ENGINEERING TRIPOS PART IA

Tuesday 13 June 2000 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **four** *may be taken from section A and not more than* **four** *from Section B.*

The **approximate** *number of marks allocated to each part of a question is indicated in the right margin.*

All questions carry the same number of marks.

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

Answer not more than **four** *questions from this section.*

1 (a) Determine the line of intersection of the planes:

$$\text{Plane A:} \quad 2x+3y-z=1$$
$$\text{Plane B:} \quad x-y+z=2$$ [6]

(b) (i) Find the equation of *one* of the planes which is distance 1 from plane A over its entire area. [6]

(ii) How many such planes are there? [2]

(c) Find the equation of *one* of the lines which is distance 1 from the two planes along its entire length. [6]

2 (a) Find:

$$\lim_{x\to 0} \frac{\sin x \cosh x - x}{x(1-\cos x)}$$ [7]

(b) Find all values of:

$$\sin^{-1}(2i)$$ [7]

(c) Simplify:

$$\{[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})]\times\mathbf{c}\}\times\mathbf{b}$$ [6]

3 (a) Find the general solution of:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = \sin x$$ [8]

(b) Find the solution of:

$$S_n - 2S_{n-1} + \left(1 - \varepsilon^2\right)S_{n-2} = 0$$

with $S_0 = 0$ and $S_1 = 1$ and where ε is a constant. [8]

(c) By considering the case $\varepsilon \to 0$ for your answer to part (b), or otherwise, find the general solution of:

$$S_n - 2\,S_{n-1} + S_{n-2} = 0$$ [4]

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4 (a) By considering the simplification formula for a vector triple product, show that, for any vector $\mathbf{x}$ and any *unit* vector $\mathbf{n}$:

$$\mathbf{x} = \mathbf{x}.\mathbf{n}\,\mathbf{n} + \mathbf{n}\times(\mathbf{x}\times\mathbf{n})$$ [4]

(b)

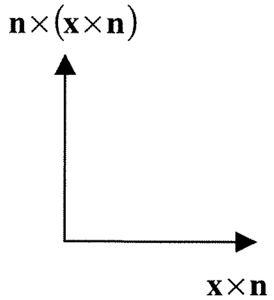

Fig. 1

Figure 1 shows the plane containing $\mathbf{x}\times\mathbf{n}$ and $\mathbf{n}\times(\mathbf{x}\times\mathbf{n})$. Explain why the angle between these two vectors is a right angle and why they are of equal length. Describe the direction, and sense, of $\mathbf{n}$ relative to this plane. [4]

(c) A matrix $\mathbf{Q}$ is to represent a rotation by α about the $\mathbf{n}$ axis. By considering the effect of this rotation on $\mathbf{x}.\mathbf{n}\,\mathbf{n}$ and on $\mathbf{n}\times(\mathbf{x}\times\mathbf{n})$, or otherwise, show that:

$$\mathbf{Qx} = \mathbf{x}.\mathbf{n}\,\mathbf{n} + \mathbf{n}\times(\mathbf{x}\times\mathbf{n})\cos\alpha - \mathbf{x}\times\mathbf{n}\sin\alpha$$

Hence show that

$$\mathbf{Qx} = \mathbf{x}.\mathbf{n}\,\mathbf{n}\,(1-\cos\alpha) + \mathbf{x}\cos\alpha - \mathbf{x}\times\mathbf{n}\sin\alpha$$ [6]

(d) Find $\mathbf{Q}$ for the case of $\alpha = \dfrac{\pi}{2}$ and $\mathbf{n}^t = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}, 0\right)$ [6]

5 (a) Find the eigenvalues and eigenvectors of the matrix:

$$\begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & -3 \end{bmatrix}$$

You should check that your answers are consistent with known properties of the eigenvalues and eigenvectors of this type of matrix. [10]

(b) The symmetric matrix $\mathbf{A}$ has eigenvalues λ_1, λ_2 and λ_3 (which are distinct) and corresponding *normalised* eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$. The non-zero matrix $\mathbf{B}$ has the property that it commutes with $\mathbf{A}$, i.e.

$$\mathbf{A\,B} = \mathbf{B\,A}$$

Show that $\mathbf{Bu}_1$ is also an eigenvector of $\mathbf{A}$, corresponding to eigenvalue λ_1 , and hence that it is parallel to $\mathbf{u}_1$. Hence deduce that $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$ are also eigenvectors of $\mathbf{B}$. [6]

(c) For the matrix $\mathbf{B}$ of part (b), find an orthogonal matrix $\mathbf{U}$ such that $\mathbf{B\,U} = \mathbf{U\,D}$, where $\mathbf{D}$ is a diagonal matrix. Hence show that $\mathbf{B}$ is symmetric. [4]

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SECTION B

Answer not more than **four** *questions from this section.*

6 (a) The function:

$$f(t)=\begin{cases} t & 0<t<\pi \\ -t & -\pi<t<0 \end{cases}$$

is periodic of period 2π.

