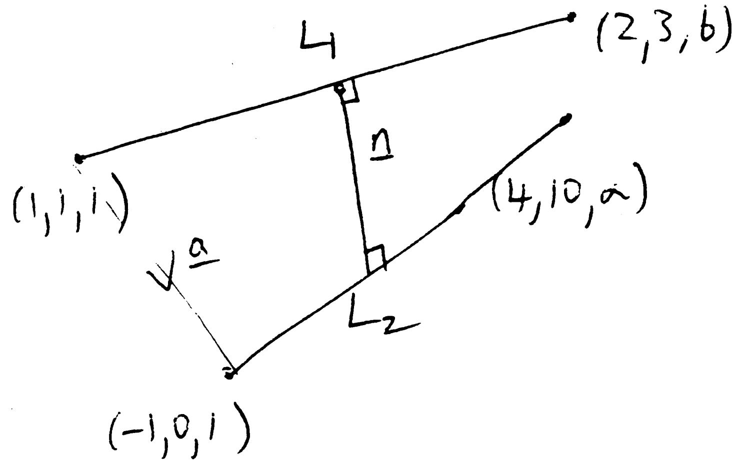
Ia Paper 4, Section A (51) 1998

i. L_1 passes thro' $(1,1,1)$ & $(2,3,6)$



line $\parallel$ to L_1 is $(1,2,5)$

line $\parallel$ to L_2 is $(5,10,\alpha-1)$

$\therefore$ line $\perp$ to both L_1 & L_2 is $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 5 \\ 5 & 10 & \alpha-1 \end{vmatrix}$

$= (2\alpha-52,\ 26-\alpha,\ 0) = \underline{n} = (\alpha-26)[2,-1,0]$

$\hat{\underline{n}} = \dfrac{[2,-1,0]}{\sqrt{5}}$

vector $\underline{a} = (-2,-1,0)$

$\underline{a}.\hat{\underline{n}} = \dfrac{-4+1}{\sqrt{5}}$ $\therefore$ shortest distance $= 3/\sqrt{5}$.

$\alpha=26$ the 2 lines are parallel $\Rightarrow$ shortest distance everywhere.

Plane P passes $(0,0,0)$, normal $(\beta,-2,1) = \underline{m}$

$\underline{r}.\underline{m} = 0$

eqn. of L_1 is $\underline{r} - (1,1,1) = \lambda(1,2,5)$.

dot with $\underline{m}$ $\quad \underline{r}.\underline{m} - [\beta-2+1] = \lambda(\beta-4+5)$

$\underline{r}.\underline{m} = 0 \quad \therefore \lambda = -\dfrac{(\beta-1)}{\beta+1}$

$\beta \neq -1 \quad \therefore$ pt of intersection is $(1,1,1) - \dfrac{(\beta-1)}{\beta+1}(1,2,5)$

$= \dfrac{(2,\ -\beta+3,\ -4\beta+6)}{\beta+1}$

$\beta=-1$ no intersection, line $\parallel$ plane

2 a) $\underset{x\to 0}{Lt}\ \frac{x^3}{x-\sin x\cos x} = \underset{x\to 0}{Lt}\ \frac{3x^2}{1-\cos^2 x+\sin^2 x}$

$= \underset{x\to 0}{Lt}\ \frac{6x}{4\sin x\cos x} = \underset{x\to 0}{Lt}\ \frac{6}{4\cos^2 x-4\sin^2 x}$

$= \frac{6}{4-0} = \underline{\underline{\frac{3}{2}}}$

b) $\cos z = 2 \qquad z = x+iy$

$\cos x\cos iy - \sin x\sin iy = 2$

$\cos x\cosh y = 2$ ① $\quad \sin x\sinh y = 0$ ②

② $\Rightarrow y=0$ or $x = n\pi$ n integer

in ① if $y=0$ $\cos x = 2$ (not possible)

if $x = n\pi$ $\quad (-1)^n\cosh y = 2 \quad \Rightarrow \cosh y = 2(-1)^n$

$\Rightarrow y = 1.32$, n even $\quad \therefore$ sol^{n}s $\underline{\underline{2n\pi + i\times 1.32}}$

no solution, n odd.

