IA JUNE 2001 P1

Q1) a) $T_2 = 60^\circ C = 333\,K$

$$\rho_2 = \frac{P_2}{RT_2} = \frac{1.5\times10^5}{287\times333} = \underline{1.570\ kg/m^3}$$

$$\dot m = \rho_2 \times \frac{V_{SWEPT}}{CYCLE} \times \frac{CYCLE}{SEC} = 1.570\times500\times10^{-6}\times100 = \underline{\underline{0.0785\ kg/s}}$$

b) $$\dot Q_{IN} - \dot Q_{COOL} - \dot W_x = \dot m(h_3 - h_2) = \dot m C_p(T_3 - T_2)$$

$$\dot Q_{IN} - 0.35\dot Q_{IN} - 0.3\dot Q_{IN} = 0.35\dot Q_{IN} = \dot m C_p(T_3 - T_2)$$

$$\dot W_x = 0.3\times2.8\times10^6\times0.0785 = \underline{\underline{65.94\ kW}}$$

$$T_3 - T_2 = \frac{0.35\dot Q_{IN}}{\dot m C_p} = \frac{0.35\times2.8\times10^6\times\cancel{\dot m}}{\cancel{\dot m}\times1010} = \underline{970.3\ K}$$

$$T_3 = T_2 + 970.3 = 333 + 970.3 = \underline{\underline{1303.3\ K}}$$

c) $$\dot W_x|_{COMP} = \dot W_x|_{TURB} = \dot m C_p(T_2 - T_1) = 0.0785\times1010\times45 = \underline{\underline{3.57\ kW}}$$

$$\Delta T|_{COMP} = \Delta T|_{TURB} = 45\,K$$

$$T_4 = T_3 - 45 = 1303.3 - 45 = \underline{\underline{1258.3\ K}}$$

d) $$\frac{P_3}{P_4} = \left(\frac{T_3}{T_4}\right)^{\frac{\gamma}{\gamma-1}} \Rightarrow P_3 = 1.0\times\left(\frac{1303.3}{1258.3}\right)^{3.5} = \underline{\underline{1.131\ bar}}$$

e) $$\rho_1 = \frac{P_1}{RT_1} = \frac{1.0\times10^5}{287\times288} = 1.210\ kg/m^3$$

$$\dot m = 1.210\times500\times10^{-6}\times100 = 0.0605\ kg/s$$

$$\dot W_x = 0.3\times2.8\times10^6\times0.0605 = \underline{\underline{50.82\ kW}}$$

Examiners Comments: Students had major problems calculating the density for part a).
Rest of question was done reasonably well. Few spotted the isentropic expansion reduced the pressure.

1A JUNE 2001 P1

Q2) a) PARALLEL JET ⇒ NO TRANSVERSE PRESSURE GRADIENT
⇒ PRESSURE IN JET IS AMBIENT $P_2 = 0$ (GAUGE)

b) MASS CONSERVATION $\rho v_1 A = \rho v_2 \alpha A \Rightarrow v_1 = \alpha v_2$

BERNOULLI: $P_1 + \frac{1}{2}\rho v_1^2 = P_2 + \frac{1}{2}\rho v_2^2$ ($P_2 = 0$ GAUGE)

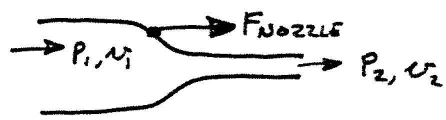
$$P_1 = \tfrac{1}{2}\rho(v_2^2 - v_1^2) = \tfrac{1}{2}\rho v_2^2(1-\alpha^2)$$

$$\frac{P_1}{\rho v_2^2} = \frac{1}{2}(1-\alpha^2)$$

c) WORK WITH GAUGE PRESSURE, SO ONLY NEED TO INCLUDE P_1.

$F_{FLOW} = -F_{NOZZLE}$

STEADY FLOW MOMENTUM EQUATION



$$\Sigma PA + F = \Sigma \dot{m} v$$

$$P_1 A - F_{NOZZLE} = (\alpha A \rho v_2)(v_2 - v_1)$$

$$F_{NOZZLE} = P_1 A - \alpha A \rho v_2^2 (1-\alpha)$$

$$= A\rho v_2^2 \tfrac{1}{2}(1-\alpha^2) - A\rho v_2^2 \alpha(1-\alpha)$$

$$= A\rho v_2^2 \left[\tfrac{1}{2}(1+\alpha) - \alpha\right](1-\alpha)$$

$$F_{NOZZLE} = \tfrac{1}{2} A\rho v_2^2 (1-\alpha)^2 \qquad \frac{F_{NOZZLE}}{A\rho v_2^2} = \frac{1}{2}(1-\alpha)^2$$

d) F_{NOZZLE} depends on $\{P_1, P_2, \alpha, v_2, v_1, \rho, A\}$

Now: $P_2 = 0$ (GAUGE)

$P_1 = f_n(\rho, v_2, \alpha)$ FROM (b)

$v_1 = f_n(\alpha, v_2)$ FROM (b)

⇒ F_{NOZZLE} depends on $\{\alpha, v_2, \rho, A\}$

$\frac{ML}{T^2}$ — $\frac{L}{T}$ $\frac{M}{L^3}$ L^2

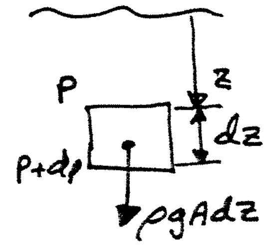
BUCKINGHAM'S Pi : 5 VAR − 3 DIM = 2 OR MORE N-D GROUPS

REARRANGE $\frac{F_{NOZZLE}}{A\rho v_2^2}$ depends on $\{\alpha\}$

Examiner's comments: a) & b) well done. Part c) was okay except algebra caused problems. Dimensional analysis was moderately attempted, most knew about Buckingham's Pi Thm.

