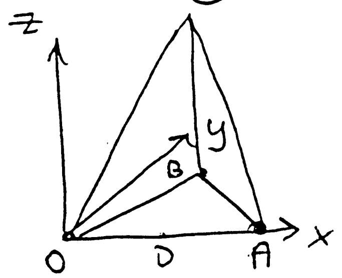
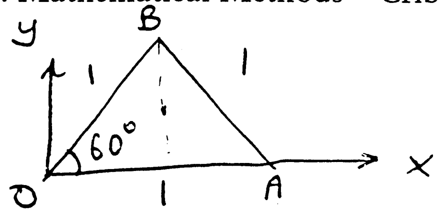
Engineering Tripos Part IA 1999 Paper 4: Mathematical Methods – Crib (1)

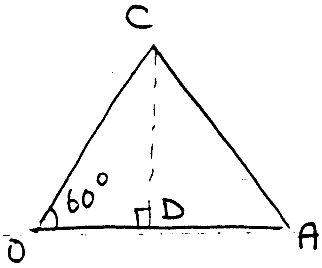
1.



(a)



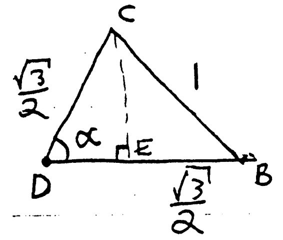
$$B = (1\cos 60^\circ, 1\sin 60^\circ, 0) = \underline{\underline{\left(\tfrac{1}{2}, \tfrac{\sqrt{3}}{2}, 0\right)}}$$



$|OD| = \tfrac{1}{2}$

$|CD| = \dfrac{\sqrt{3}}{2}$

$|DB| = \dfrac{\sqrt{3}}{2}$



so by cosine rule $\cos\alpha = \dfrac{\frac{3}{4} + \frac{3}{4} - 1}{2 \cdot \frac{3}{4}} = \dfrac{1}{3} \Rightarrow \sin\alpha = \sqrt{1 - \frac{1}{9}} = \dfrac{2\sqrt{2}}{3}$

$\therefore |CE| = \dfrac{\sqrt{3}}{2}\sin\alpha = \dfrac{\sqrt{3}}{2} \cdot \dfrac{2\sqrt{2}}{3} = \underline{\underline{\sqrt{\tfrac{2}{3}}}} = z$ coord., $|DE| = \dfrac{\sqrt{3}}{2}\cos\alpha = \underline{\underline{\dfrac{1}{2\sqrt{3}}}}$

$= y$ coord.

b) (i) plane containing $\underline{a}, \underline{b}, \underline{c}$ is $(\underline{r} - \underline{a}) \cdot [(\underline{b} - \underline{a}) \wedge (\underline{c} - \underline{a})] = 0$

$$\rightarrow (x-1, y, z) \cdot \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2\sqrt{3}} & \sqrt{\frac{2}{3}} \end{vmatrix} = (x-1, y, z) \cdot \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{6}}, \frac{1}{2\sqrt{3}}\right) = 0$$

$$\frac{(x-1)}{\sqrt{2}} + \frac{y}{\sqrt{6}} + \frac{z}{2\sqrt{3}} = 0 \qquad \underline{\underline{\sqrt{3}x + y + \frac{z}{\sqrt{2}} = \sqrt{3}}}$$

(ii) normal to this plane is $\parallel \left(\sqrt{3}, 1, \frac{1}{\sqrt{2}}\right)$

$\therefore$ line thro' O is $\underline{\underline{\dfrac{x}{\sqrt{3}} = y = \sqrt{2}z}}$

(iii) thro' C $\perp$ OAB is $\underline{\underline{x = \tfrac{1}{2},\ y = \tfrac{1}{2\sqrt{3}}}}$ from (a)

(c) $x = \frac{1}{2}, y = \frac{1}{2\sqrt{3}}$ in (ii) $\Rightarrow z = \dfrac{x}{\sqrt{3}} = \underline{\underline{\dfrac{1}{2\sqrt{6}}}}$

$\therefore$ pt of intersection is $\underline{\underline{\left(\tfrac{1}{2}, \tfrac{1}{2\sqrt{3}}, \tfrac{1}{2\sqrt{6}}\right)}}$

[This is the centre of mass, which is just the average of the pts OABC]

Engineering Tripos Part IA 1999 Paper 4: Mathematical Methods – Crib (2)

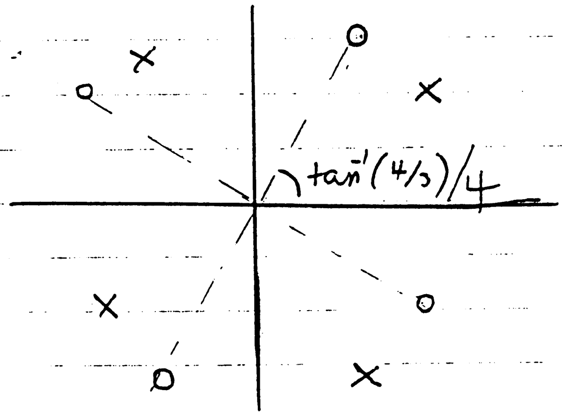
2 (a) $(3+4i) = 5e^{i\theta + 2n\pi i}$ where $\theta = \tan^{-1} 4/3$