c) $\frac{1-\cos x}{e^x(1+x)} = e^{-x}\left(\frac{x^2}{2!}-\frac{x^4}{4!}+\dots\right)\left(1-x+x^2+\dots -x^3\dots\right)$

$= e^{-x}\left\{\frac{x^2}{2!}-\frac{x^4}{4!}-\frac{x^3}{2!}+\frac{x^4}{2!}\right\}$

$= \left(1-x+\frac{x^2}{2}\right)\left(\frac{x^2}{2}-\frac{x^4}{24}-\frac{x^3}{2}+\frac{x^4}{2!}\right)$

$= \frac{x^2}{2}-\frac{x^4}{24}-\frac{x^3}{2}-\frac{x^3}{2}+\frac{x^4}{2}+\frac{x^4}{2}+\frac{x^4}{2} = \underline{\underline{\frac{x^2}{2}-x^3+\frac{29x^4}{24}}}$

3 a)

$$\sin x \frac{dy}{dx} + y\cos x = x\cos x$$

$$\frac{d}{dx}(y\sin x) = x\cos x$$

$$y\sin x + c = \int x\cos x = x\sin x - \int \sin x$$

$$= x\sin x + \cos x$$

when $x = \pi/2$ $\quad y = 1 \quad \therefore 1 + c = \frac{\pi}{2}$

$$c = \frac{\pi}{2} - 1$$

$$\therefore \underline{y = \frac{x\sin x + \cos x - \frac{\pi}{2} + 1}{\sin x}}$$

b) $y = e^{\lambda t}$ $\quad \lambda^2 - 2\lambda - 24 = 0 \Rightarrow \lambda = 6$ or -4

CF $\quad y = \alpha e^{6t} + \beta e^{-4t}$

PI try $\quad y = \gamma t e^{6t}$ $\quad y' = \gamma e^{6t} + 6\gamma t e^{6t}$

$$y'' = 12\gamma e^{6t} + 36\gamma t e^{6t}$$

$$\therefore \cancel{36\gamma t e^{6t}} + 12\gamma e^{6t} - 2\gamma e^{6t} - 12\gamma t e^{6t} - \cancel{24\gamma t e^{6t}} = e^{6t}$$

$$10\gamma = 1 \Rightarrow \gamma = \frac{1}{10}$$

$\therefore$ general sol^n $\quad \underline{\underline{y = \alpha e^{6t} + \beta e^{-4t} + \frac{te^{6t}}{10}}}$

4 a) $\quad 2y_{n+1} - 13y_n + 6y_{n-1} = 0$

$y_n = \lambda^n \qquad 2\lambda^2 - 13\lambda + 6 = 0$

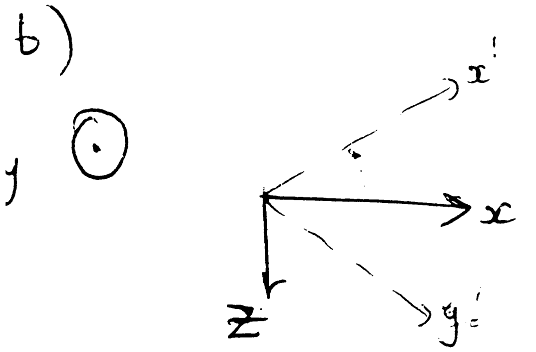
$$(2\lambda - 1)(\lambda - 6) = 0$$

$\lambda = 6$ or $1/2$

$$\therefore\ y_n = \alpha 6^n + \beta 2^{-n}$$

$y_n \to 0$ as $n \to \infty$ $\therefore \alpha = 0 \qquad y_0 = 1 \therefore \beta = 1 \qquad \underline{y_n = 2^{-n}}$.

b)



$\begin{pmatrix}1\\0\\0\end{pmatrix} \Rightarrow \begin{pmatrix}\frac{1}{\sqrt{2}}\\0\\\frac{1}{\sqrt{2}}\end{pmatrix}$ relative to new axes

