)

$$x = \frac{-10603 + \sqrt{10603^2 + 4 \times 100 \times 4.241 \times 10^6}}{200}$$

$$\underline{\underline{x = 159.6 \text{ mm}}}$$

ENGINEERING TRIPOS PART IA

PAPER 2: STRUCTURES AND MATERIALS (JUNE 1998)

SOLUTIONS TO SECTION B: MATERIALS

Note: Examiner's comments are given at the end of the crib

Question 6

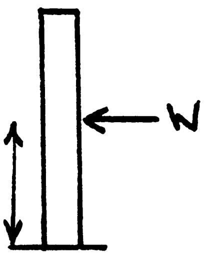
(a) Young's modulus is defined as the initial gradient of the (linear) elastic stress-strain response of an engineering material under uniform tensile or compressive loading.

Three methods for measuring Young's modulus:

- measure extension of a tensile specimen as a function of load (or deflection of a beam under a simple bending configuration)
- measure natural frequency of vibration of a uniform beam
- measure speed of sound in material using piezo-electric transducers

The deflection methods are difficult to do accurately, due to errors in measuring small deflections; the natural frequency method is better, if beams can be made with precise dimensions; the speed of sound is most accurate of all.

(b) (i)



Maximum moment $M_{max} = \frac{Wl}{2}$

$\frac{\sigma}{R} = \frac{M}{I}$ where $I = \frac{\pi R^4}{4}$

$\therefore \ \sigma = \frac{2Wl}{\pi R^3}$

(ii) Mass $m = \pi R^2 l \rho$ (objective)

From (i), at failure $\therefore \ \sigma_f = \frac{2Wl}{\pi R^3}$ (constraint)

Substitute for R (free variable): Mass $m = \pi^{1/3} l^{5/3} (2W)^{2/3} \frac{\rho}{\sigma_f^{2/3}}$

Performance index: $M_1 = \frac{\sigma_f^{2/3}}{\rho}$ (maximise for minimum mass)

Material	M_1
CFRP	28.0
Wood	35.9
Steel	6.5
Al	18.4
Concrete	8.8

So CFRP and wood appear the best.

(iii) $\sigma_f = \frac{2Wl}{\pi R^3}$, so for $R \leq R_{max}$, $\sigma_f \geq \frac{2Wl}{\pi R_{max}^3}$

For $W = 40$ kN, $l = 0.6$ m, $R \leq 0.05$m: $\sigma_f \geq 122$ MPa.

This rules out wood (and concrete), so the best two materials are now CFRP and Al alloy.

(c) Other factors:

Cost - CFRP or Al alloy
Corrosion - Al alloy
Difficulty with fabrication - CFRP
Fracture toughness - CFRP (and perhaps Al alloy)
(Stiffness should not be an issue for either CFRP or Al alloy).

<u>Question 7</u>

(a) Nominal stress in a tensile test is the applied force divided by the original cross-sectional area, while true stress is the force over the current cross sectional area.

Nominal strain is the change in length of the specimen divided by the original length, while true strain is the incremental extensions per unit length integrated from the initial to the current length.

$$\sigma_t = \frac{F}{A},\ \sigma_n = \frac{F}{A_o},\ \varepsilon_t = \ln\left(\frac{l}{l_o}\right),\ \varepsilon_n = \frac{l - l_o}{l_o} = \frac{l}{l_o} - 1$$