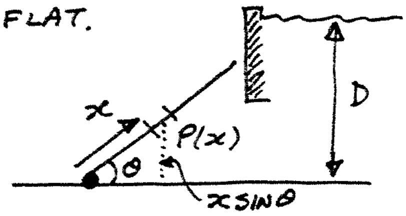
1A JUNE 2001 P1

Q3) a) R↑)



$$(P+dP)A = \rho g A\,dz + PA$$
$$dP = \rho g\,dz$$
$$\underline{\underline{\frac{dP}{dz} = \rho g}}$$

b) PRESSURE ACTS EQUALLY IN ALL DIRECTIONS. HENCE PRESSURE FORCE IS PERPENDICULAR TO SURFACE AND SURFACE IS FLAT.



c) $P(x) = (D - x\sin\theta)\rho g$

$$F = \int_0^L P\,dx = \int_0^L (D - x\sin\theta)\rho g\,dx$$

$$F = \rho g\left[Dx - \tfrac{1}{2}x^2\sin\theta\right]_0^L = \underline{\underline{\rho g L\left(D - \tfrac{1}{2}L\sin\theta\right)}}$$

[ALT: COULD USE NET HORIZONTAL FORCE, OR JUSTIFY LINEAR PRESSURE VARIATION AND STATE AVERAGE PRESSURE × AREA.]

d) MOMENTS ABOUT A: $FX = \text{Mom} = \int_0^L xP\,dx$

$$FX = \int_0^L (Dx - x^2\sin\theta)\rho g\,dx = \rho g\left[\tfrac{1}{2}Dx^2 - \tfrac{1}{3}x^3\sin\theta\right]_0^L = \rho g L^2\left(\tfrac{1}{2}D - \tfrac{1}{3}L\sin\theta\right)$$

$$X = \rho g\left(L^2\left(\tfrac{1}{2}D - \tfrac{1}{3}L\sin\theta\right)\right) / \rho g L\left(D - \tfrac{1}{2}L\sin\theta\right)$$

$$\underline{\underline{X = L\frac{\left(\tfrac{1}{2}D - \tfrac{1}{3}L\sin\theta\right)}{\left(D - \tfrac{1}{2}L\sin\theta\right)}}}$$

e) MOMENTS ABOUT A $\quad FX = \tfrac{1}{2}LMg\cos\theta$

$$\rho g L^2\left(\tfrac{1}{2}D - \tfrac{1}{3}L\sin\theta\right) \geq \tfrac{1}{2}LMg\cos\theta$$
$$\tfrac{1}{2}D - \tfrac{1}{3}L\sin\theta \geq \tfrac{1}{2}(M/L\rho)\cos\theta$$
$$\tfrac{1}{2}D \geq \tfrac{1}{2}(M/L\rho)\cos\theta + \tfrac{1}{3}L\sin\theta$$
$$\underline{\underline{D \geq \frac{M\cos\theta}{\rho L} + \frac{2}{3}L\sin\theta}}$$

Examiner's comment: generally well done, most students understood hydrostatics but failed to take moments correctly.

IA JUNE 2001 P1

Q4) a) $H_2 + \tfrac{1}{2}(1+x)(O_2 + \tfrac{79}{21} N_2) = H_2O + \tfrac{1}{2} x O_2 + \tfrac{1}{2}(1+x)\tfrac{79}{21} N_2$

CONSIDER $1\ \text{kmol } H_2 \equiv 2\ \text{kg } H_2$ $\quad$ LCV = 120 MJ/kg

SFEE : $\dot{Q} - \dot{W}_x = \dot{m}(h_2 - h_1) = \sum M_2 \bar{h}_2 - \sum M_1 \bar{h}_1$ $\quad \left(\begin{array}{l} \bar{h} \equiv \text{J/kmol} \\ M \equiv \text{kmol} \end{array} \right)$

$$0 = \underbrace{\sum M_2 \left(\bar{h}_2\Big|_{T_2} - \bar{h}_2\Big|_{25^\circ C} \right)}_{\text{TABLE BELOW}} + \underbrace{\sum \left(M_2 \bar{h}_2\Big|_{25^\circ C} - M_1 \bar{h}_1\Big|_{25^\circ C} \right)}_{-2\,\text{kg} \times 120\,\text{MJ/kg}} + \underbrace{\sum M_1 \left(\bar{h}_1\Big|_{25} - \bar{h}\Big|_{T_1} \right)}_{=0}$$

1 kmol H_2 (kmol)	M (kmol)	$\bar{h}_2\vert_{1500}$ (MJ/kmol)	$\bar{h}_2\vert_{25^\circ C}$ (MJ/kmol)	$\bar{h}_2\vert_{1500} - \bar{h}_2\vert_{25}$ (MJ/kmol)	$\bar{m}$ (kg/kmol)
H_2O	1	58.05	9.90	48.15	18
O_2	$\tfrac{1}{2}x$	49.27	8.66	40.61	32
N_2	$\tfrac{1}{2}(1+x)\tfrac{79}{21}$	47.09	8.67	38.42	28

$$0 = 1 \times 48.15 + \tfrac{1}{2}x \times 40.61 + \tfrac{1}{2}(1+x)\tfrac{79}{21} \times 38.42 - 2 \times 120 \qquad [\text{MJ}]$$

$$x\left(\tfrac{1}{2} \times 40.61 + \tfrac{1}{2} \times \tfrac{79}{21} \times 38.42\right) = 2 \times 120 - 1 \times 48.15 - \tfrac{1}{2} \times \tfrac{79}{21} \times 38.42$$

$$92.57\,x = 119.58 \qquad \underline{\underline{x = 1.29}} \quad \text{(129\% EXCESS AIR)}$$

b) WET BASIS (INC WATER) $\sum M_2 = 1 + \tfrac{1}{2}x + \tfrac{1}{2}(1+x)\tfrac{79}{21} = 1 + 0.645 + 4.307 = 5.95$ kmol

O_2 MOLAR FRACTION $= 0.645 / 5.95 = \underline{\underline{0.108 \quad (10.8\%)}}$

DRY BASIS (IGNORE WATER) $\sum M_2 = \tfrac{1}{2}x + \tfrac{1}{2}(1+x)\tfrac{79}{21} = 0.645 + 4.307 = 4.95$ kmol

O_2 MOLAR FRACTION $= 0.645 / 4.95 = \underline{\underline{0.130 \quad (13\%)}}$

c) MASS PRODUCTS $= 18 \times 1 + 32 \times \tfrac{1}{2}x + 28 \times \tfrac{1}{2}(1+x)\tfrac{79}{21}$