$\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\end{pmatrix}$

$\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}$

$\underset{\sim}{v}' = R\underset{\sim}{v} \qquad R = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$

$\underset{\sim}{y} = A\underset{\sim}{x} \qquad R\underset{\sim}{y} = RA\underset{\sim}{x} = RAR^TR\underset{\sim}{x} \qquad y' = A'\underset{\sim}{x}$

where $A' = RAR^T$

$$A' = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}2 & 1 & 2\\ 3 & 0 & 6\\ 2 & 6 & -1\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}0 & \frac{-5}{\sqrt{2}} & \frac{3}{\sqrt{2}}\\ 3 & 0 & 6\\ \frac{4}{\sqrt{2}} & 7/\sqrt{2} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}1/\sqrt{2} & 0 & 1/\sqrt{2}\\ 0 & 1 & 0\\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\end{pmatrix}$$

$$= \begin{pmatrix}-3/2 & -5/\sqrt{2} & 3/2\\ -3/\sqrt{2} & 0 & 9/\sqrt{2}\\ 3/2 & 7/\sqrt{2} & 5/2\end{pmatrix} \qquad D$$

5. a) $\lambda_1^n, \lambda_2^n, \lambda_3^n \quad \underline{e}_1, \underline{e}_2, \underline{e}_3$

b) $\lambda_1^{-1}, \lambda_2^{-1}, \lambda_3^{-1} \quad \underline{e}_1, \underline{e}_2, \underline{e}_3$

$\lambda = 10$ or $\begin{vmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{vmatrix} = 0 \quad (\lambda-3)^2 - 4 = 0$

$\lambda - 3 = \pm 2$

$\lambda = 5$ or 1.

$\lambda = 10 \quad \begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 10\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

e vector $\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}$

$\lambda = 5 \quad x = 0 \quad \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = 5\begin{pmatrix} y \\ z \end{pmatrix}$

$3y - 2z = 5y \quad y = -z \quad \therefore \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$

$\lambda = 1 \quad x = 0 \quad \begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y \\ z \end{pmatrix}$

$3y - 2z = y \quad y = z \quad \therefore \underline{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}}$

$\underset{\sim}{x} = (1,1,2) = (1\,0\,0) + \frac{3}{2}(0\,1\,1) - \frac{1}{2}(0,1,-1)$

$A^{10}\underset{\sim}{x} \approx \underline{10^{10}(1,0,0)}$

$(A^{-1})^{10}\underset{\sim}{x} \approx \underline{+\frac{3}{2}(0,1,-1)}$

1a Mathematics 1998 Section B – Cis.

6 (a) If the system response to $x(t)=h(t)$ is $y(t)=t$ and if the system is linear and stationary the " " " " $x(t)=h(t)$ $=$ $y(t) = \frac{d}{dt}(t) = \underline{\underline{1}}$ [because $\delta(t) = \frac{d}{dt}h(t)$]

(b) By convolution $y(t) = \int_{-\infty}^{t} f(\tau)\, g(t-\tau)\, d\tau$

where $f(t)$ = the input and $g(t)$ = the impulse response, or vice versa.

Assuming the latter:

$$y(t) = \int_0^t 1 \, . \, (t-\tau)^2 \, d\tau$$
$$= \int_0^t (t^2 - 2t\tau + \tau^2) . \, d\tau$$
$$= \left[t^2\tau - t\tau^2 + \frac{\tau^3}{3} \right]_0^t$$
$$= \underline{\underline{t^3/3}}$$

(c)

$$y(t) = \int_0^t 1 . \, e^{-(t-\tau)} \, d\tau$$
$$= e^{-t} \int_0^t e^{\tau} \, d\tau$$
$$= \underline{\underline{1 - e^{-t}}}$$

7 (i)

