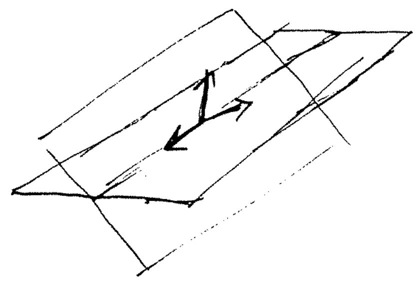
Q1 $\quad A: \; 2x + 3y - z = 1$

$\quad\quad B: \; x - y + z = 2$

The line of intersection must be pependicular to both normals



Equation of plane $\underline{r} \cdot \underline{n} = p$

so direction of normals $(2, 3, -1)$ and $(1, -1, 1)$

$$\begin{vmatrix} i & j & k \\ 2 & 3 & -1 \\ 1 & -1 & 1 \end{vmatrix} = (3-1)i - (2 - -1)j + (-2-3)k$$
$$= (2, -3, -5)^T$$

So line is of form $\quad a + \lambda \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$

set $\lambda = 0$ and add A and B $\Rightarrow 3x + 2y = 3$

must cross $y = 0$ somewhere $\Rightarrow x = 1, \; z = 1$;

Line is $\quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$

1b) plane A in $\underline{r}.\underline{n} = p$ form

$$r.\frac{1}{\sqrt{2^2+3^2+1^2}}\begin{pmatrix}2\\3\\1\end{pmatrix} = r.\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} = \frac{1}{\sqrt{14}}$$

p is distance from origin, so new $p = \frac{1}{\sqrt{14}} \pm 1$,

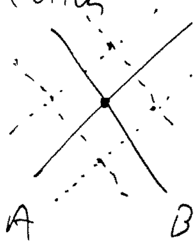
proof (for $\cdot 1$)

$$\begin{pmatrix}x'\\y'\\z'\end{pmatrix} = \begin{pmatrix}x\\y\\z\end{pmatrix} + \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix}$$

$$\left(\begin{pmatrix}x\\y\\z\end{pmatrix} + \frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix}\right)\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix} = \underbrace{\begin{pmatrix}x\\y\\z\end{pmatrix}\frac{1}{\sqrt{14}}\begin{pmatrix}2\\3\\1\end{pmatrix}}_{=\frac{1}{\sqrt{14}}} + \underbrace{\frac{2^2+3^2+1^2}{\sqrt{14}.\sqrt{14}}}_{1}$$

1b) two planes $\left(\frac{1}{\sqrt{14}} \pm 1\right)$

1c) To be distance 1 from both planes must be parallel to the direction in 1a). Looking in that direction



so line is $a + \lambda\begin{pmatrix}2\\3\\5\end{pmatrix}$

From (1b), planes are (with + sign)

$$A: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{14}} \pm 1 \qquad 2x + 3y - z = 1 + \sqrt{14}$$

$$B: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \frac{2}{\sqrt{3}} \pm 1 \qquad x - y + z = 2 + \sqrt{3}$$

adding $\quad 3x - 2y = 3 + \sqrt{14} + \sqrt{3}$

Again, must go through $y = 0 \Rightarrow x = 1 + \frac{\sqrt{14}}{3} + \frac{\sqrt{3}}{3}$

$$z = 2 + \sqrt{3} - 1 - \frac{\sqrt{14}}{3} - \frac{1}{\sqrt{3}}$$
$$= 1 + \sqrt{3} - \frac{\sqrt{14}}{3} - \frac{1}{\sqrt{3}}$$

giving solution

$$\begin{pmatrix} 1 + \frac{\sqrt{14}}{3} + \frac{\sqrt{3}}{3} \\ 0 \\ 1 + \sqrt{3} - \frac{\sqrt{14}}{3} - \frac{1}{\sqrt{3}} \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ -5 \end{pmatrix}$$