$[5 = \sqrt{3^2+4^2}]$

$$\therefore (3+4i)^{0.25} = \underline{\underline{5^{1/4} e^{i\theta/4}}} \times [1, i, -1, -i] \quad (*)$$

$$z^8 - 6z^4 + 25 = 0 \Rightarrow z^4 = \frac{6 \pm \sqrt{36-100}}{2} = 3 \pm 4i$$

so the values of z corresponding to $z^4 = 3+4i$ are exactly those given above, those to $z^4 = +3-4i$ are precisely the complex conjugates of those, ie $5^{1/4} e^{-i\theta/4} [1, -i, 1, i]$ $(**)$



b) $$\frac{(e^{x^2}-1)\sin x}{x^3} = \frac{\left(1 + x^2 + \frac{x^4}{2!} + \cdots - 1\right)\left(x - \frac{x^3}{3!} + \cdots\right)}{x^3} = \frac{(x^2 + \cdots)(x + \cdots)}{x^3}$$

$$\therefore \underline{\underline{\text{Lt} = 1}}$$

c) $$\cosh x = \frac{e^x + e^{-x}}{2} = \frac{1 + x + \frac{x^2}{2!} + \cdots + (1 - x + \frac{x^2}{2!})}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

$$\frac{1}{\cosh x} = \frac{1}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots} = 1 - \left(\frac{x^2}{2!} + \frac{x^4}{4!}\right) + \left(\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^4}{2!^2} + \cdots$$

$$\underline{\underline{\frac{1}{\cosh x} = 1 - \frac{x^2}{2} + \frac{5}{24}x^4}}$$

3 (a) CF $y = e^{mx}$

$\Rightarrow m^2 + m - 12 = 0$ or $(m-3)(m+4) = 0$

$\therefore y = \alpha e^{-4x} + \beta e^{3x}$

PI in 2 bits: (i) to get $2e^{-2x}$ try $y = Ae^{-2x}$

$4A - 2A - 12A = 2. \Rightarrow A = -1/5$

(ii) to get e^{-4x} need $y = Bxe^{-4x}$

$\frac{dy}{dx} = Be^{-4x} - 4Bxe^{-4x}$

$\frac{d^2y}{dx^2} = -8Be^{-4x} + 16Bxe^{-4x}$

$\Rightarrow -8Be^{-4x} + \cancel{16Bxe^{-4x}} + Be^{-4x} - \cancel{4Bxe^{-4x}} - \cancel{12Bxe^{-4x}} = e^{-4x}$

$-7B = 1 \quad \therefore B = -1/7$

$\therefore$ G.S is $y = \alpha e^{-4x} + \beta e^{3x} - \frac{e^{-2x}}{5} - \frac{xe^{-4x}}{7}$

bounded as $x \to \infty \Rightarrow \beta = 0$

$y(0) = 0 \Rightarrow \alpha = \frac{1}{5}$ $\quad \therefore y = \frac{e^{-4x}}{5} - \frac{e^{-2x}}{5} - \frac{xe^{-4x}}{7}$

(b) $y_n \sim \lambda^n \Rightarrow 2\lambda^2 - 3\lambda - 2 = 0$

$(2\lambda + 1)(\lambda - 2) = 0$

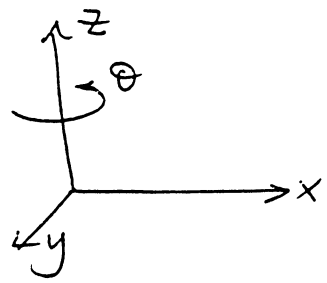
$\therefore y_n = \alpha 2^n + \beta\left(-\frac{1}{2}\right)^n$ $\quad y_0 = 1 \Rightarrow \alpha + \beta = 1$

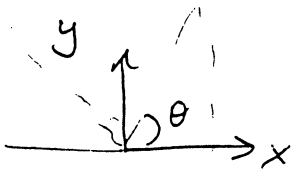
$y_1 = 1 \Rightarrow 2\alpha - \frac{\beta}{2} = 1$

$\Rightarrow \alpha = \frac{3}{5}, \beta = \frac{2}{5} \quad \therefore y_n = \frac{3}{5}2^n + \frac{2}{5}\left(-\frac{1}{2}\right)^n$

Engineering Tripos Part IA 1999 Paper 4: Mathematical Methods – Crib (4)

4 (a)