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{3+in}{n^2+1} e^{int}$$

Let $\bar{c}_n = \frac{3+in}{n^2+1}$ $\Rightarrow$

$$y(t) = \bar{c}_0 + \sum_{n=1}^{\infty} \left[(\bar{c}_n + \bar{c}_{-n}) \cos nt + \frac{1}{i}(\bar{c}_n - \bar{c}_n) \sin nt \right]$$
$$= 3 + \sum_{n=1}^{\infty} \left[\frac{6}{n^2+1} \cos nt + \frac{1}{i}\left(\frac{-in}{n^2+1} - \frac{in}{n^2+1} \right) \sin nt \right]$$
$$= \underline{\underline{3 + \sum_{n=1}^{\infty} \left[\frac{6}{n^2+1} \cos nt - \frac{2n}{n^2+1} \sin nt \right]}}$$

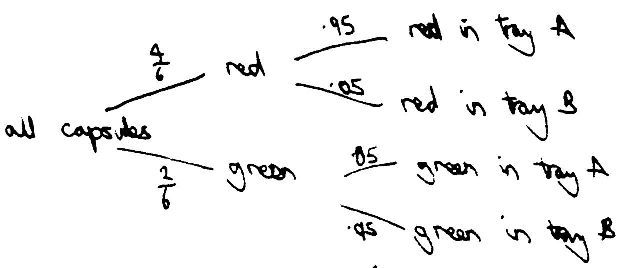
(ii)

$$\bar{c}_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \cosh t \; e^{-int} \, dt$$
$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} (e^{t} + e^{-t}) e^{-int} \, dt$$
$$= \frac{1}{4\pi} \int_{-\pi}^{\pi} \left(e^{(1-in)t} + e^{(-1-in)t} \right) t$$
$$= \frac{1}{4\pi} \left[\frac{e^{(1-in)t}}{1-in} - \frac{e^{(-1-in)t}}{1+in} \right]_{-\pi}^{\pi}$$
$$= \frac{1}{4\pi} \left[\frac{e^{in\pi}\, 2\sinh\pi}{1-in} - \frac{e^{in\pi}\, 2\sinh(-\pi)}{1+in} \right]$$
$$= \frac{\sinh\pi}{\pi} \cdot \frac{2e^{in\pi}}{1+n^2}$$
$$= \underline{\underline{\frac{2(-1)^n \sinh\pi}{\pi(1+n^2)}}}$$

So

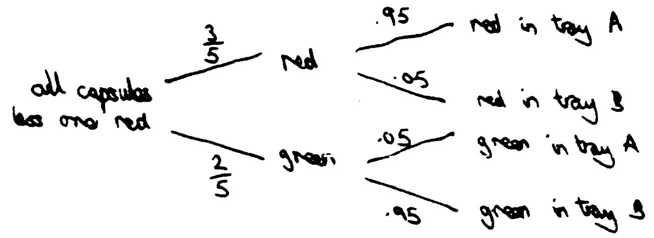
$$y(t) = \underline{\underline{\sum_{-\infty}^{\infty} \frac{2(-1)^n \sinh\pi}{\pi(1+n^2)} e^{int}}}$$

8. (a) $p(\text{red}) = \frac{4}{6}$



$$p(\text{Red} \,|\, \text{Tray B}) = \frac{\cdot 05 \times \frac{4}{6}}{\cdot 05 \times \frac{4}{6} + \cdot 95 \times \frac{2}{6}} = \underline{0.0952}$$

(b)



$$p(\text{Red} \mid \text{Tray A}) = \frac{.95 \times \frac{3}{5}}{.95 \times \frac{3}{5} + .05 \times \frac{2}{5}} = 0.966$$

$$p(\text{both capsules are red}) = 0.966 \times 0.0952 = \underline{\underline{0.092}}$$

(c) $p(\text{tray B contains only red capsules}) = p(\text{both green capsules are in tray A})$

$$= 0.05^2 = \underline{\underline{0.0025}}$$

[Strictly speaking, it is $\dfrac{0.05^2}{1-(.05)^2(.95)^4-(.95)^2(.05)^4}$ because there is already one capsule in each tray.]