Q2)

$$\lim_{x\to 0} \frac{\sin x \cosh x - x}{x(1-\cos x)}$$

Expansions from data book

$$= \frac{\left(x - \frac{x^3}{3!} + O(x^5)\right)\left(1 + \frac{x^2}{2!} + O(x^4)\right) - x}{x\left(1 - \left(1 - \frac{x^2}{2!} + O(x^4)\right)\right)}$$

$$= \frac{x - \frac{x^3}{3!} + \frac{x^3}{2!} - x + O(x^5)}{x^2/2! + O(x^5)}$$

$$= \frac{1/3 + O(x^2)}{1/2 + O(x^2)} = \frac{2}{3} + O(x)$$

$$\to 2/3$$

2b) $x + iy = \sin^{-1}(2i)$

$$2i = \sin(x+iy) = \sin(x)\cos(iy) + \cos(x)\sin(iy)$$
$$= \sin x \cosh y + (\cos x \sinh y)i$$

data book

Equating Real and Imaginary $\sin x \cosh y = 0$, but $\cosh y > 0$

so $\sin x = 0 \Rightarrow x = n\pi$

$2i = \cos n\pi \sinh y\, i \Rightarrow 2 = (-1)^n \sinh y \Rightarrow y = \sinh^{-1} 2\,(-1)^n$

so $\sin^{-1}(2i) = n\pi + (-1)^n \sinh^{-1} 2$

$\left(= n\pi + (-1)^n \ln(2+\sqrt{5}) = n\pi + (-1)^n 1.444\right)$

2c) $\{[a\times(b\times c)]\times c\}\times b$

$= ([(a.c)b - (a.b)c]\times c)\times b$

$= ((a.c)b\times c - (a.b)c\times c)\times b$ $\qquad c\times c = 0$

$= (a.c)(b\times c)\times b$

$= (a.c)((b.b)c - (c.b)b)$

Q3 $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = \sin x$

Find C.F., propose $y = e^{\lambda x}$

$\lambda^2 - 2\lambda + 1 = 0$

$(1-\lambda)^2 = 0 \quad \Rightarrow \lambda = 1$ is one solution

For the other try $y = (A + Bx)e^x$ to find standard solution C.F. $(A+Bx)e^x$

Next Find P.I. Data book suggests $y = a\sin x + b\cos x$

$\dfrac{dy}{dx} = a\cos x - b\sin x$

$\dfrac{d^2y}{dx^2} = -a\sin x - b\cos x$

Substitute in

$$-a\sin x - b\cos x - 2a\cos x + 2b\sin x + a\sin x + b\cos x = \sin x$$

sin terms $\quad -a + 2b + a = 1 \quad \Rightarrow \quad b = \frac{1}{2}$

cos terms $\quad -b - 2a + b = 0 \quad \Rightarrow \quad a = 0$

Hence general solution is

$$y = (A + Bx)e^{x} + \frac{1}{2}\cos x$$

3b) $\quad S_n - 2S_{n-1} + (1-\varepsilon^2)S_{n-2} = 0$

Try C.F. $\quad S_n = \lambda^n S_0$

$$S_0\lambda^n - 2S_0\lambda^{n-1} + (1-\epsilon^2)\lambda^{n-2}S_0 = 0$$

$$\lambda^2 - 2\lambda + (1-\varepsilon^2) = 0$$

$$(\lambda - 1)^2 = \varepsilon^2$$

$$\lambda = 1 \pm \varepsilon$$

General solution

$$S_n = A(1+\varepsilon)^n + B(1-\varepsilon)^n$$

Boundary conditions $S_0 = 0$, $S_1 = 1$

$$0 = A + B \Rightarrow A = -B$$

$$1 = A(1+\varepsilon) + B(1-\varepsilon)$$

$$1 = A + A\varepsilon - A + A\varepsilon \Rightarrow A = \frac{1}{2\varepsilon} \quad B = \frac{-1}{2\varepsilon}$$

$$S_n = \frac{1}{2\varepsilon}(1+\varepsilon)^n - \frac{1}{2\varepsilon}(1-\varepsilon)^n$$

3c) Binomial expansion $(1+\varepsilon)^n = 1 + n\varepsilon + O(\varepsilon^2)$

$(1-\varepsilon)^n = 1 - n\varepsilon + O(\varepsilon^2)$

So $S_n = \frac{1}{2\varepsilon}\left(1 + n\varepsilon - (1 - n\varepsilon) + O(\varepsilon^2)\right)$

$= \frac{1}{2\varepsilon}(2n\varepsilon + O(\varepsilon^2))$

$= n + O(\varepsilon)$

$\lim_{\varepsilon \to 0} S_n = n$

General solution for $\varepsilon = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = 1$ repeated

$S_n = (A + nB)\lambda^n$

as in parts (a) and (b)