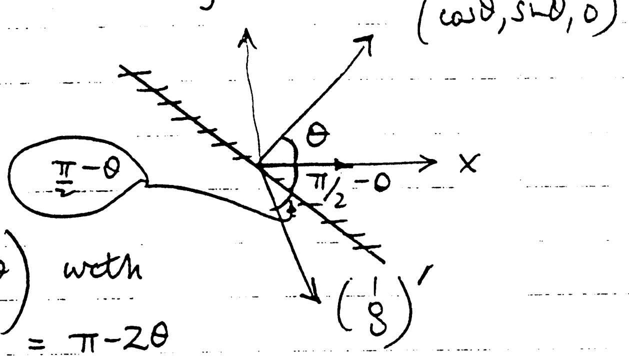


$$\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}\cos\theta\\ \sin\theta\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}-\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$$

$$\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$\therefore Q(\theta) = \begin{pmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{pmatrix}$$



(b) $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$

$\begin{pmatrix}1\\0\\0\end{pmatrix}'$ makes an angle $2\left(\frac{\pi}{2}-\theta\right)$ with x axis $= \pi - 2\theta$

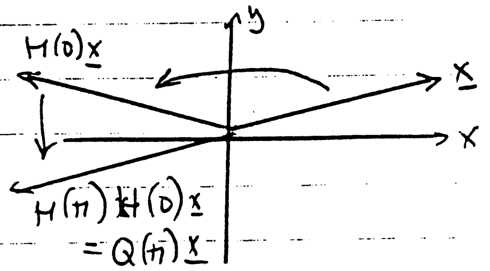
$$\begin{pmatrix}1\\0\\0\end{pmatrix}' = \begin{pmatrix}\cos(\pi-2\theta)\\ \sin(\pi-2\theta)\\ 0\end{pmatrix} = \begin{pmatrix}-\cos 2\theta\\ -\sin 2\theta\\ 0\end{pmatrix}$$

$\begin{pmatrix}0\\1\\0\end{pmatrix}'$ makes angle $\pi-2\theta$ with y axis $\begin{pmatrix}0\\1\\0\end{pmatrix}' = \begin{pmatrix}-\sin 2\theta\\ \cos 2\theta\\ 0\end{pmatrix}$

Hence result.

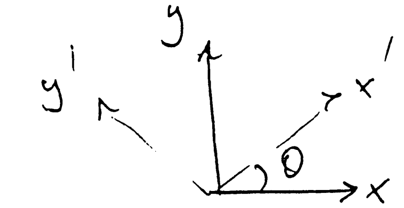
(c) $$H\left(\tfrac{\pi}{2}\right)H(0) = \begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$= Q(\pi)$ ✓



Consider vector $\underset{\sim}{x}$

the 2 reflections are same as rotation by 180° about z axis.

(d) 

x' axis is aligned with normal

∴ in new coords

$$\begin{pmatrix}1\\0\\0\end{pmatrix} \longrightarrow \begin{pmatrix}-1\\0\\0\end{pmatrix}$$

~~z~~ y' axis unaffected, as it lies in the plane $\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix}$

z' " " " " " " " " $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix}$

∴ have determined images of new axes, hence result.

5 (a) A has eigenvalues $\lambda_1, \lambda_2, \lambda_3$, corresponding orthonormal eigenvectors $\underline{e}_1, \underline{e}_2, \underline{e}_3$.

$$\Lambda = \begin{pmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{pmatrix} \qquad U = \begin{pmatrix}\uparrow & \uparrow & \uparrow\\ \underline{e}_1 & \underline{e}_2 & \underline{e}_3\\ \downarrow & \downarrow & \downarrow\end{pmatrix}$$

b) $$\det(A-\lambda I) = \begin{vmatrix}2-\lambda & 0 & 0\\ 0 & -\lambda & 1\\ 0 & 1 & -\lambda\end{vmatrix} = (2-\lambda)(\lambda^2-1) = 0$$

$\therefore \lambda = 2,\ \lambda = \pm 1$

$\lambda = 2$ $$\begin{pmatrix}2&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = 2\begin{pmatrix}x\\y\\z\end{pmatrix} \rightarrow \left.\begin{matrix}2x = 2x\\ z = 2y\\ y = 2z\end{matrix}\right\} y = z = 0 \qquad \underline{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$$

$\lambda = 1$ $$\begin{pmatrix}2&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} \rightarrow \left.\begin{matrix}2x = x\\ z = y\\ y = z\end{matrix}\right\} y = z,\ x = 0 \qquad \underline{e}_2 = \begin{pmatrix}0\\ \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{pmatrix}$$

$\lambda = -1$ $$\begin{matrix}2x = -x\\ z = -y\\ y = -z\end{matrix} \qquad \underline{e}_3 = \begin{pmatrix}0\\ 1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix} \qquad \therefore \Lambda = \begin{pmatrix}2&0&0\\0&1&0\\0&0&-1\end{pmatrix} \quad U = \begin{pmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\end{pmatrix}$$