$= 18 \times 1 + 32 \times 0.645 + 28 \times 4.307 = \underline{\underline{159.2\ \text{kg}}}$

MASS REACTANTS (CHECK) $= 2 \times 1 + 137.33 \times \tfrac{1}{2}(1+x)$

$= 2 \times 1 + 137.33 \times 1.145 = \underline{\underline{159.2\ \text{kg}}}$

PER kmol H_2

d) SFEE WITH K.E. $\sum M_2 \bar{h}_2\big|_{1500} = \sum M_2 \bar{h}_2\big|_{1300} + \tfrac{1}{2} \dot{m} V^2$

$\tfrac{1}{2} \times 159.2 \times V^2 = 1 \times (58.05 - 48.84) + 0.645(49.27 - 42.01) + 4.307(47.09 - 40.19)$[M

$= [1 \times 9.21 + 0.645 \times 7.26 + 4.307 \times 6.90] \times 10^6$

$= 43.61 \times 10^6$

$V^2 = 43.61 \times 10^6 / (\tfrac{1}{2} \times 159.2) \qquad \underline{\underline{V = 755.3\ \text{m/s}}}$

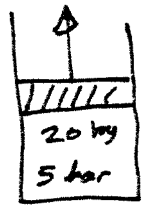
Examiner's comment: Quite well done, many students got the correct excess air. However most students confused over wet & dry basis.

1A JUNE 2001 P1

Q5) a) SYSTEM : $Q - W = \Delta E = \Delta(U + \tfrac{1}{2}mV^2 + mgz)$ (FIXED MASS)

CONTROL VOL : $\dot{Q} - \dot{W}_x = \dot{m}\,\Delta(h + \tfrac{1}{2}V^2 + gz)$ (FIXED BOUNDARY)

b) $T_1 = 15°C = 288\text{ K}$ $\quad V_1 = \dfrac{mRT_1}{P_1} = \dfrac{20 \times 287 \times 288}{5 \times 10^5} = 3.306\text{ m}^3$



$T_2 = 500°C = 773\text{ K}$ $\quad V_2 = \dfrac{mRT_2}{P_2} = \dfrac{20 \times 287 \times 773}{5 \times 10^5} = 8.874\text{ m}^3$

CONSTANT PRESSURE $Q = mC_p\Delta T = 20 \times 1010 \times (773 - 288) = \underline{9797\text{ kJ}}$

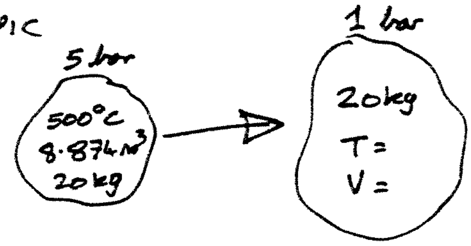
[NOTE : $Q = W + \Delta U = 2784 \times 10^3 + 20 \times 287 \times (773 - 288) = 9768\text{ kJ}$]

DISP. WORK = $P\Delta V = 5 \times 10^5 (8.874 - 3.306) = \underline{\underline{2784\text{ kJ}}}$

c)i) EXPANSION IS ADIABATIC & REVERSIBLE ⇒ ISENTROPIC

$T = T_2 \left(\dfrac{P}{P_2}\right)^{\frac{\gamma-1}{\gamma}} = 773 \left(\dfrac{1}{5}\right)^{1/3.5} = \underline{488.1\text{ K}}$

$V = V_2 \left(\dfrac{P_2}{P}\right)^{1/\gamma} = 8.874 \left(\dfrac{5}{1}\right)^{1/1.4} = \underline{28.015\text{ m}^3}$



MASS REMAINING IN CYL : $M_{LEFT} = \dfrac{PV_{CY}}{RT} = \dfrac{1 \times 10^5 \times 8.874}{287 \times 488.1} = \underline{6.335\text{ kg}}$

MASS ESCAPED $= 20 - 6.335 = \underline{\underline{13.665\text{ kg}}}$

TOTAL WORK $(Q - W = \Delta U)$ $\quad Q = 0 \Rightarrow$ TOTAL WORK $= -\Delta U$

TOTAL WORK $= -mC_v\Delta T = -20 \times 720 \times (488.1 - 773) = \underline{\underline{4103\text{ kJ}}}$

