ENGINEERING TRIPOS PART 1A

Tuesday 8 June 1999 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **four** *may be taken from Section A and not more than* **four** *from Section B.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

Answer not more than **four** *questions from this section*

1 The regular tetrahedron $OABC$ has its vertex O at the origin, vertex A at (1, 0, 0), vertex B in the plane $z = 0$ with $y > 0$, and vertex C in $z > 0$.

(a) Show that the coordinates of B and C are

$$\left(\frac{1}{2},\frac{\sqrt{3}}{2},0\right) \quad \text{and} \quad \left(\frac{1}{2},\frac{1}{2\sqrt{3}},\sqrt{\frac{2}{3}}\right)$$

respectively. [6]

(b) Determine the equation of:

(i) the plane containing ABC;

(ii) the line through O perpendicular to the plane containing ABC;

(iii) the line through C perpendicular to the plane containing OAB [10]

(c) Where does the line described in (ii) intersect the line described in (iii)? [4]

2 (a) Determine all possible values of

$$(3+4i)^{0.25} \quad .$$

Solve the equation

$$z^8 - 6z^4 + 25 = 0$$

and draw a diagram indicating the position of all its roots in the complex z plane. [7]

(b) Calculate the limit as $x \rightarrow 0$ of

$$\frac{\left(e^{x^2}-1\right)\sin x}{x^3} \quad .$$ [6]

(c) Calculate the terms up to and including $O\left(x^4\right)$ in the expansion about $x = 0$ of

$$\frac{1}{\cosh x} \quad .$$ [7]

3 (a) Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 12y = e^{-4x} + 2e^{-2x}$$

with $y(0) = 0$ and $y(x)$ bounded as $x \to \infty$. [12]

(b) Solve

$$2y_{n+1} - 3y_n - 2y_{n-1} = 0$$

with $y_0 = 1$ and $y_1 = 1$. [8]

4 (a) Find the matrix $Q(\theta)$ which describes a rotation of a vector by an angle θ about the z axis. [4]

(b) The matrix $H(\theta)$ describes reflection of a vector in the plane through the origin with normal vector $(\cos\theta, \sin\theta, 0)$. Show that

$$H(\theta) = \begin{pmatrix} -\cos 2\theta & -\sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$ [6]

(c) Show that

$$H\left(\frac{\pi}{2}\right)H(0) = Q(\pi) \quad ,$$

and draw a diagram to demonstrate the geometrical meaning of this result. [5]

(d) The xyz axes are rotated by an angle θ about the z axis. By considering the effects of the reflection described in (b) on vectors parallel to the new axes, or otherwise, show that relative to the new axes $H(\theta)$ takes the form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$ [5]

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5 (a) Explain briefly how the real, symmetric matrix A can be expressed in the form

$$A = U \Lambda U^t, \qquad (1)$$

where U is an orthogonal matrix and Λ is a diagonal matrix. [3]

(b) Determine the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$ [8]

and hence write down expressions for U and Λ in this case.

(c) Verify that Λ found in (b) satisfies

$$\Lambda^3 - 2\Lambda^2 - \Lambda + 2I = 0 , \qquad (2)$$

where I is the identity matrix. [3]

(d) Use equations (1) and (2) to calculate the matrix

$$A^3 - 2A^2 .$$ [6]

SECTION B

Answer not more than **four** *questions from this section.*

6 (a) Explain why the response of a system to a general input can be written in the form

$$y(t) = \int_0^t x(\tau) g(t-\tau) d\tau$$

where x is the input to the system, y is the output from the system and g is the impulse response. Describe the conditions which must apply to the system for this equation to be valid. [5]

(b) The impulse response for a system is as shown in Fig. 1. Find and sketch the step response of the system, showing that, for sufficiently large values of t, this response is $\frac{T}{2}$. [3]

(c) The input to the system is

$$x(t) = \begin{cases} t & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) For t in the range $0 \leq t \leq T$, find the response of the system. [7]

(ii) Find $y(2T)$. [5]

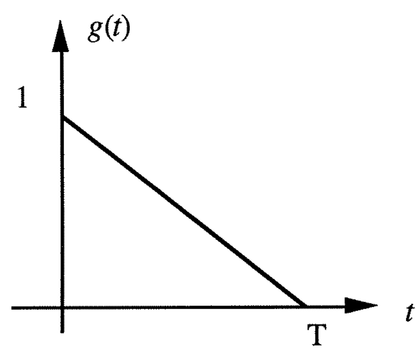

Fig. 1

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7 (a) Starting with one of the series tabulated in the Electrical Databook, find a real Fourier series representation over the range $0 \leq \theta \leq 2\pi$ of the function $g(\theta)$ shown in Fig. 2(a). Explain, using symmetry arguments, which coefficients must be zero. [5]