(c) $$\Lambda^3 = \begin{pmatrix}2^3 & 0 & 0\\ 0 & 1^3 & 0\\ 0 & 0 & -1^3\end{pmatrix} = \begin{pmatrix}8 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -1\end{pmatrix} \text{ etc}$$

$$\Lambda^3 - 2\Lambda^2 - \Lambda + 2I = \begin{pmatrix}8-2(4)-2+2 & 0 & 0\\ 0 & 1-2-1+2 & 0\\ 0 & 0 & -1-2+1+2\end{pmatrix}$$

$$= \underline{\text{zero}}\ \checkmark$$

(d) $\Lambda^3 - 2\Lambda^2 - \Lambda + 2I = 0$

$A = U\Lambda U^t \Rightarrow U^tAU = U^tU\Lambda U^tU = \Lambda$ since $UU^t = I$

$\therefore \Lambda^3 = U^tA\underbrace{U \cdot U^t}_{=I}A\underbrace{U \cdot U^t}_{=I}AU = U^tAAAU = U^tA^3U$ etc

$\therefore$ equation (2) $\rightarrow U^tA^3U - 2U^tA^2U - U^tAU + I = 0$

premultiply by U, postmultiply by U^t

$$A^3 - 2A^2 - A + 2I = 0$$ (ie A satises (2) as well)

$$A^3 - 2A^2 = A - 2I$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{pmatrix}$$

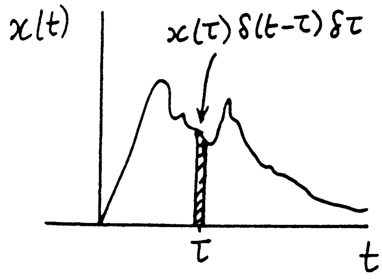
Section B

Examiner's Note.

Poor manipulative skills prevented many candidates from doing themselves justice on this section.

6.

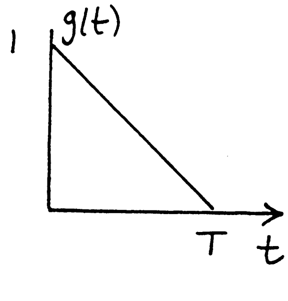
(a) The system must be linear, stationary and at rest prior to the input. Under these conditions, since any input, $x(t)$, can be written as a sum (integral) of impulses, the corresponding output can be synthesised from the results obtained from a suitable combination of impulse responses.



$$x(t) = \int_0^t x(\tau)\,\delta(t-\tau)\,d\tau$$

Input	Output	
$\delta(t)$	$g(t)$	
$\delta(t-\tau)$	$g(t-\tau)$	(assumes system stationary)
$x(\tau)\,\delta(t-\tau)\,d\tau$	$x(\tau)\,g(t-\tau)\,d\tau$	(" " linear)
$\int_0^t x(\tau)\,\delta(t-\tau)\,d\tau$	$\int_0^t x(\tau)\,g(t-\tau)\,d\tau$	(" " ")

(b)



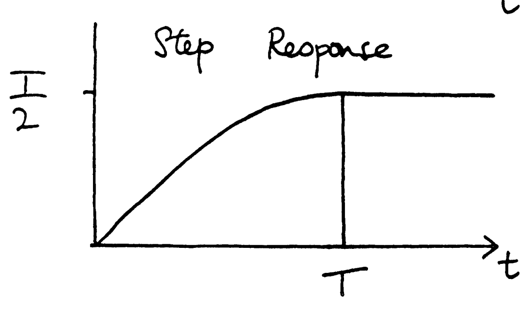
$$\text{Step Response} = \int \text{impulse response} = \int_0^t g(t')\,dt'$$

$t < 0$ step response $= 0$

$0 < t < T$ " " $= \int_0^t \left(1 - \frac{t'}{T}\right) dt' = t - \frac{t^2}{2T}$

$t > T$ " " $= \int_0^T \left(1 - \frac{t'}{T}\right) dt' = \left[t - \frac{t^2}{2T}\right]_0^T$

$$= \frac{T}{2}$$



(c) (i) $y(t) = \int_0^t x(\tau)\, g(t-\tau)\, d\tau$

If $0 \leqslant t \leqslant T$ then, throughout the range of integration, $t-\tau \leqslant T$

i.e. g is given by $1 - \frac{t-\tau}{T}$.