c)ii) TOTAL WORK $W = W_{DISP} + W_x$

$W_{DISP} = \int p\,dV = 1 \times 10^5 \times (V_{ESCAPE} - 0) = m_{ESCAPE}RT$

$W_{DISP} = 13.665 \times 287 \times 488.1 = 1914\text{ kJ}$

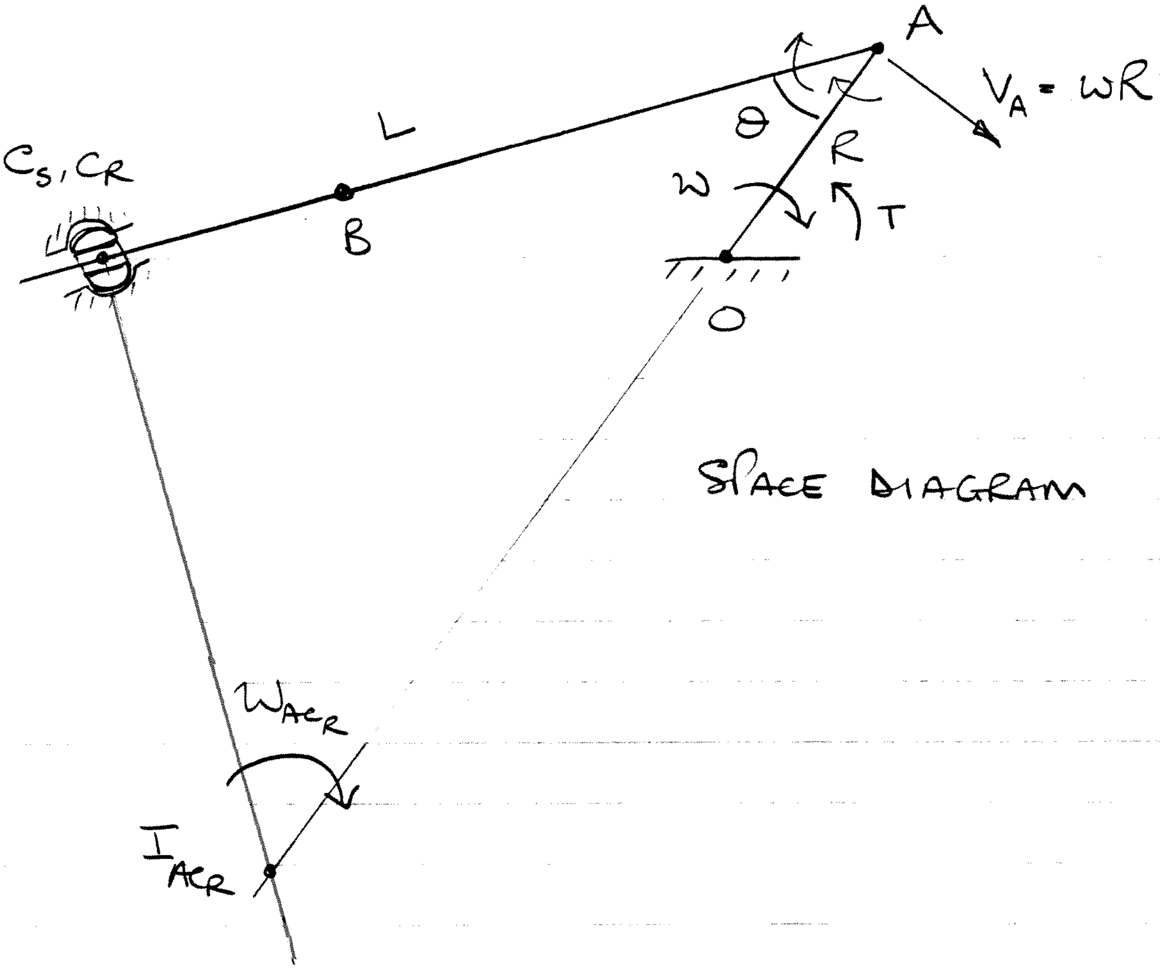
TURBINE SHAFT WORK $W_x = W - W_{DISP} = 4103 - 1914 = \underline{\underline{2189\text{ kJ}}}$

d) ASSUMING NO HEAT TRANSFER THEN VOLUME OUT OF CYLINDER IS SAME AS VOLUME ENTERING ATMOSPHERE. HENCE NO NET VOLUME CHANGE WHEN PURGING, SO <u>NO</u> ADDITIONAL WORK.

Examiner's comment: Very badly done, students did not appreciate difference between total work W and displacement work $\int p\,dV$.

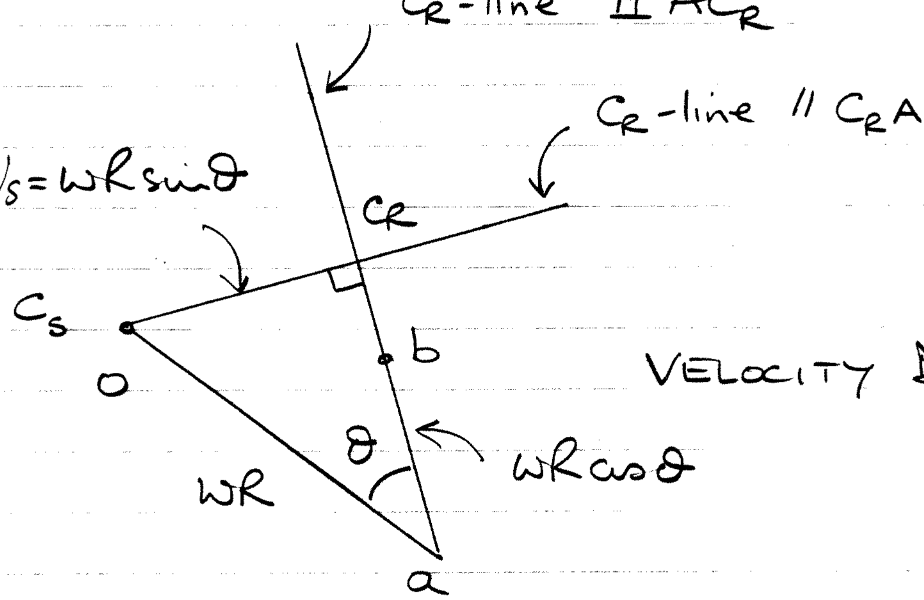
ENGINEERING TRIPOS PART IA 2001 - CRIB
PART B: MECHANICS & VIBRATIONS

6



SPACE DIAGRAM

(a)



VELOCITY DIAGRAM

(b) $\omega_{AB} = \omega_{AC_R} = \frac{a_{C_R}}{L} = \frac{\omega R \cos\theta}{L}$

Sliding velocity $= C_S C_R$

$= \omega R \sin\theta$

(c) $\sum$ Instantaneous power $= 0$

(i) or Input power = Output power (no friction)

$F_{(i)}$ (compt of vel of B in dirn of F) $= T\omega$

$C_S C_R$

$$F_{(i)} = \frac{T\omega}{\omega R \sin\theta} = \frac{T}{R\sin\theta}$$

(ii) Input Power = Output Power + Frictional power

Frictional power at C : $Q\omega_{AC_R} = \frac{Q\omega R\cos\theta}{L}$

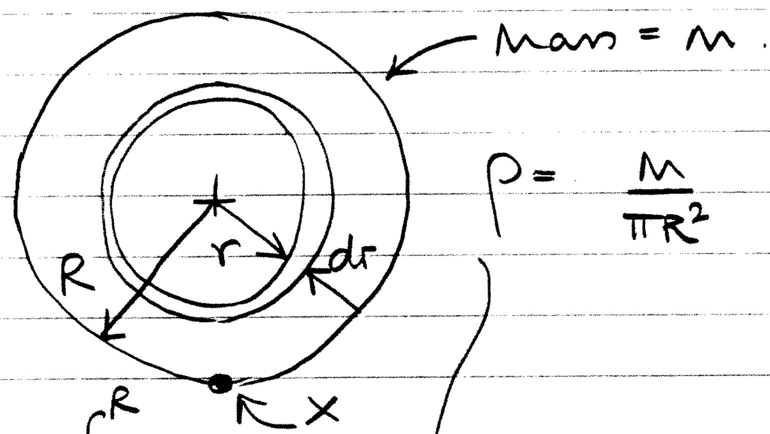
" " " A : Since the angular velocities are in the same sense – subtract : $Q\left(\omega - \frac{\omega R\cos\theta}{L}\right)$