$\lambda = 1$ so general solution is

$S_n = A + nB$

4) Vector triple product (From data book)

$$n \times (x \times n) = (n \cdot n)\, x - (n \cdot x)\, n$$

n is a unit vector $|n| = 1$, $n \cdot n = 1$, $x \cdot n = n \cdot x$

$$n \times (x \times n) = x - (x \cdot n)\, n$$

$$\Rightarrow \quad n = (x \cdot n)\, n + n \times (x \times n)$$

4b)

$n \times (x \times n)$ is $\perp$ to both n and $(x \times n)$ by definition

$|n| = 1 \qquad a \times b = |a|\,|b| \sin\theta\, n$

$\theta = 90°$, $\sin\theta = 1$, so $|x \times n| = |x|$

and $|n \times (x \times n)| = |x|$

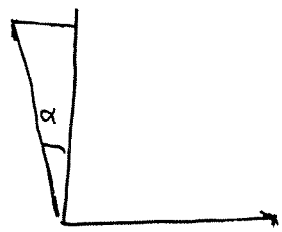
i.e. all have equal length, $|x|$.

Vector product definition is that a, b, $a \times b$ are a right handed set $\Rightarrow$ n comes out of paper.

4c) From 4b), rotation about $\underline{n}$ affects only $n \times (x \times n)$ and $(x \times n)$

General rotation $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

Apply



$$n \times (x \times n) \rightarrow n \times (x \times n) \cos\alpha - x \times n \sin\theta$$

$$(x.n)n \rightarrow (x.n)n$$

hence

$$Qx = (x.n)n + n \times (x \times n) \cos\alpha - x \times n \sin\theta$$

Using equation (4a)

$$Qx = (x.n)n + (x - (x.n)x) \cos\alpha - x \times n \sin\theta$$

$$= (x.n)n(1 - \cos\alpha) + x\cos\alpha - x \times n \sin\theta$$

4d) $\alpha = \pi/2$ $\cos\alpha = 0$ $\sin\alpha = 1$

$$Qx = (x.n)x - x \times n$$

let $x = (x_1 \ x_2 \ x_3)^T$

$$x.n = \frac{x_1}{\sqrt{2}} + \frac{x_2}{\sqrt{2}}$$

$$(x.n)n = \frac{x_1 + x_2}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)^T = \frac{x_1 + x_2}{2}(1, 1, 0)^T$$

$$x \times n = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{vmatrix} = -\frac{x_3 i}{\sqrt{2}} + \frac{x_3 j}{\sqrt{2}} + \left(\frac{x_1}{\sqrt{2}} - \frac{x_2}{\sqrt{2}}\right) k$$

$$= \left(\frac{-x_3}{\sqrt{2}}, \frac{x_3}{\sqrt{2}}, \frac{x_1 - x_2}{\sqrt{2}} \right)^T$$

Hence $Qx = \begin{pmatrix} \frac{x_1}{2} + \frac{x_1}{2} \\ \frac{x_1}{2} + \frac{x_2}{2} \\ 0 \end{pmatrix} - \begin{pmatrix} \frac{-x_3}{\sqrt{2}} \\ \frac{x_3}{\sqrt{2}} \\ \frac{x_1 - x_2}{\sqrt{2}} \end{pmatrix}$

$$= \begin{pmatrix} \frac{x_1}{2} + \frac{x_2}{2} + \frac{x_3}{\sqrt{2}} \\ \frac{x_1}{2} + \frac{x_2}{2} + \frac{-x_3}{\sqrt{2}} \\ \frac{x_1}{\sqrt{2}} + \frac{-x_2}{\sqrt{2}} + 0 \end{pmatrix}$$

and $Q = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ 1/2 & 1/2 & -1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$

Q5) Eigenvalues:

$$\begin{vmatrix} 3-\lambda & 0 & 4 \\ 0 & 2-\lambda & 0 \\ 4 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\left[(2-\lambda)(-3-\lambda)\right] + 4\left[-4(2-\lambda)\right] = 0$$

$$(2-\lambda)\left[(3-\lambda)(-3-\lambda) - 16\right] = 0$$

$$(2-\lambda)\left[\lambda^2 - 25\right] = 0$$

$$(2-\lambda)(5-\lambda)(-5-\lambda) = 0$$

$$\lambda = 2,\ 5,\ -5$$

Eigenvectors

$\lambda = 2$
$$\begin{pmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$3x + 4z = 2x, \quad 2y = 2y, \quad 4x - 3z = 2z$$

$$\therefore x = -4z, \quad -16z - 3z = 2z$$

$x = 0$, $z = 0$, $y = 1$, eigenvector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$