Thus $y(t) = \int_0^t \tau\left(1 - \frac{t-\tau}{T}\right) d\tau = \int_0^t \left(\tau - \frac{t\tau}{T} + \frac{\tau^2}{T}\right) d\tau$

$$= \left[\frac{\tau^2}{2} - \frac{t\tau^2}{2T} + \frac{\tau^3}{3T}\right]_0^t = \frac{t^2}{2} - \frac{t^3}{2T} + \frac{t^3}{3T} = \frac{t^2}{2} - \frac{t^3}{6T}.$$

(ii) Since g falls to zero after T, when $t > T$

$$y(t) = \int_{t-T}^{t} x(\tau)\, g(t-\tau)\, d\tau$$

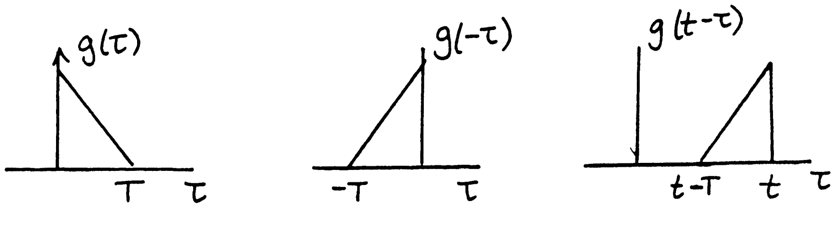
(i.e. y is not affected by what x was doing more than T ago)

Thus $y(2T) = \int_T^{2T} \tau\left(1 - \frac{-\tau}{T}\right) d\tau = \int_T^{2T} \left(-\tau + \frac{\tau^2}{T}\right) d\tau$

$$= \left[-\frac{\tau^2}{2} + \frac{\tau^3}{3T}\right]_T^{2T} = -2T^2 + \frac{8T^2}{3} - \left(-\frac{T^2}{2} + \frac{T^2}{3}\right)$$

$$= T^2\left[-2 + \frac{8}{3} + \frac{1}{2} - \frac{1}{3}\right] = \frac{5T^2}{6}$$

Aliter

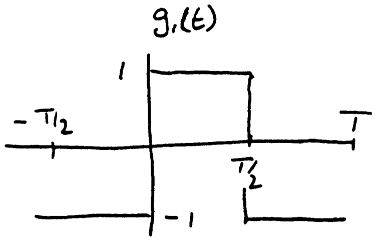


$\therefore\ y(t) = \int_{t-T}^{t} x(\tau) g(t-\tau)\, d\tau$

Examiner's Note.

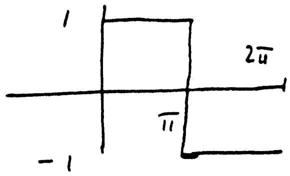
Most candidates did not appreciate the need to be careful of the need to use $g = 0$ when its argument is bigger than T, and simply put $t = 2T$ in the expression derived in part (c)(i). The rest of the question was well answered.

7.(a) Databook has



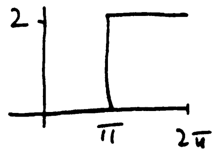
$$g_1(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\omega_0 t}{2n-1}$$

Now $T \to 2\pi$ $\quad t \to \theta \quad \omega_0 = \frac{2\pi}{T} \to 1 \Rightarrow$

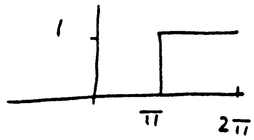


$$g_1(\theta) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}$$

$1 - g_1(\theta) =$



$\therefore \frac{1}{2}(1 - g_1(\theta)) =$



$$\therefore \quad g(\theta) = \frac{1}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\theta}{2n-1}$$

Apart from a_0 (the d.c. level) no cosine terms, since g is an odd fn.
No $\sin n\theta$ for n even since g_1 is antisymmetric about π.

(b) $\quad h(\theta) = 0 \quad 0 < \theta < \pi \qquad h(\theta) = \frac{\theta - \pi}{\pi} \quad \pi < \theta < 2\pi$

$$\therefore a_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{\theta-\pi}{\pi}\cos n\theta \, d\theta$$
$$= \frac{1}{\pi^2}\left[(\theta-\pi)\frac{\sin n\theta}{n}\right]_{\pi}^{2\pi} - \frac{1}{\pi^2}\int_{\pi}^{2\pi} \frac{\sin n\theta}{n} d\theta$$
$$= \frac{1}{\pi^2}\left[\frac{\cos n\theta}{n^2}\right]_{\pi}^{2\pi}$$
$$= \frac{1}{n^2\pi^2}(\cos 2n\pi - \cos n\pi) = \frac{1-(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{\theta-\pi}{\pi}\sin n\theta \, d\theta$$
$$= \frac{1}{\pi^2}\left[-\frac{(\theta-\pi)\cos n\theta}{n}\right]_{\pi}^{2\pi} + \frac{1}{\pi^2}\int_{\pi}^{2\pi} \frac{\cos n\theta}{n} d\theta$$
$$= \frac{1}{\pi^2}\left(-\frac{\pi}{n}\cos 2n\pi\right) + \frac{1}{\pi^2}\left[\frac{\sin n\theta}{n^2}\right]_{\pi}^{2\pi}$$
$$= -\frac{1}{n\pi}$$

and $\quad \frac{a_0}{2} = $ average value $\Rightarrow \quad a_0 = \frac{1}{4}$

(c) Convergence of series is determined by how "smooth" a function is.
A function with a simple discontinuity (e.g. g or h) has a series with converges like $\frac{1}{n}$. (value at $0 \neq$ value at 2π counts as a discontinuity).
A function continuous but 1st derivative discontinuous has a $\frac{1}{n^2}$ convergence
In general, if r is the highest order continuous derivative, series converges like $\frac{1}{n^{r+2}}$

(d) For the series for F to converge as quickly as possible, λ and μ should be chosen to cancel the discontinuities in f.