$\omega > \omega_{AC_R}$

$$F_{(i)}\, \omega R\sin\theta = T\omega + Q\omega\left(\cancel{\frac{R\cos\theta}{L}} + 1 - \cancel{\frac{R\cos\theta}{L}}\right)$$

$$F_{(ii)} = \frac{T+Q}{R\sin\theta}$$

7 (a)



$\rho = \dfrac{m}{\pi R^2}$ (mass/unit area)

$$J = \int_0^R r^2\,dm \qquad dm = \rho\, 2\pi r\,dr$$

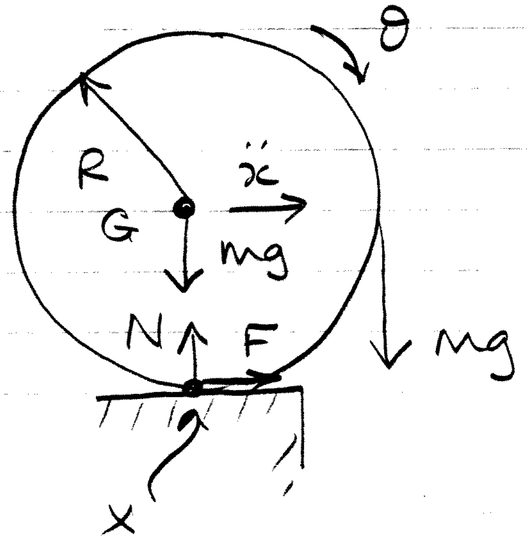
$$J = 2\rho\pi \int_0^R r^3\,dr = 2\rho\pi \frac{R^4}{4}$$

$$J = \frac{m}{\cancel{\pi} R^2} \cdot \frac{\cancel{\pi} R^4}{2} = \underline{\underline{\frac{mR^2}{2}}}$$

Parallel axes theorem $J_x = J_G + mR^2$

$$\therefore\ \underline{\underline{J_x = \frac{3}{2} mR^2}}$$

(b)
(i)



Take moments about X (+)

$$mgR = J_x \ddot{\theta}$$

$$= \frac{3}{2} mR^2 \ddot{\theta}$$

$$\rightarrow \underline{\ddot{\theta} = \frac{2}{3}\frac{g}{R}}$$

From the geometry, $\ddot{x} = R\ddot{\theta}$

$$\therefore \ddot{x} = \frac{2}{3}g$$

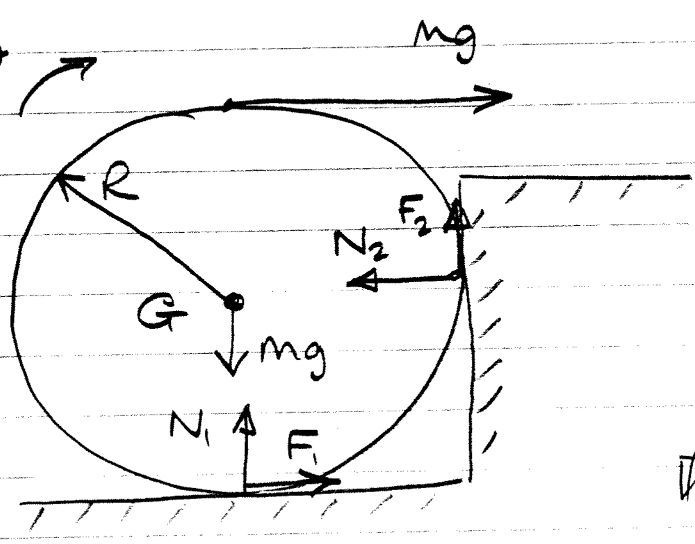
(ii) Equations of motion:

$\xrightarrow{+}$ $$F = m\ddot{x} = \frac{2}{3}mg$$

$\uparrow +$ $$N - mg - mg = m(0) \rightarrow N = 2mg$$

$$\mu \geq \frac{F}{N} = \frac{1}{3}$$

(c)



5 UNKNOWNS
$(F_1, F_2, N_1, N_2, \ddot{\theta})$
5 EQUATIONS

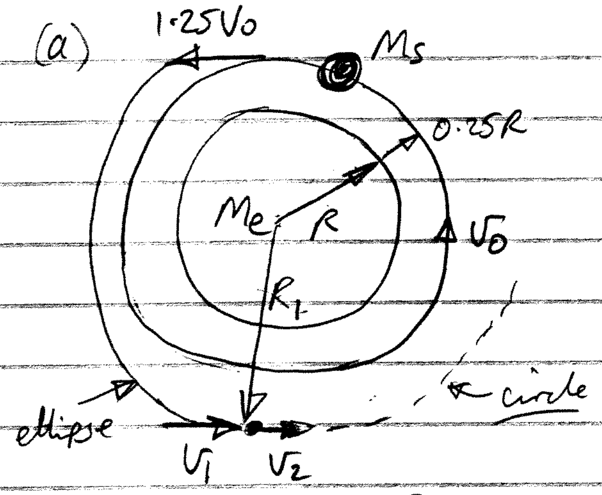
Friction : $\dfrac{F_1}{N_1} = \dfrac{F_2}{N_2} = \mu.$ —— (1)

$\Sigma T = J\ddot{\theta}$: About G $\circlearrowright +$ $\quad mgR - F_1R - F_2R = J_G\ddot{\theta} = \dfrac{mR^2}{2}\ddot{\theta}$ —— (2)

$\Sigma F = ma$: $\xrightarrow{+}$: $\quad F_1 + mg - N_2 = m(0)$ —— (3)