$\lambda = 5$
$$3x + 4z = 5x, \quad 2y = 5y, \quad 4x - 3z = 5z$$

$\therefore\ y = 0$, $x = 1$, $z = \frac{1}{2}$ eigenvector $\frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = -5$

$3x + 4z = -5x, \quad 2y = -5y, \quad 4x - 3z = -5z$

Again $y = 0$, $\quad x = 1, \quad z = -2$, eigenvector $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\0\\-2\end{pmatrix}$

A is symmetric so expect real eigenvectors (true) and orthogonal eigenvalues.

$$\begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}2\\0\\1\end{pmatrix} = 0 \qquad \begin{pmatrix}0\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} = 0 \qquad \begin{pmatrix}2\\0\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-2\end{pmatrix} = 0$$

confirmed.

5b) $\quad A u_1 = \lambda_1 u_1 \quad$ by def^{n}

$B A u_1 = B \lambda_1 u_1 \quad$ pre multiply by B

$A(B u_1) = \lambda_1 (B u_1) \quad$ $AB = BA$, λ_1 scalar

hence Bu_1 is an eigenvector with eigenvalue λ_1

Bu_1 must be in the same direction as u_1 as both are eigenvectors corresponding to λ_1

Hence $\quad B u_1 = \mu_1 u_1$

which is the definition of an eigenvector, so u_1 is also an eigenvector of B.

Repeat for u_2, u_3.

5c) Let $U = (u_1 \ \ u_2 \ \ u_3)$

already shown to be orthogonal, ie. $U^T = U^{-1}$

Can choose $D = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_3 \end{pmatrix}$ from (5b)

$B = U D U^T$ (data book)

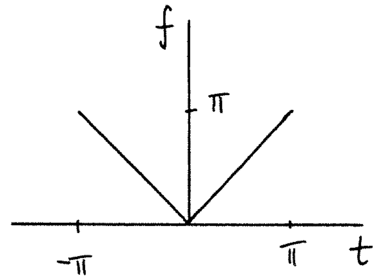
$B^t = (U D U^t)^t = U D^t U^t$ $(D^t = D)$

$= B$

Hence B is symmetric.

6 a) (i)

$$f(t) = \begin{cases} t & 0<t<\pi \\ -t & -\pi<t<0 \end{cases}$$



$f(t)$ is an even function of t

$\therefore$ Only cosines necessary

$$f(t) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nt \quad \text{where} \quad a_n = \frac{2}{\pi}\int_0^\pi f(t)\cos nt\,dt$$

$$\therefore\ a_n = \frac{2}{\pi}\left[\frac{t\sin nt}{n}\right]_0^\pi - \frac{2}{\pi}\int_0^\pi 1.\frac{\sin nt}{n}\,dt = -\frac{2}{n\pi}\left[-\frac{\cos nt}{n}\right]_0^\pi$$

$$= \frac{2}{n^2\pi^2}\left\{\cos n\pi - 1\right\} = \frac{2}{n^2\pi^2}\{(-1)^n - 1\} = -\frac{4}{n^2\pi^2} \quad n \text{ odd}$$

$$= 0 \quad n \text{ even}$$

Also $a_0 = \frac{2}{\pi}\int_0^\pi t\,dt = \frac{2}{\pi}\cdot\frac{\pi^2}{2} = \pi$

$$\therefore\ f(t) = \frac{\pi}{2} - \sum_{\substack{n=1 \\ n \text{ odd}}}^\infty \frac{4}{n^2\pi}\cos nt$$

(ii) Rate of convergence is determined by how "smooth" a function is: $a_n, b_n = O\left(\frac{1}{n^{r+2}}\right)$ where r = highest order continuous derivative. For this function, function is continuous but 1st derivative is not i.e. $r=0$ $\Rightarrow$ coeffs $\propto \frac{1}{n^2}$.