Total jump at π

$$[F] = [f] - \lambda[g] - \mu[h] = 1 - \lambda$$

Total jump at $0/2\pi$

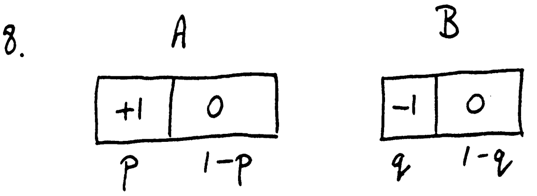
$$[F] = [f] - \lambda[g] - \mu[h] = 1.5 - \lambda - \mu$$

These jumps will be zero with $\lambda = 1$, $\mu = \frac{1}{2}$

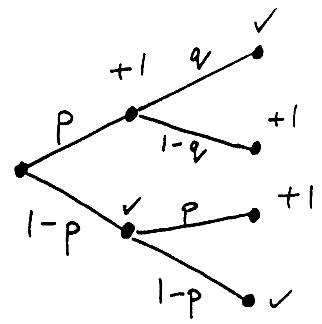
Examiner's Note

Very few candidates managed to find a_n & b_n for h by successfully integrating by parts or to discuss why even n are absent for the series for g. Other parts of the question were well answered.

8.



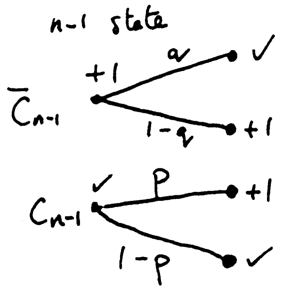
(a)



Using branching diagram

$$P(\checkmark) = pq + (1-p)^2$$

(b) Consignment can never be underweight or overweight by more than one kg. i.e. There are only two possible states, C or $+1 = \bar{C}$.



$$P(C_n) = q\,P(\bar{C}_{n-1}) + (1-p)P(C_{n-1})$$

and $P(\bar{C}_{n-1}) = 1 - P(C_{n-1})$

$$\therefore\ P(C_n) = q(1 - P(C_{n-1})) + (1-p)P(C_{n-1})$$
$$= (1-p-q)P(C_{n-1}) + q$$

Method 1 Solve difference equation.

"C.F." Try λ^n $\Rightarrow$ $\lambda = 1-p-q$

"P.I." Try c $\Rightarrow$ $c = (1-p-q)c+q$ $\Rightarrow$ $c = \dfrac{q}{p+q}$

$$\therefore \text{ G.S. } \quad P(C_n) = A(1-p-q)^n + \frac{q}{p+q}$$

$$P(C_1) = 1-p \;\Rightarrow\; A(1-p-q) + \frac{q}{p+q} = 1-p \;\Rightarrow\; A = \frac{p}{p+q}$$

Hence $\underline{P(C_n) = \dfrac{p}{p+q}(1-p-q)^n + \dfrac{q}{p+q}}$.

Method 2 Induction.

Assume true for $n-1$ i.e. $P(C_{n-1}) = \dfrac{p}{p+q}(1-p-q)^{n-1} + \dfrac{q}{p+q}$

Then
$$\begin{aligned} P(C_n) &= (1-p-q)P(C_{n-1}) + q \\ &= \frac{p}{p+q}(1-p-q)^n + \frac{(1-p-q)q}{p+q} + q \\ &= \frac{p}{p+q}(1-p-q)^n + \frac{q}{p+q} \end{aligned}$$
i.e. True for n.

But $P(C_1) = 1-p = \dfrac{p}{p+q}(1-p-q) + \dfrac{q}{p+q}$ Hence true for all n

Method 3 Direct calculation.

$$\begin{aligned} P(C_n) &= (1-p-q)P(C_{n-1}) + q = (1-p-q)\left[(1-p-q)P(C_{n-2}) + q\right] + q \\ &= (1-p-q)^{n-1}P(C_1) + q\left[1 + (1-p-q) + (1-p-q)^2 + \dots + (1-p-q)^{n-2}\right] \end{aligned}$$

But $P(C_1) = 1-p$ and $q(1 + 1-p-q + (1-p-q)^2 + \dots + (1-p-q)^{n-2})$ is G.P

with sum $q\dfrac{(1-(1-p-q)^{n-1})}{1-(1-p-q)}$.

Hence
$$\begin{aligned} P(C_n) &= (1-p-q)^{n-1}(1-p) + \frac{q}{p+q}\left(1-(1-p-q)^{n-1}\right) \\ &= \frac{p}{p+q}(1-p-q)^n + \frac{q}{p+q}. \end{aligned}$$

(c) A is used if the weight is correct, B otherwise.

i.e. prob of picking from A for the n'th bag is $P(C_n)$.