$\Sigma F = ma$: $\uparrow +$: $\quad N_1 + F_2 - mg = m(0)$ —— (4)

8 (a)



Weight on earth:

$$\frac{G M_e \cancel{M_s}}{R^2} = \cancel{M_s} g$$

$$\therefore \frac{G M_e}{R} = gR \qquad (1)$$

Circular orbit: $F = ma$: $\frac{G M_e \cancel{M_s}}{(1.25R)^2} = \frac{\cancel{M_s} V_0^2}{\cancel{1.25R}}$

$(1) \rightarrow$ $\therefore \frac{gR}{1.25} = V_0^2 \Rightarrow V_0 = \sqrt{\frac{gR}{1.25}} \qquad (2)$

(b) Transfer orbit: Energy & moment of momentum about centre of earth are conserved. At max height, R_1, V_1:

Energy: $\frac{1}{2}\cancel{M_s}(1.25V_0)^2 - \frac{G M_e \cancel{M_s}}{1.25R} = \frac{1}{2}\cancel{M_s}V_1^2 - \frac{G M_e \cancel{M_s}}{R_1}$

using (1) & (2) $\Rightarrow 1.25^{2}\frac{gR}{\cancel{1.25}} - \frac{2gR}{1.25} = V_1^2 - \frac{2gR^2}{R_1}$

ie $-0.35\,gR = V_1^2 - \frac{2gR^2}{R_1} \qquad (3)$

M. of. M: $1.25R(1.25V_0)\cancel{M_e} = R_1 V_1 \cancel{M_e} \qquad (4)$

$(1) \rightarrow \therefore V_1 = (1.25)^2 \frac{R}{R_1}\sqrt{\frac{gR}{1.25}} \qquad (5)$

(5) into (3) $\rightarrow$ $-0.35\,gR = (1.25)^{\cancel{4}\,3}\frac{R^2}{R_1^2}\frac{gR}{\cancel{1.25}} - \frac{2gR^2}{R_1}$

$\times \frac{R_1^2}{gR}$ $\therefore -0.35R_1^2 = \frac{(1.25)^3 R^2 \cancel{gR}}{\cancel{gR}} - \frac{2\cancel{g}R^{\cancel{2}}R_1}{\cancel{g}\cancel{R}}$

$\therefore -0.35R_1^2 + 2RR_1 - (1.25)^3R^2 = 0$

8 cont

$$R_1 = \frac{\cancel{+2}R \pm \sqrt{4R^2 - 4(-0.35)(-1.25)^3R^2}}{+\cancel{2}(0.35)}$$

$$= \frac{R}{0.35} \pm \frac{R\sqrt{1 - 0.5469}}{0.35}$$

$$\frac{R}{0.35}\left(1 \pm 0.5625\right) = 1.25R \ \& \ \underline{4.464R}$$

Max height $= \underline{\underline{3.364R}}$

(c) Impulse

At R_1, (5) $\Rightarrow$ $U_1 = (1.25)^2 \frac{\cancel{R}}{4.464\cancel{R}}\sqrt{\frac{gR}{1.25}}$

$$U_1 = 0.35\sqrt{\frac{gR}{1.25}} = \underline{\underline{0.313\sqrt{gR}}}$$

For a circular orbit at this radius, the necessary speed is given by

$$\frac{GM_e \cancel{m_s}}{(4.464R)^{\cancel{2}}} = \frac{\cancel{m_s} V_2^2}{\cancel{4.464R}}$$

ie $V_2 = \sqrt{\frac{gR}{4.464}}$

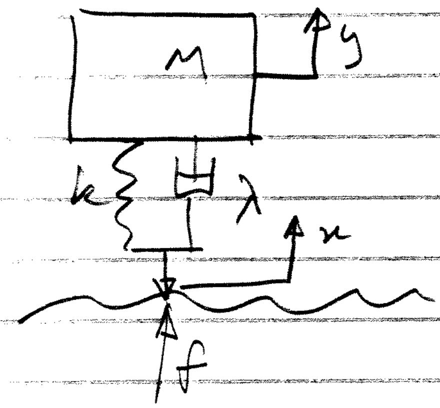
So the necessary impulse applied to the satellite <u>per unit mass</u> is

$$I = 1.(U_2 - U_1) = \sqrt{gR}\left(\frac{1}{\sqrt{4.464}} - \frac{0.35}{\sqrt{1.25}}\right)$$

$$= \underline{\underline{0.16\sqrt{gR}}}$$

tangential to the path in the direction of motion.

9. (a)



$$m\ddot{y} + \lambda\dot{y} + ky = \lambda\dot{x} + kx$$

$$f = m\ddot{y}$$

$$\zeta = \frac{\lambda}{2\sqrt{km}} \qquad \omega_n^2 = k/m$$

$$\frac{\ddot{y}}{\omega_n^2} + \frac{2\zeta}{\omega_n}\dot{y} + y = \frac{2\zeta}{\omega_n}\dot{x} + x$$

(b) Sinusoidal input let $x = He^{i\omega t}$ & $y = Ye^{i\omega t}$

$$\Rightarrow \left(-\frac{\omega^2}{\omega_n^2} + \frac{i2\zeta\omega}{\omega_n} + 1\right)Y = \left(\frac{i2\zeta\omega}{\omega_n} + 1\right)H$$

ie
$$\frac{Y}{H} = \frac{i2\zeta\omega/\omega_n + 1}{1 - \omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}$$

& $F = -\omega^2 mY = -\frac{\omega^2}{\omega_n^2}kY \Rightarrow \frac{F}{kH} = -\frac{\frac{\omega^2}{\omega_n^2}\left(i2\zeta\omega/\omega_n + 1\right)}{1 - \omega^2/\omega_n^2 + i2\zeta\omega/\omega_n}$

(This is case (c) in DB multiplied by ω^2/ω_n^2) with $\omega = \frac{2\pi V}{L}$

$$\frac{|F|}{k|H|} = \frac{\omega^2/\omega_n^2\sqrt{1 + (2\zeta\omega/\omega_n)^2}}{\sqrt{\left(1 - \omega^2/\omega_n^2\right)^2 + \left(2\zeta\omega/\omega_n\right)^2}} \qquad \text{with } \omega = 2\pi V/L$$