b) (i) $f_1(t) = \sum_{-\infty}^{\infty} c_n e^{int}$ and $f_2(t) = f_1(t-\pi)$ (or $f_1(t+\pi)$)

$$\therefore\ f_2(t) = \sum_{-\infty}^{\infty} c_n e^{int-in\pi} = \sum_{-\infty}^{\infty} d_n e^{int} \quad \text{where} \quad d_n = c_n e^{-in\pi}$$

$$\left(= (-1)^n c_n = e^{in\pi} c_n\right)$$

(ii) $f_3(t) = f_1(t) + f_2(t) \Rightarrow f_3(t) = \sum_{-\infty}^{\infty} e_n e^{int}$ where $e_n = c_n + d_n$

i.e. $e_n = c_n(1+(-1)^n)$

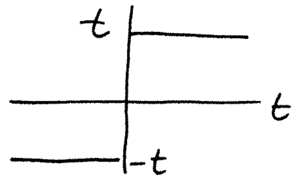
6b contd)

(iii) $f_3(t)$ has periodicity π, so only those n with this periodicity will be present. i.e. only even n.

This can also be seen from (ii) since $1+(-1)^n = 0$ for n odd.

Examiner's Note

6a(i) Many candidates sketched $f(t)$ as



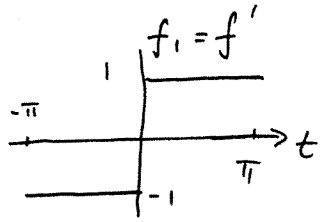
and then said it was an odd function.

The inability to integrate by parts was a serious handicap to a significant number.

6b(iii) Many candidates confused the fact that $f_3(t)$ was an even function with having something to do with only even n being present.

6a(i) Aliter

$f(t)$ is the integral of $f_1(t)$ =



$f'(t)$ odd fn $\Rightarrow$ only sines and $f'(t) = \sum_1^\infty b_n \sin nt \qquad b_n = \frac{2}{\pi}\int_0^\pi f_1 \sin nt\,dt$

$$b_n = \frac{2}{\pi}\int_0^\pi \sin nt\,dt = \frac{2}{n\pi}\left[-\frac{\cos nt}{n}\right]_0^\pi = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

$$\therefore \quad f'(t) = \sum_1^\infty \frac{2}{n\pi}\left[1-(-1)^n\right]\sin nt$$

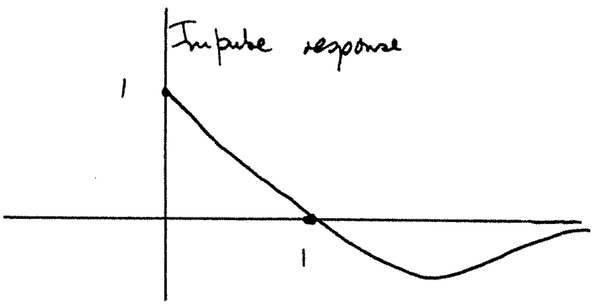
d.c. value $= \frac{\pi}{2}$

$$\text{Integrating} \qquad f(t) = -\sum_1^\infty \frac{2}{n^2\pi}(1-(-1)^n)\cos nt + \text{const}$$

$$\text{i.e.} \qquad f(t) = \frac{\pi}{2} - \sum_{\substack{1 \\ n\text{ odd}}}^\infty \frac{4}{n^2\pi}\cos nt$$

7 a) Step response $= \begin{cases} te^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$

Impulse response $= \frac{d}{dt}$ step response $= \begin{cases} e^{-t} - te^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$



$$g(t) = \begin{cases} (1-t)e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

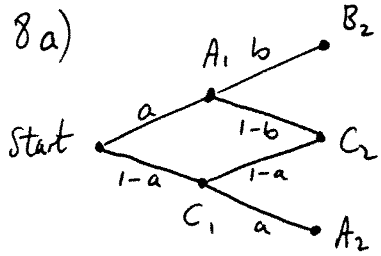
(b)
$$y(t) = \int_0^t x(\tau)\, g(t-\tau)\, d\tau = \int_0^t \tau(1-t+\tau)\, e^{-(t-\tau)}\, d\tau$$
$$= e^{-t}\int_0^t (\tau - \tau t + \tau^2)\, e^{\tau}\, d\tau$$
$$= e^{-t}\left[(\tau-\tau t+\tau^2)e^{\tau}\right]_0^t - e^{-t}\int_0^t (1-t+2\tau)e^{\tau}\, d\tau$$
$$= e^{-t}(te^{t}-0) - e^{-t}\left[(1-t+2\tau)e^{\tau}\right]_0^t + e^{-t}\int_0^t 2e^{\tau}\, d\tau$$
$$= t - e^{-t}\{(1+t)e^{t} - (1-t)\} + 2e^{-t}\left[e^{\tau}\right]_0^t$$
$$= t - (1+t) + (1-t)e^{-t} + 2e^{-t}(e^{t}-1)$$
$$= 1 - e^{-t} - te^{-t}$$

