ENGINEERING TRIPOS PART IA

Tuesday 9 June 1998 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **four** *may be taken from Section A and mot more than* **four** *from Section B.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

Answer not more than **four** *questions from this section*

1 Find, explaining carefully each step in your argument, the shortest distance between the line L_1, which passes through the points (1,1,1) and (2,3,6), and the line L_2, which passes through the points (−1,0,1) and (4,10, α). Explain what happens when $\alpha = 26$. [10]

The plane P passes through the origin and has normal vector parallel to $(\beta,-2,1)$. Determine the point of intersection of P and L_1 when $\beta \neq -1$, and explain what happens when $\beta = -1$. [10]

2 (a) Find the limit as $x \to 0$ of

$$\frac{x^3}{x - \sin x \cos x}.$$

[6]

(b) Determine all values of z such that

$$\cos z = 2.$$

[6]

(c) Determine the power series expansion, up to and including terms in x^4, of

$$\frac{1-\cos x}{e^x(1+x)}.$$

[8]

3 (a) Solve

$$\sin x \frac{dy}{dx} + y \cos x = x \cos x$$

with y = 1 when $x = \pi/2$. [10]

(b) Determine the general solution of

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} - 24y = e^{6t}.$$

[10]

4 (a) Solve

$$2y_{n+1} - 13y_n + 6y_{n-1} = 0$$ [8]

when $y_0 = 1$ and y_n tends to zero as $n \to \infty$.

(b) The $x'y'z'$ axes are obtained by rotating the xyz axes by an angle $45°$ about the y axis. Determine the matrix R such that the vector $\underline{v}$ relative to the xyz axes takes the form $R\underline{v}$ relative to the $x'y'z'$ axes. Relative to the xyz axes the matrix A takes the form

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 3 & 0 & 6 \\ 2 & 6 & -1 \end{pmatrix}.$$

Explain why A takes the form $A' = RAR^T$ relative to the $x'y'z'$ axes, and calculate A'.

[12]

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5 The 3×3 matrix A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$ and corresponding eigenvectors $\underline{e_1}, \underline{e_2}, \underline{e_3}$. Determine the eigenvalues and corresponding eigenvectors of

(a) A^n, where $n > 0$ is an integer;

(b) A^{-1}.

[6]

Calculate $\lambda_1, \lambda_2, \lambda_3$ and $\underline{e_1}, \underline{e_2}, \underline{e_3}$ for

$$A = \begin{pmatrix} 10 & 0 & 0 \\ 0 & 3 & -2 \\ 0 & -2 & 3 \end{pmatrix}.$$

Given that $\underline{x} = (1,1,2)^T$, calculate **approximate** expressions for $A^{10}\underline{x}$ and $\left(A^{-1}\right)^{10}\underline{x}$. [14]

SECTION B

Answer not more than **four** *questions from this section*

6 Let the input $x(t)$ and the output $y(t)$ of a linear, stationary system be zero for $t < 0$. If $x(t) = 1$ for $t \geq 0$ then $y(t) = t$ for $t \geq 0$. What is $y(t)$ for the following choices of input?

(a) $x(t) = \delta(t)$ [6]

(b) $x(t) = t^2$ [7]

(c) $x(t) = e^{-t}$ [7]

7 The complex Fourier series of a function $y(t)$ is

$$y(t) = \sum_{n=-\infty}^{\infty} \frac{3+in}{n^2+1} e^{int} .$$

Express the Fourier series of $y(t)$ in real notation. [8]

Express $x(t)$ as a complex Fourier series where

$x(t) = \cosh t$ in the range $-\pi$ to π. [12]

8 (a) A tray contains six capsules which are identical except that four are red, and two are green. A robot is intended to place the red capsules in tray A, and the green ones in tray B, but makes errors with a 5% probability. The robot randomly picks a capsule and places it in tray B. Find the probability that the capsule is red. [6]

(b) This first capsule remains in tray B. The robot now picks a second capsule and places it in tray A. Find the joint probability that this capsule and the one previously placed are both red. [7]

(c) The remaining capsules are picked and placed. What is the probability that tray B contains only red capsules? [7]

9 Find $y(t)$ for each of the following equations:

(a) $\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 1$ where $\dot{y}(0) = y(0) = 0$ [8]

(b) $\ddot{y}(t) + 2\dot{y}(t) + y(t) = \sin(t)$ where $\dot{y}(0) = y(0) = 1$ [12]

10 Sketch the surface $z = (x^2 + y^2)$. [6]

Calculate

$$\frac{\partial \underline{r}}{\partial x} \text{ and } \frac{\partial \underline{r}}{\partial y}$$

where $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$. Hence find an expression for the normal to the surface at all points. [14]

END OF PAPER