$\omega/\omega_n \to 0$, $\frac{|F|}{k|H|} \to 0$ as ω/ω_n^2

$\omega/\omega_n \to \infty$, $\frac{|F|}{k|H|} \to 2\zeta\omega/\omega_n$

9 $\omega_n^2 = k/m = \dfrac{32\times10^3}{800} \Rightarrow \omega_n = 6.32$ rad/s $= 1.0$ Hz

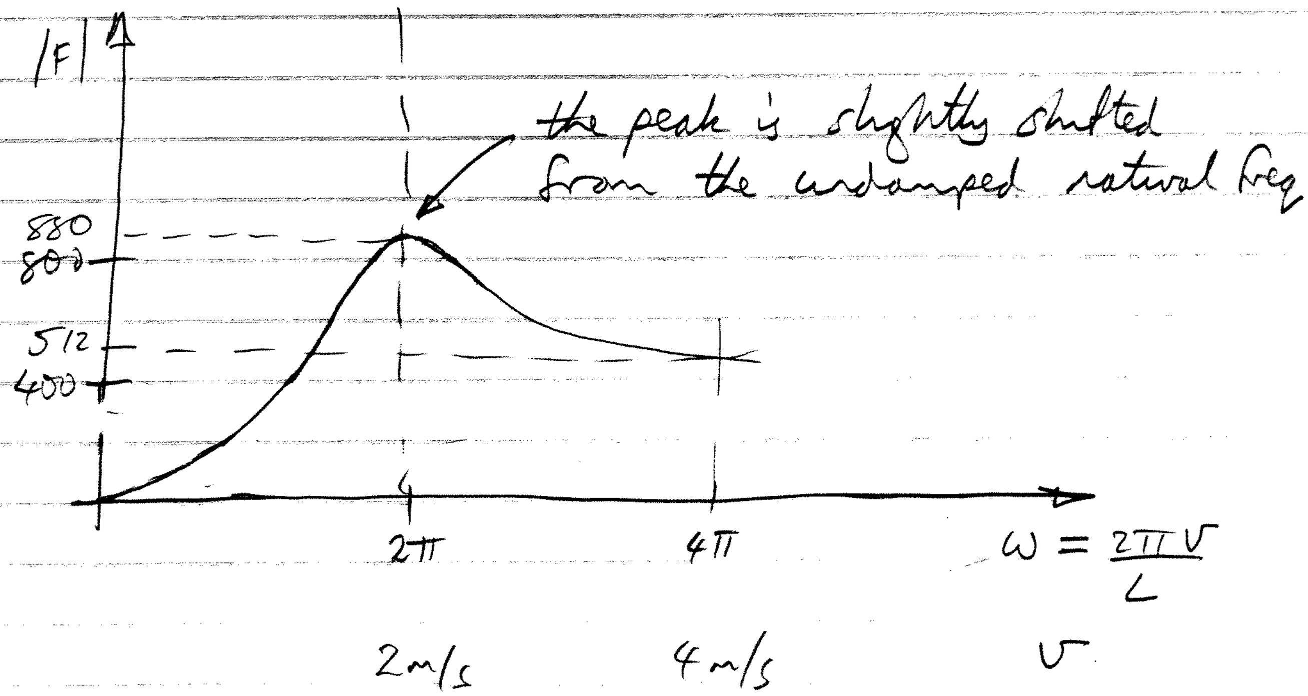
$\zeta = \dfrac{\lambda}{2\sqrt{km}} = \dfrac{2000}{2\sqrt{32\times10^3\times800}} \approx 0.2$

from data book, at peak:

$$|F|_{max} \approx \frac{k|H|}{2\zeta}\left(1+\tfrac{5}{2}\zeta^2\right)\left(\frac{\omega}{\omega_n}\right)^2$$

$$= \frac{(32\times10^3)(0.01)}{2(0.2)}\left(1+\tfrac{5}{2}(0.2)^2\right)$$

$= \underline{\underline{880\,N}}$ (ie about 11% of weight)



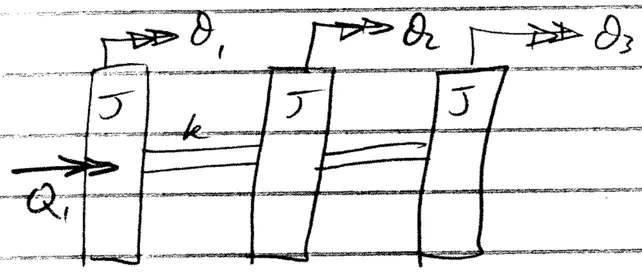
at $\omega/\omega_n = 2$, databook case (c) with $\zeta = 0.2$ gives

$\dfrac{|Y|}{|X|} \approx 0.4$ ∴ $|F| \approx k|H|(2)^2(0.4)$

$= 32\times10^3 \times 0.01\ (4)(0.4) = \underline{512\,N}$

(d) At high freqs, mass becomes "inertial", force is due to deflection of suspension only & $|F/_{ku}| \to 2\zeta\omega/_{\omega_n}$

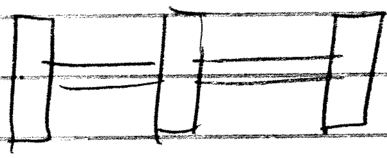
10.