(c) Since $x(t)$ = integral of step, would expect $y(t) = \int$ step response

(d) $\frac{d}{dt}y(t) = e^{-t} - e^{-t} + te^{-t} = te^{-t}$ = step response.

Examiner's Note: Many candidates had great difficulty integrating by parts correctly. They then had little hope of getting part (d) correct.

8 a)



$P(A_1) = a, \quad P(B_1) = 0$ (must hit A first)

$P(C_1) = 1-a$

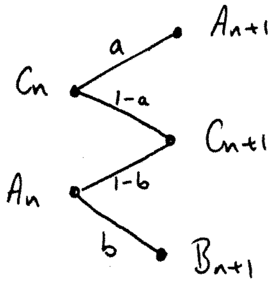
$P(A_2) = a(1-a) \qquad P(B_2) = ba$

$P(C_2) = a(1-b) + (1-a)^2$

b) If player hits B for a throw numbered less than n, then the turn finishes & the n'th throw may not happen

$$P(A_n) + P(B_n) + P(C_n) + P(\text{already finished}) = 1$$

c)



$P(C_{n+1}) = (1-a)P(C_n) + (1-b)P(A_n)$

$P(B_{n+1}) = b\,P(A_n)$

$P(A_{n+1}) = a\,P(C_n) \qquad (\Rightarrow P(A_n) = a\,P(C_{n-1}))$

$\therefore P(C_{n+1}) = (1-a)P(C_n) + (1-b)a\,P(C_{n-1}).$

Since $P(B_{n+1}) = \text{const} \times P(A_n) = \text{const}\, P(C_{n-1})$ then $P(B_{n+1})$ satisfies same eq^n as $P(C_{n-1})$ i.e. equation (1). Similarly for $P(A_n)$

d) Try $\lambda^n \Rightarrow \lambda^2 - (1-a)\lambda - a(1-b) = 0 \Rightarrow \lambda^2 - \frac{3\lambda}{4} - \frac{7}{64} = 0$

$\therefore 64\lambda^2 - 48\lambda - 7 = 0 \quad \Rightarrow (8\lambda - 7)(8\lambda + 1) = 0 \quad \Rightarrow \lambda = 7/8 \text{ or } -1/8$

$\therefore$ Gen. solution $= \alpha\left(\frac{7}{8}\right)^n + \beta\left(-\frac{1}{8}\right)^n$, α, β constants.

e) $P(B_n) = \alpha\left(\frac{7}{8}\right)^n + \beta\left(-\frac{1}{8}\right)^n$ and $P(B_1) = 0 \quad P(B_2) = \frac{9}{64}$

$\therefore \frac{7\alpha}{8} - \frac{\beta}{8} = 0 \quad \& \quad \alpha\left(\frac{7}{8}\right)^2 + \beta\left(\frac{1}{8}\right)^2 = \frac{9}{64}$

$\therefore \beta = 7\alpha \quad \& \quad 49\alpha + \beta = 9 \quad \Rightarrow \alpha = \frac{9}{56}$

$$\therefore P(B_n) = \frac{9}{56}\left[\left(\frac{7}{8}\right)^n + 7\left(-\frac{1}{8}\right)^n\right] \qquad n \geq 1.$$