Assume constant volume: $Al = A_o l_o$

$$\sigma_t = \frac{Fl}{l_o A_o} = \sigma_n (1 + \varepsilon_n)$$

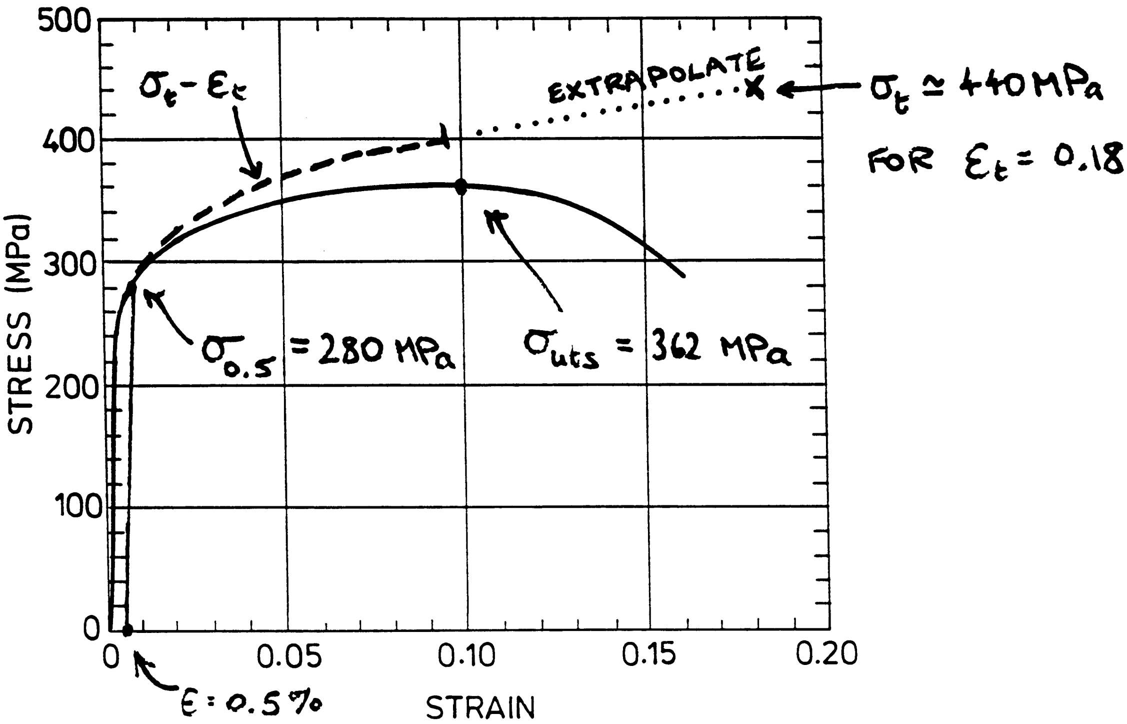
$$\varepsilon_t = \ln(1 + \varepsilon_n)$$

For $\frac{\sigma_t}{\sigma_n} \leq 1.01$ (i.e. 1% difference), $\varepsilon_n \leq 0.01$

(b) (i) See figure: 0.5% proof stress = 280MPa
Tensile strength = 362 MPa

(ii)

ε_n	σ_n	ε_t	σ_t
0.02	320	0.0198	326
0.04	340	0.0392	353
0.06	355	0.058	376
0.08	360	0.077	389
0.10	362	0.095	398

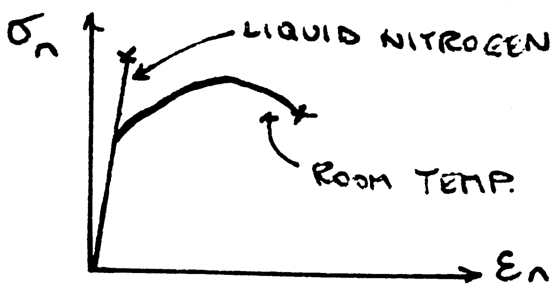


(iii) No change in width of rolled strip, therefore for conservation of volume: $t\,l = t_o\,l_o$ where t is the strip thickness.

Nominal strain (elongation) $= \frac{l}{l_o} - 1 = \frac{t_o}{t} - 1 = 0.2$, and true strain = 0.18.

The true strain in rolling the strip is larger than can be achieved in tension (rolling avoids failure by necking), so the true stress after rolling cannot be found accurately from the graph. By extrapolating the true stress-strain curve to a true strain of 0.18, a rough estimate is $\sigma_t \approx 440$ MPa, and thus an estimate for hardness $H = 3\sigma_t \approx 1320$ MPa

(c) A fairly ductile stress-strain curve is obtained at room temp in medium carbon steel, showing work hardening before necking, similar to the aluminium alloy. In liquid nitrogen, brittle fracture occurs with elastic loading virtually up to the point of fracture.



This occurs because slip is more difficult as temperature falls in BCC iron, whereas FCC aluminium is always ductile.

Question 8

(a) Welding often introduces crack-like defects and these may grow in fatigue due to applied cyclic stresses. The situation is worse if welding also causes tensile residual stress. Carbon steels can form brittle microstructures, which are particularly susceptible to fracture in the presence of hyrdrogen, so cracks can quickly grow by fatigue to critical size for fast fracture.

(b) Closed-loop central heating systems do not rust as the dissolved oxygen is expelled from the water in the system after a short period of use. Leaks cause fresh water to be brought in continuously - this water contains oxygen and enables the corrosion of the iron in the radiators.

(c) Gold is stable in metallic form as it has a positive energy of oxide formation. Iron has a negative energy of oxide formation, and therefore oxidises. Aluminium has a higher negative energy of oxide formation than iron, so reacts strongly with oxygen. However, the oxide formed is protective, once a thin film of alumina has formed, it acts as an effective barrier to inward oxygen diffusion and prevents further oxidation of the metal.

(d) Asperities on the surface make contact and junctions grow by elastic flattening (and yielding if the load is sufficiently high) until the load is supported. The required contact area to carry the loads typically applied in engineering applications is only a small proportion of the apparent area of contact.