9. (a)

$$(s^2+3s+2)\,\bar{y}(s) = \frac{1}{s}$$

$$\bar{y}(s) = \frac{1}{s(s+2)(s+1)}$$

$$= \frac{1/2}{s} + \frac{1/2}{s+2} - \frac{1}{s+1}$$

$$\underline{\underline{y(t) = \tfrac{1}{2} + \tfrac{1}{2}e^{-2t} - e^{-t}}}$$

(b)

$$s^2\bar{y}(s) - s - 1 + 2s\bar{y}(s) - 2 + \bar{y}(s) = \frac{1}{s^2+1}$$

$$(s^2+2s+1)\,\bar{y}(s) = \frac{1+(s+3)(s^2+1)}{s^2+1}$$

$$\bar{y}(s) = \frac{s^3+3s^2+s+4}{(s+1)^2(s^2+1)}$$

$$= \frac{5/2}{(s+1)^2} + \frac{B}{s+1} + \frac{Cs+D}{s^2+1}$$

So $s^3+3s^2+s+4 = \frac{5}{2}(s^2+1) + B(s+1)(s^2+1) + (Cs+D)(s^2+2s+1)$

Compare Coeff of $s^3 \Rightarrow B + C = 1$

Let $s = 0 \Rightarrow 4 = \frac{5}{2} + B + D$

Let $s = 1 \Rightarrow 9 = 5 + 4B + (C+D)4$

Hence $D = 0$, $B = 3/2$, $C = -1/2$

So

$$\bar{y}(s) = \frac{5/2}{(s+1)^2} + \frac{3/2}{s+1} - \frac{\frac{1}{2}s}{s^2+1}$$

$$\underline{\underline{y(t) = \tfrac{5}{2}te^{-t} + \tfrac{3}{2}e^{-t} - \tfrac{1}{2}\cos t}}$$

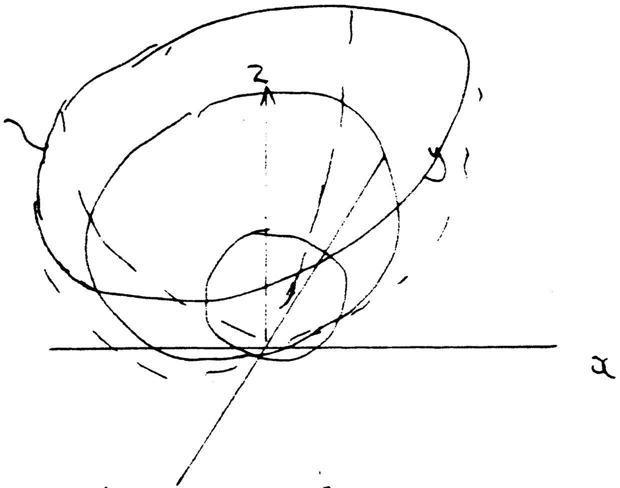
Check :

$$\dot{y}(t) = \tfrac{5}{2}e^{-t} - \tfrac{5}{2}te^{-t} - \tfrac{3}{2}e^{-t} + \tfrac{1}{2}\sin t$$

$$\ddot{y}(t) = -\tfrac{5}{2}e^{-t} - \tfrac{5}{2}e^{-t} + \tfrac{5}{2}te^{-t} + \tfrac{3}{2}e^{-t} + \tfrac{1}{2}\cos t$$

$$\ddot{y} + 2\dot{y} + y = 0 + 0 + \sin t.$$

10 (i) circles of constant radius



(ii)

$$\underline{r} = x\,\underline{i} + y\,\underline{j} + z\,\underline{k}$$

$$\frac{\partial \underline{r}}{\partial x} = \underline{i} + 2x\,\underline{k}$$

$$\frac{\partial \underline{r}}{\partial y} = \underline{j} + 2y\,\underline{k}$$

Normal to surface $= \dfrac{\partial \underline{r}}{\partial x} \times \dfrac{\partial \underline{r}}{\partial y} = \begin{pmatrix} 1 \\ 0 \\ 2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2y \end{pmatrix} = \underline{\underline{\begin{pmatrix} -2x \\ -2y \\ 1 \end{pmatrix}}}$