Now $-1 \le 1-p-q \le 1$ (since $p+q \le 2$ & $p>0, q>0$)

$\therefore (1-p-q)^n \to 0$ as $n \to \infty$

$\therefore$ For large n $P(\text{using } A) = P(C_n) \simeq \dfrac{q}{p+q}$

$\therefore$ Proportion from A approaches $\dfrac{q}{p+q}$.

Aliter

Let proportion from A be a, proportion from B be b

i.e. number from $A = na$ number " $B = nb$

For large n number of those from A overweight $\simeq nap$

" " " " B underweight $\simeq nbq$

But consignment is either correct weight or at most 1 overweight

i.e. for large n $nap \simeq nbq$ (since the 1 is negligible)

i.e. $b \simeq \dfrac{ap}{q}$

But $n = na + nb \Rightarrow a + \dfrac{ap}{q} \simeq 1$ i.e. $a \simeq \dfrac{q}{p+q}$

Examiner's note.

The commonest mistake was not to use $P(\text{overweight}) = 1 - P(\text{correct})$ and hence to fail to prove part (b). A significant number of candidates also failed to prove the general formula for $P(C_n)$. Otherwise well done by most candidates

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9. (a) $L(\dot{x}) = sX - x(0) = sX - 1 \qquad L(\dot{y}) = sY - y(0) = sY$

$$\therefore \quad sX - 1 + X + sY - Y = \frac{s}{s^2+1} \Rightarrow (s+1)X + (s-1)Y = \frac{s}{s^2+1} + 1 \quad ①$$

$$sY + 2Y - sX + 1 = \frac{1}{s^2+1} \Rightarrow -sX + (s+2)Y = \frac{1}{s^2+1} - 1 \quad ②$$

Eliminating X between ① & ②

$$\Rightarrow \left[s(s-1) + (s+1)(s+2)\right]Y = \frac{s^2}{s^2+1} + s + \frac{s+1}{s^2+1} - s - 1 = \frac{s}{s^2+1}$$

i.e. $$Y = \frac{s}{(s^2+1)(2s^2+2s+2)} = \frac{As+B}{2(s^2+1)} + \frac{Cs+D}{2(s^2+s+1)} \quad \text{(say)}$$

$$\Rightarrow (As+B)(s^2+s+1) + (Cs+D)(s^2+1) = s$$

Coeff s^3 : $A + C = 0 \qquad s^2$: $A + B + D = 0 \qquad s$: $A + B + C = 1$

s^0 : $B + D = 0 \qquad$ i.e. $B = 1 \quad D = -1 \quad A = 0 \quad C = 0$

$$\therefore \quad Y = \frac{1}{2(s^2+1)} - \frac{1}{2(s^2+s+1)} \longleftarrow \frac{1}{\sqrt{3}} \frac{\frac{\sqrt{3}}{2}}{\left[(s+\frac{1}{2})^2 + \frac{3}{4}\right]}$$

$$\Rightarrow \quad \underline{y = \frac{1}{2}\sin t - \frac{1}{\sqrt{3}} e^{-t/2} \sin \frac{\sqrt{3}t}{2}}$$

Returning to ① & ②, eliminating Y gives

$$\left[(s+2)(s+1) + (s-1)s\right] X = \frac{s(s+2)}{s^2+1} + s + 2 - \frac{s-1}{s^2+1} + s - 1$$

$$\Rightarrow 2(s^2+s+1)X = \frac{s^2+s+1}{s^2+1} + 2s + 1$$

$$\therefore \quad X = \frac{1}{2(s^2+1)} + \frac{s+\frac{1}{2}}{s^2+s+1} \qquad \frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2 + \frac{3}{4}}$$

$$\Rightarrow \quad \underline{x = \frac{1}{2}\sin t + e^{-t/2}\cos \frac{\sqrt{3}}{2}t}$$

(b) Putting $t = 0$ and $x(0) = 1$ & $y(0) = 0$ into the diff eq^{n}s gives

$$\dot{x}(0) + 1 + \dot{y}(0) = 1 \qquad \text{i.e.} \quad \underline{\dot{x}(0) = \dot{y}(0) = 0}$$

and $$\dot{y}(0) - \dot{x}(0) = 0$$

(c) From the solution obtained above

$x(0) = 1 \qquad \dot{x}(t) = \frac{1}{2}\cos t - \frac{1}{2}e^{-t/2}\cos\frac{\sqrt{3}t}{2} - \frac{\sqrt{3}}{2}e^{-t/2}\sin\frac{\sqrt{3}t}{2}$

$\Rightarrow \dot{x}(0) = \frac{1}{2} - \frac{1}{2} = 0$

$y(0) = 0 \qquad \dot{y}(t) = \frac{1}{2}\cos t + \frac{1}{2\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}t}{2} - \frac{1}{2}e^{-t/2}\cos\frac{\sqrt{3}t}{2}$

$\Rightarrow \dot{y}(0) = \frac{1}{2} - \frac{1}{2} = 0$ as required

Examiner's Note

With hindsight, this question was too long. Finding y or x alone would have been more appropriate.