(e) Oil is drawn into the sliding surfaces of a bearing and keeps the metal surfaces apart, reducing local surface stresses, and preventing wear mechanisms from operating. The forging of the bearing is conducted at high temperatures, well above the temperature at which lubricating oils decompose.

Question 9

(a) A thermally activated process is one in which the mechanism controlling the process involves diffusion at the atomic level, i.e. it depends on the rate at which atoms can jump to new positions. Atomic mobility is strongly determined by temperature, following Arrenhius' Law: rate $\propto$ exp(-Q/RT).

Creep of metallic alloys occurs at low stresses (provided temperature is greater than about half the melting point) by "diffusional flow". Atoms diffuse from grain boundaries perpendicular to the applied stress to those normal to the applied stress, giving continuous elongation at a low overall strain rate. The overall rate depends on the rate of atomic diffusion, either along grain boundaries or through the bulk of the grains.

(b) (i) Reading from graph for $\sigma = 40$ MPa, $T = 538$ °C:

$\dot{\varepsilon} \approx 0.24\%/1000$ h. so strain in 5000 hours $= 5 \times 0.24\% = 1.2\ \%$

Change in length $= 0.012 \times 750\,\text{mm} = 9\,\text{mm}$

S.

(ii) To find n: use 2 conditions at constant T

$\dot{\varepsilon} = B\sigma^n$ (where $B = A \exp -Q/RT$)

$\log \dot{\varepsilon} = \log B + n \log \sigma$

$\log \sigma = \frac{1}{n} \log \dot{\varepsilon} - \frac{1}{n} \log B$

Slope of const. T lines on log graph of σ vs. $\dot{\varepsilon}$ is $1/n$

e.g. for T = 538 °C: $\dot{\varepsilon} = 10^{-2}$, $\sigma = 20$ MPa
$\dot{\varepsilon} = 1$, $\sigma = 50$ MPa

$\Delta(\log \dot{\varepsilon}) = 2,\ 1,\ \Delta(\log \sigma) = 0.4 \Rightarrow \frac{1}{n} = \frac{0.4}{2} \Rightarrow n = 5$

To find Q: use conditions at const. $\dot{\varepsilon}$

$\ln \dot{\varepsilon} = \ln A + n \ln \sigma - \frac{Q}{RT}$

$\therefore\ n \ln \sigma - \frac{Q}{RT} = \ln \dot{\varepsilon} - \ln A$ (= constant)

$\therefore\ n(\ln \sigma_1 - \ln \sigma_2) = \frac{Q}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$ (temperature in *Kelvin*)

From graph, for $\dot{\varepsilon} = 10^{-2}$: $\sigma_1 = 20$ MPa at $T_1 = 538$ °C = 811 K
$\sigma_2 = 50$ MPa at $T_2 = 427$ °C = 700 K

R = 8.314, n = 5 $\therefore$ $Q = 195$ kJ / mol

(c) (i) $\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2} = \left(\frac{\sigma_1}{\sigma_2}\right)^n = \left(\frac{P_1}{P_2}\right)^n$

$\frac{\dot{\varepsilon}_1}{\dot{\varepsilon}_2} = 1.5$ (50% higher strain in given time)

$\left(\frac{P_1}{P_2}\right)^5 = 1.5 \Rightarrow P_2 = 6.51$ kN

(ii) New $\dot{\varepsilon} = \dot{\varepsilon}_3$ at temp. $T_3 = 430$ °C = 703 K
Old $\dot{\varepsilon} = \dot{\varepsilon}_2$ at temp. $T_2 = 450$ °C = 723 K (both with load P = 6.51 kN)

$\frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_2} = \exp -\frac{Q}{R}\left(\frac{1}{T_3} - \frac{1}{T_2}\right) \Rightarrow \frac{\dot{\varepsilon}_3}{\dot{\varepsilon}_2} = 0.4$

Hence same extension will occur in 2.5 times the time which it took at the original temperature, i.e. 25, 000 hours. This is less than the required 40,000 hours so the drop in temperature is not sufficient.

(d) Metallurgical route: impede dislocation motion by alloying to give solid solution hardening, and precipitation hardening.

Processing route: cast single crystal blades to eliminate fast diffusion paths along grain boundaries.

Question 10

(a) Fatigue failure in metals means crack growth under cyclic loading leading to a critical crack length which causes fast fracture. Crack growth may occur from a pre-eixsting flaw, or a crack may first initiate under cyclic loading.

Low cycle fatigue means failure under cyclic loading in a relatively small number of cycles (< 10^3 approx.) Failure is caused by reversed plastic strain in the whole component on each cycle, leading to internal damage and failure. The number of cycles to failure is controlled by the plastic strain range in each cycle.