(i) Find a Fourier series representation for this function. [6]

(ii) Describe which properties of a function influence the rate at which a Fourier series representation of that function converges. Illustrate this using the Fourier series representation of $f(t)$. [4]

(b) The half-wave rectified cosine wave, $f_1(t)$, shown in Fig. 2(a), is periodic of period 2π and has a complex Fourier series given by:

$$f_1(t)=\sum_{n=-\infty}^{\infty} c_n e^{int}$$

(i) By making a substitution for t in this equation, or otherwise, determine, in terms of c_n, the coefficients d_n in a complex Fourier series representation of the function $f_2(t)$, shown in Fig. 2(b), which is also periodic of period 2π. [3]

(ii) Hence find, in terms of c_n, the complex Fourier series representation of the full-wave rectified cosine wave, $f_3(t)$, shown in Fig. 2(c). [3]

(iii) Explain carefully why some values of n are not present in the representation for $f_3(t)$ and identify these values of n. [4]

[NOTE: For part (b) of this question, there is *no need to evaluate the Fourier coefficients*.]

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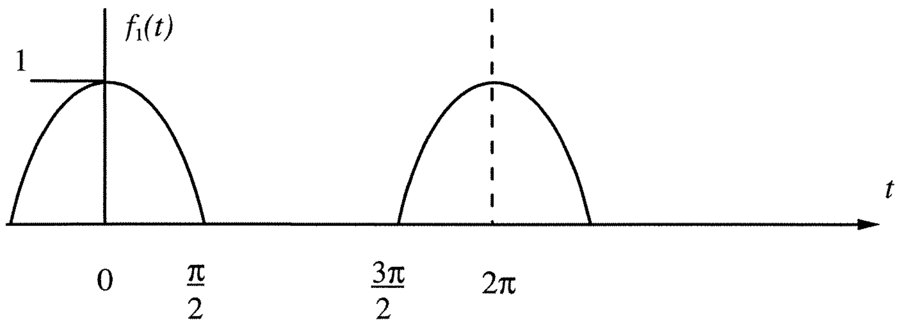

Fig. 2 (a)

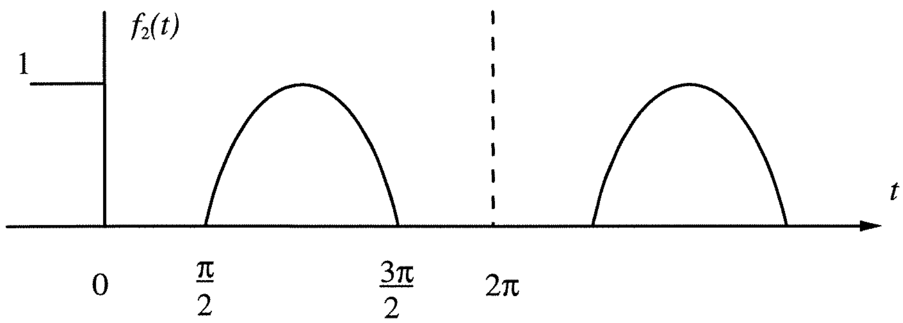

Fig. 2 (b)

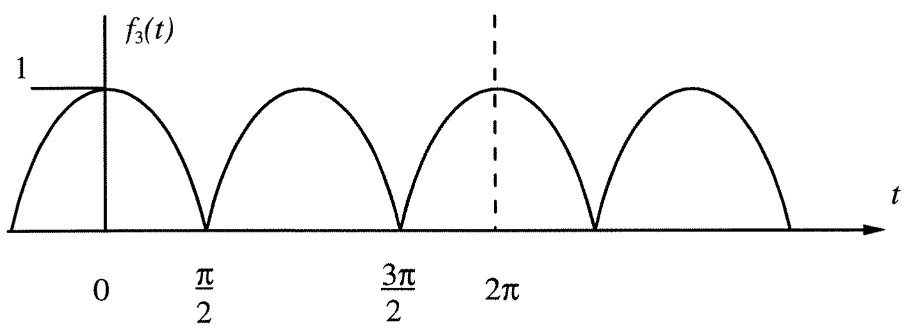

Fig. 2 (c)