Very many candidates became bogged down in the algebra for this question having difficulty solving the simultaneous equations for X and Y, or failing to split the expression for Y into partial fractions. Those candidates who had practised their manipulative skills found it easy.

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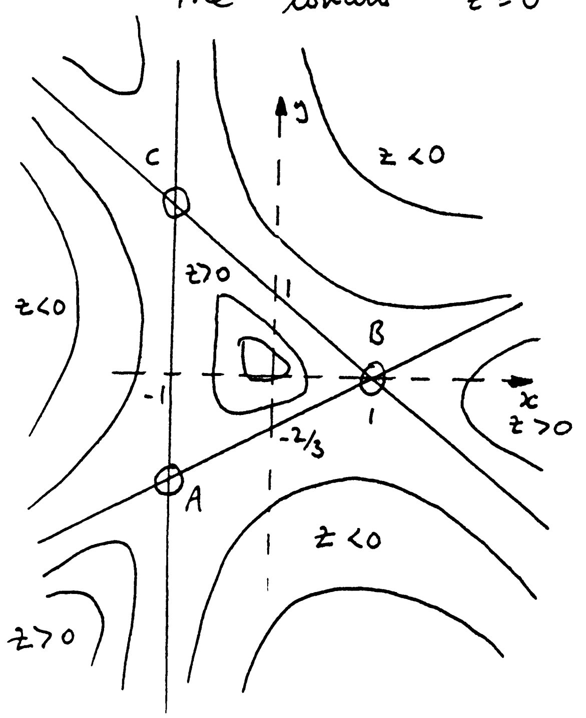
10. $z = (x+1)(x+y-1)(2x-3y-2)$

The contour $z=0$ is $x+1=0$ and $x+y-1=0$ and $2x-3y-2=0$



From the diagram it is clear that A, B, C are saddle points

$A = (-1, -4/3) \quad B = (1,0) \quad C = (-1,2)$

It is also clear that there is one further maximum inside ABC.

At a stationary point $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$

These equations should, therefore, have four roots.

$$z = (x^2+xy-x+x+y-1)(2x-3y-2)$$
$$= (x^2+xy+y-1)(2x-3y-2)$$
$$= 2x^3-3x^2y-2x^2+2x^2y-3xy^2-2xy+2xy-3y^2-2y-2x+3y+2$$
$$= 2x^3-x^2y-3xy^2-2x^2-3y^2+y-2x+2$$

(check $x=-1 \Rightarrow z = -2-y+3y^2-2-3y^2+y+2+2 = 0$ ✓)

$$\frac{\partial z}{\partial x} = 6x^2-2xy-3y^2-4x-2$$
$$\frac{\partial z}{\partial y} = -x^2-6xy-6y+1$$

(check $x=1$ $y=0 \Rightarrow \frac{\partial z}{\partial x} = \frac{\partial z}{\partial y} = 0$ ✓)

Now $\frac{\partial z}{\partial y} = 0 \Rightarrow 1-x^2 = 6(x+1)y$ i.e. (i) $x+1=0$ or (ii) $1-x=6y$

(i) $x+1=0 \Rightarrow \frac{\partial z}{\partial x} = 6+2y-3y^2+4-2$, $\frac{\partial z}{\partial x}=0 \Rightarrow 3y^2-2y-8=0$

i.e. $(3y+4)(y-2)=0$ i.e. $y = 2$ or $-\frac{4}{3}$

$\therefore$ $\underline{x} = (-1,2)$ or $(-1,-4/3)$ <u>as expected</u>

(ii) $\frac{\partial z}{\partial x}=0 \Rightarrow 6x^2 - 2x\frac{(1-x)}{6} - 3\frac{(1-x)^2}{6^2} - 4x - 2 = 0$

$$\Rightarrow x^2\left[6+\frac{1}{3}-\frac{1}{12}\right] + x\left[-\frac{1}{3}+\frac{1}{6}-4\right] + \left(-2-\frac{1}{12}\right) = 0$$

i.e. $x^2\,\frac{72+4-1}{12} + x\,\frac{-2+1-24}{6} + \left(\frac{-25}{12}\right) = 0$

i.e. $75x^2-50x-25=0$ or $3x^2-2x-1=0$

$\therefore$ $(3x+1)(x-1)=0$ $x=1$ $(\Rightarrow y=0)$

or $x=-1/3$ $\left(\Rightarrow y = \frac{1+\frac{1}{3}}{6} = \frac{2}{9}\right)$

$\therefore$ $\underline{x = (1,0)\ \text{(saddle)}\ \text{and}\ (-1/3, 2/9)}$ maximum

Examiner's Note

A distressing number of candidates failed to use the information from the diagram about number, position and type of stationary points and thus became entangled in algebraic difficulties.