High cycle fatigue means failure under cyclic loading in a large number of cycles (> 10^4 approx.) Failure is caused by localised reversed slip in favourably oriented grains leading to crack nucleation, and propagation to failure. The number of cycles to failure is dominated by nucleation, and is controlled by the applied stress range.

(b) (i) Goodman's rule: $\Delta\sigma = \Delta\sigma_o \left(1 - \frac{\sigma_m}{\sigma_{uts}}\right)$

$\sigma_m = 150$ MPa, $\sigma_m = 150$ MPa, $\therefore \frac{\sigma_m}{\sigma_{uts}} = 0.25$

From test 1, $N_f = 10^4$ cycles $\Delta\sigma_o = \frac{630}{1 - 0.25} = 840$ MPa

(ii) Miner's rule: $\sum_i \frac{N_i}{N_{fi}} = 1$

For first part of each test $\Delta\sigma_1 = 630$ MPa, $N_{f1} = 10^4$, hence calculate N_1 / N_{f1}

Then find N_2 / N_{f2} $(= 1 - N_1 / N_{f1})$; N_2 is given in the Table, hence find N_{f2}.

Test	N_1 / N_{f1}	N_2 / N_{f2}	N_2	N_{f2}
1	1.0	0	--	--
2	0.5	0.5	5 x 10^5	10^6
3	0.5	0.5	1.2 x 10^5	2.4 x 10^5
4	0.25	0.75	4.4 x 10^4	5.9 x 10^4

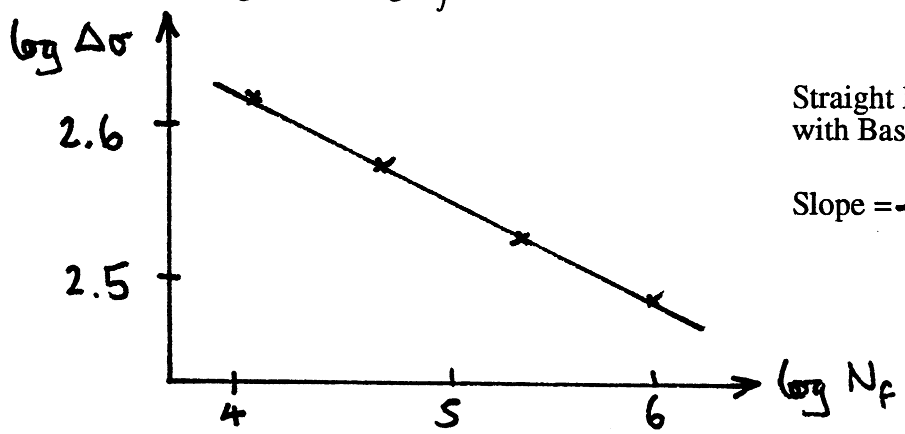
(iii) Convert $\Delta\sigma$ to equivalent with zero mean stress by dividing by (1 - 0.25) as in part (i):

Test	$\Delta\sigma$	$\Delta\sigma_0$	N_f
1	630	840	10^4
2	460	614	10^6
3	510	680	2.4×10^5
4	560	746	5.9×10^4

(iv) Basquin's Law: $\Delta\sigma\, N_f^{\alpha} = C_1$

$$\log \Delta\sigma = \alpha \log N_f + \log C_1$$

Plot $\log \Delta\sigma \rightarrow \log N_f$



Straight line, so data consistent with Basquin's Law.

Slope = $-\alpha$ = 0.065

(c) Engineering components which fail in fatigue:

Bicycle components: spokes, wheel axles
Rotating machine components: shafts, bearings

(c) $I = \int y^2 dA$

$$= \int_0^{159.6} y^2 . 200dy + 10603 \times (400-159.6)^2$$

$$= {}_0^{159.6}\left[\frac{200y^3}{3}\right] + 612.8 \times 10^6$$

$$= 883.8 \times 10^6 \ mm^4$$

E of concrete = 14000 N/mm^2

$\therefore EI = 12.37 \times 10^{12}$ Nmm^2

curvature

'modulus ratio'

(d) Stress in steel $\sigma_s = E y . k . 15$

$\therefore \sigma_s = 400$ N/mm^2 (yield)

$$400 = 14000 \times (400-159.6) . 15 . k$$

$$k = 7.29 \times 10^{-6} \ mm^{-1}$$

Max stress in concrete

$$\sigma_c = E . y . k$$

at $\sigma_c = 40$ N/mm^2 (failure)

$$40 = 14000 \times 159.6 \times k$$

$$k = 17.90 \times 10^{-6} \ mm^{-1}$$

$\therefore$ Steel will yield first at a curvature of $\underline{\underline{7.29 \times 10^{-6} \ mm^{-1}}}$