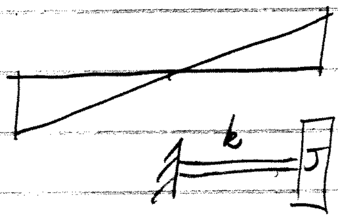
$$J\ddot{\theta}_1 + k(\theta_1-\theta_2) = Q_1$$
$$J\ddot{\theta}_2 + k(\theta_2-\theta_1) + k(\theta_2-\theta_3) = 0$$
$$J\ddot{\theta}_3 + k(\theta_3-\theta_2) = 0$$

$$\begin{bmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J \end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\end{Bmatrix} + \underbrace{k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}}_{[K]}\begin{Bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{Bmatrix} = \begin{Bmatrix}Q_1\\ 0\\ 0\end{Bmatrix} \quad (1)$$

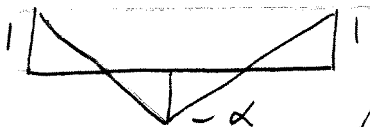
Mode shapes



Rigid body mode $\omega_1 = 0$



Anti symmetric mode
Node on centre disc
$\omega_2 = \sqrt{k/J}$



Symmetric mode $\begin{Bmatrix}1\\ -\alpha\\ 1\end{Bmatrix}$
$\omega_3 = ?$

Eigenvalue problem is $([K] - \omega^2[M])\underset{\sim}{\phi} = \underset{\sim}{0}$

$$\begin{vmatrix} k-\omega^2 J & -k & 0 \\ -k & 2k-\omega^2 J & -k \\ 0 & -k & k-\omega^2 J \end{vmatrix} = 0$$

10 cont

Determinant is

$$(k-\omega^2J)\left((2k-\omega^2J)(k-\omega^2J)-k^2\right)+k(-k)(k-\omega^2J)=0$$

ie $$(k-\omega^2J)\left(\cancel{2k^2}-3k\omega^2J+\omega^4J^2-\cancel{2k^2}\right)=0$$

ie $$J\omega^2(k-\omega^2J)(\omega^2J-3k)=0 \qquad (2)$$

$$\Rightarrow \quad \omega_1^2=0, \quad \omega_2^2=k/J, \quad \omega_3^2=3k/J$$

Harmonic response

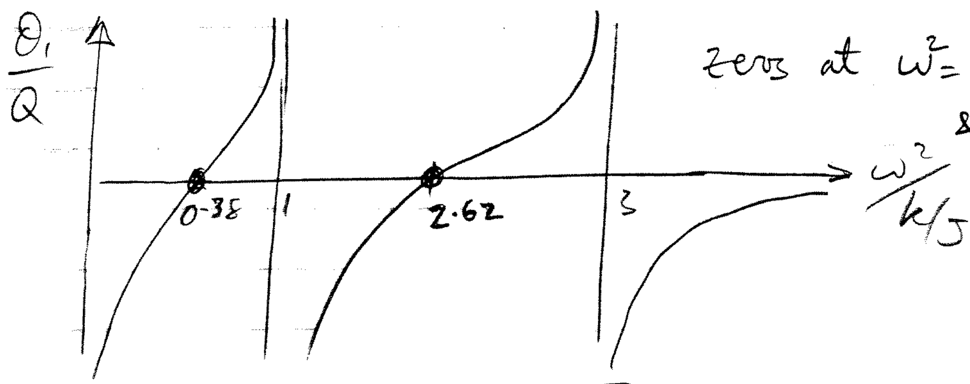
let $\underset{\sim}{Q}=\begin{Bmatrix}Qe^{i\omega t}\\0\\0\end{Bmatrix}$ & $\underset{\sim}{q}=\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix}e^{i\omega t}$

(1) becomes

$$\begin{bmatrix}k-\omega^2J & -k & 0\\ -k & 2k-\omega^2J & -k\\ 0 & -k & k-\omega^2J\end{bmatrix}\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix}\cancel{e^{i\omega t}}=\begin{Bmatrix}Q\\0\\0\end{Bmatrix}\cancel{e^{i\omega t}}$$

$$\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix}=\frac{\begin{bmatrix}\begin{vmatrix}2k-\omega^2J & -k\\ -k & k-\omega^2J\end{vmatrix} & -\begin{vmatrix}-k & -k\\ 0 & k-\omega^2J\end{vmatrix} & 0\\ & etc & \end{bmatrix}}{J\omega^2(k-\omega^2J)(\omega^2J-3k)}\begin{Bmatrix}Q\\0\\0\end{Bmatrix}$$

$$\therefore \frac{\theta_1}{Q}=\frac{(2k-\omega^2J)(k-\omega^2J)-k^2}{J\omega^2(k-\omega^2J)(\omega^2J-3k)}=\frac{(0.38k-\omega^2J)(2.62k-\omega^2J)}{J\omega^2(k-\omega^2J)(\omega J-3k)}$$



zeros at $\omega^2=0.38\,k/J$

& $2.62\,k/J$

\\

ENGINEERING TRIPOS PART IA – 2001

Part B: Mechanics and Vibrations

ANSWERS

6. (b) $\omega_{AB} = \dfrac{\omega R \mathrm{Cos}\,\theta}{L}$; $v_{\text{sliding}} = \omega R \sin\theta$

(c) $F_{(i)} = \dfrac{T}{R\sin\theta}$; $F_{(ii)} = \dfrac{T+Q}{R\sin\theta}$

7. (a) $J_X = \dfrac{3}{2}mR^2$

(b)(ii) $\mu \geq \dfrac{1}{3}$

8. (b) Height $= 3.364R$

(c) $I = 0.16\sqrt{gR}$ tangential to the path, in the direction of motion.

9. (a) $\dfrac{\ddot{y}}{\omega_n^2} + \dfrac{2\zeta}{\omega_n}\dot{y} + y = \dfrac{2\zeta}{\omega_n}\dot{x} + x$, with $\zeta = \dfrac{\lambda}{2\sqrt{km}}$, $\omega_n^2 = \dfrac{k}{m}$

(b) $$\left|\frac{F}{kH}\right| = \frac{\dfrac{\omega^2}{\omega_n^2}\sqrt{1+\left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}{\sqrt{\left(1-\dfrac{\omega^2}{\omega_n^2}\right)^2+\left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}, \quad \text{with} \quad \omega = \frac{2\pi v}{L}$$

(c) (i) $|F_{\max}| \approx 880\text{N}$, at $v \approx 2\text{m/s}$; (ii) 512N

(d) At high frequencies mass becomes inertial, and $\left|\dfrac{F}{kH}\right| \to 2\zeta\dfrac{\omega}{\omega_n}$

10. (c) 0, $\sqrt{\dfrac{k}{J}}$, $\sqrt{3\dfrac{k}{J}}$

(d) $\sqrt{0.38\dfrac{k}{J}}$, $\sqrt{2.62\dfrac{k}{J}}$