(b) Determine, from first principles, a Fourier series representation over the range $0 \leq \theta \leq 2\pi$ of the function $h(\theta)$ shown in Fig. 2(b). [5]

(c) Explain what factors influence the rate of convergence of a Fourier series. [4]

(d) Determine the values of the constants λ and μ which make the Fourier series for the function

$$F(\theta) = f(\theta) - \lambda g(\theta) - \mu h(\theta)$$

converge most rapidly, where $f(\theta)$ is as shown in Fig. 2(c). [6]

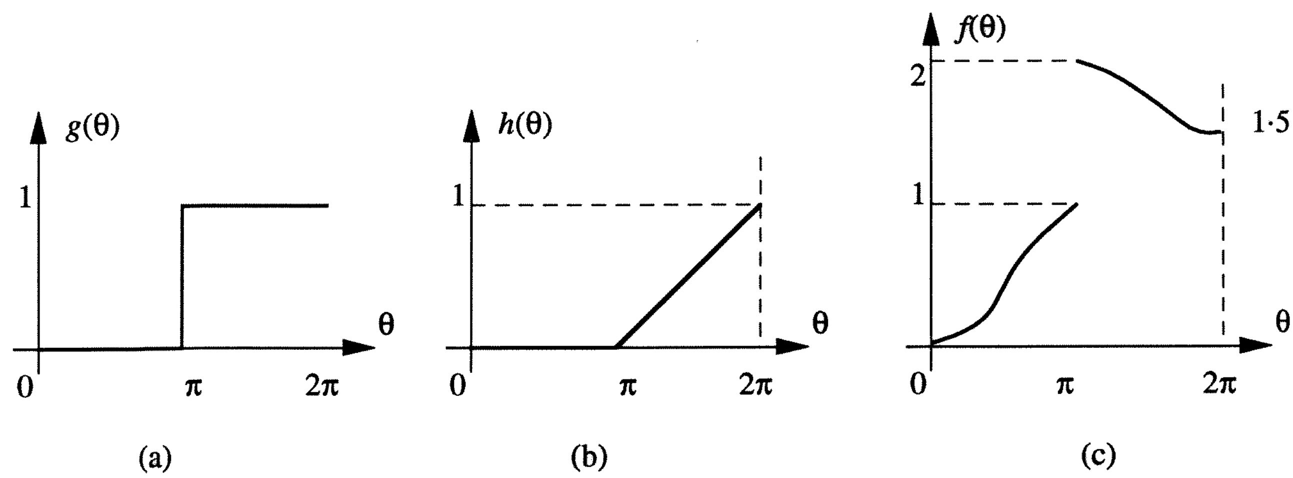

Fig. 2

8 1 kg bags of sugar are produced continuously by two machines. Machine A produces bags which are either the correct weight or are 1 g overweight, while machine B produces bags which are either the correct weight or 1 g underweight. The proportion of bags from machine A that are overweight is p, while the proportion from machine B that are underweight is q.

A consignment of n bags is formed by taking bags from A and B according to the rule:-

(i) Take the first bag from machine A

(ii) If, at any stage, the consignment is overweight, take a bag from B; otherwise take a bag from A.

(a) Show that the probability that a consignment of 2 bags will be the correct weight is

$$p q + (1-p)^2$$ [5]

(b) If the event - a consignment of n bags is the correct weight - is denoted by C_n, show that

$$P(C_n) = (1-p-q)\,P(C_{n-1}) + q\ .$$

and hence that

$$P(C_n) = \frac{p}{p+q}(1-p-q)^n + \frac{q}{p+q}$$ [10]

(c) For large consignments (i.e. large n), estimate the proportion of bags that have come from machine A. [5]

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9 (a) Find, using Laplace Transforms, $x(t)$ and $y(t)$ where

$$\dot{x} + x + \dot{y} - y = \cos t$$

$$\dot{y} + 2y - \dot{x} = \sin t$$

where $x(0) = 1$ and $y(0) = 0$. [14]

(b) Determine, using the differential equations, the values of $\dot{x}(0)$ and $\dot{y}(0)$. [2]

(c) Confirm that your solutions give the correct values of $x(0)$, $y(0)$, $\dot{x}(0)$ and $\dot{y}(0)$. [4]

10 A function is defined by the equation

$$z = (x+1)(x+y-1)(2x-3y-2)$$

(a) Sketch contours of constant z. [6]

(b) Find and classify the stationary points of z, and mark these on the sketch. [14]

END OF PAPER