9

a) $L(\dot{y}) = sY - y(0) \qquad L(\ddot{y}) = s^2 Y - sy(0) - \dot{y}(0)$

$$\ddot{y} + 4\dot{y} + 3y = e^{-t} \Rightarrow (s^2 + 4s + 3)Y - s - 4 = \frac{1}{s+1}$$

$$\therefore \quad Y = \frac{1}{(s+3)(s+1)}\left[s+4 + \frac{1}{s+1}\right]$$

$$\therefore \quad Y = \frac{A}{s+3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \Rightarrow A(s+1)^2 + B(s+1)(s+3) + C(s+3) = (s+4)(s+1) + 1$$

$$\Rightarrow \quad A + B = 1; \quad 2A + 4B + C = 5; \quad A + 3B + 3C = 5$$

$$\therefore \quad 5A + 9B = 10 \quad \Rightarrow \quad B = \frac{5}{4} \quad A = -\frac{1}{4} \quad C = \frac{1}{2}$$

$$\therefore \quad \underline{y(t) = -\frac{e^{-3t}}{4} + \frac{5}{4}e^{-t} + \frac{t}{2}e^{-t}} \qquad y(0) = 1 \quad \dot{y}(0) = 0$$

b) $L(f * g) = L(f)\,L(g)$ i.e. $L(\text{convolution})$ = Product of transforms

c) Taking Laplace transforms

$$Y = \frac{1}{s} + Y\frac{1}{s^2+1}$$

$$\therefore \quad Y(s)\left[1 - \frac{1}{s^2+1}\right] = \frac{1}{s} \quad \text{or} \quad Y(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

$$\therefore \quad \underline{y(t) = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}}$$

Examiner's Note:

N.B. Many algebraic errors would have been detected if initial conditions for part (a) had been checked.

For a repeated root $(s+1)^2$ of the denominator for part (a)

Need either $\dfrac{B}{s+1} + \dfrac{C}{(s+1)^2}$ or $\dfrac{Bs+C}{(s+1)^2}$ NOT $\dfrac{B}{s+1} + \dfrac{Cs+D}{(s+1)^2}$.

10 a) (i) $P\,dx + Q\,dy$ is exact if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(ii) $P = 2xy^3 - \dfrac{2}{x} \Rightarrow \dfrac{\partial P}{\partial y} = 6xy^2$

$Q = 3x^2y^2 + 9y^2 \Rightarrow \dfrac{\partial Q}{\partial x} = 6xy^2$ $\Big\} \Rightarrow$ differential exact

$$\frac{\partial f}{\partial x} = P = 2xy^3 - \frac{2}{x} \Rightarrow f = x^2y^3 - 2\ln x + g(y)$$

$$\frac{\partial f}{\partial y} = Q = 3x^2y^2 + 9y^2 \Rightarrow f = x^2y^3 + 3y^3 + h(x)$$

$$\therefore f = x^2y^3 - 2\ln x + 3y^3 \quad (+\text{ const})$$

b) (i) When $x+y-1=0$, $f = (x-a)^2 + (1-x-b)^2$

$\dfrac{df}{dx} = 2(x-a) - 2(1-x-b)$. At minimum $\dfrac{df}{dx} = 0$

$\dfrac{d^2f}{dx^2} = 2+2 = 4 > 0$ $\qquad \Rightarrow x = \dfrac{a+1-b}{2}$

$$\therefore f_{min} = \left(\frac{1-b-a}{2}\right)^2 + \left(\frac{1-b-a}{2}\right)^2 = \frac{(1-b-a)^2}{2}$$

(ii) $dg = \dfrac{\partial g}{\partial x}dx + \dfrac{\partial g}{\partial y} = 0$. When $g =$ const $dg = 0 \Rightarrow dy = -\dfrac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}\,dx$

$\therefore$ For changes in f $\quad df = \dfrac{\partial f}{\partial x}dx + \dfrac{\partial f}{\partial y}dy = \left[\dfrac{\partial f}{\partial x} - \dfrac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial y}}\dfrac{\partial f}{\partial y}\right]dx$

(iii) At stationary values of f (such that $g=0$) $df = 0 \Rightarrow \dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} = \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}$

(iv) For equation (2) $\dfrac{\partial f}{\partial x} = 2(x-a)$, $\dfrac{\partial f}{\partial y} = 2(y-b)$, $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial y} = 1$

$\dfrac{\partial f}{\partial x}\dfrac{\partial g}{\partial y} = \dfrac{\partial f}{\partial y}\dfrac{\partial g}{\partial x}$ only if $2(x-a) = 2(y-b) = 2(1-x-b)$ since $x+y=1$

$$\Rightarrow x = \frac{a+1-b}{2} \text{ as required.}$$

Examiner's Note:

With hindsight, part (b) was probably too difficult. It was marked very leniently to produce a reasonable average mark.