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7 (a) A linear system has a step response given by:

$$\begin{array}{ll} te^{-t} & t \geq 0 \\ 0 & \text{otherwise} \end{array}$$

Determine and sketch the impulse response. [5]

(b) Using a convolution integral, find the response $y(t)$ of the above system to an input $x(t)$ given by:

$$\begin{array}{ll} t & t \geq 0 \\ 0 & \text{otherwise} \end{array}$$

[7]

(c) How would you expect $y(t)$ to be related to the step response of the system? [3]

(d) Verify your answer to part (c). [5]

8 A game consists of players throwing balls in succession in order to hit one of two targets - target A and target B . The object of the game is to hit target A and then target B with successive balls. A player's turn thus consists of continuing to throw balls until target A is hit. Once this is achieved, the player then aims for target B. A successful hit of B attained on the throw immediately following the hit of A ends the player's turn. If B is missed on the throw immediately following the hit of A, then the player must continue throwing to achieve another hit of A before a throw at B can again be attempted. If a hit of A is accidentally achieved when aiming for B, this does not count. The process is repeated until the player is successful.

A certain player is successful at hitting A with a single ball (when aiming for A) with probability a and at hitting B (when aiming for B) with probability b.

(a) Show that:

$$P(A_1) = a, \qquad P(B_1) = 0, \qquad P(C_1) = (1-a),$$

$$P(A_2) = a(1-a), \qquad P(B_2) = ab \quad \text{and} \quad P(C_2) = (1-a)^2 + a(1-b)$$

where the probability of hitting A with the n'th ball is denoted by $P(A_n)$, that of hitting B by $P(B_n)$, and that of missing the aimed-for target by $P(C_n)$. [4]

(b) Explain why $P(A_n) + P(B_n) + P(C_n) < 1$ for $n > 2$. [2]

(c) Write down expressions for $P(A_{n+1}), P(B_{n+1})$ and $P(C_{n+1})$ for $n \geq 2$, and hence show that:

$$P(C_{n+1}) = (1-a)P(C_n) + (1-b)aP(C_{n-1}) \qquad (1)$$

and explain why $P(A_{n+1})$ and $P(B_{n+1})$ also satisfy the same difference equation. [5]

(d) For the case $a = \frac{1}{4}$ and $b = \frac{9}{16}$, show that the general solution of eqn. (1) is

$$\alpha\left(\frac{7}{8}\right)^n + \beta\left(-\frac{1}{8}\right)^n$$

where α and β are constants. [5]

(e) Find $P(B_n)$. [4]

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9 (a) Solve using Laplace transforms:

$$\ddot{y} + 4\dot{y} + 3y = e^{-t}$$

where $y(0) = 1$ and $\dot{y}(0) = 0$. [10]

(b) Describe how the Laplace transform of the convolution of two functions is related to the transforms of the individual functions. [3]

(c) By taking Laplace transforms, or otherwise, find the function $y(t)$, where:

$$y(t) = 1 + \int_0^t y(\tau)\sin(t - \tau)\,\mathrm{d}\tau$$ [7]

10 (a) (i) State the conditions under which $P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y$ is an exact differential. [4]

(ii) For the case:

$$P = 2xy^3 - \frac{2}{x} \quad \text{and} \quad Q = 3x^2y^2 + 9y^2$$

show that $P\,\mathrm{d}x + Q\,\mathrm{d}y$ is an exact differential and find a function $f(x,y)$ such that $\mathrm{d}f = P\,\mathrm{d}x + Q\,\mathrm{d}y$. [4]

(b) (i) Find the minimum value of the function:

$$f(x, y) = (x - a)^2 + (y - b)^2 \qquad (2)$$

when x and y lie on the plane $x + y - 1 = 0$. [3]

(ii) Find a relationship between the possible changes $\mathrm{d}x$ and $\mathrm{d}y$ as x and y vary over a surface $g(x,y) = 0$. Hence show that changes in a function $f(x,y)$, when x and y are constrained to vary over the surface $g(x,y) = 0$, satisfy:

$$\mathrm{d}f = \mathrm{d}x \left(\frac{\partial f}{\partial x} - \frac{\dfrac{\partial g}{\partial x}}{\dfrac{\partial g}{\partial y}} \frac{\partial f}{\partial y} \right)$$

[3]

(iii) Hence find a condition for f to have a stationary value when x and y vary over the surface $g(x,y) = 0$. [2]

(iv) Verify that this condition is satisfied for the minimum of the function defined by eqn. (2) as x and y vary over the plane $x + y - 1 = 0$. [4]

END OF PAPER