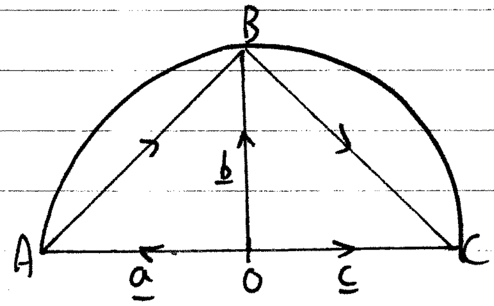
Part IA Mathematical Methods (Paper 4) Solutions

Section A

Q1.

(a) Vector geometry (Euclid)



$$\overrightarrow{AB} = (\underline{b} - \underline{a})$$
$$\overrightarrow{BC} = (\underline{c} - \underline{b}) = (-\underline{a} - \underline{b}) \quad \text{since } \underline{a} = -\underline{c}$$

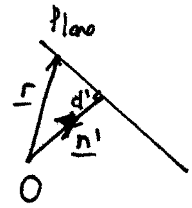
$$\therefore \overrightarrow{AB} \cdot \overrightarrow{BC} = (\underline{b} - \underline{a}) \cdot (-\underline{a} - \underline{b}) = -\underline{a}.\underline{b} + \underline{a}.\underline{b} + |a|^2 - |b|^2$$
$$= 0 \quad \text{since } |a| = |b| = |c| \text{ since on circle}$$

(6) $\therefore$ Angle subtended at B is $90°$. (QED)

(b) Vector equations of lines and planes.

(i) $ax + by + cz = d$ can be rewritten as $\underline{r} . \underline{n}' = d'$

where $\underline{n}' = \dfrac{(a, b, c)}{\sqrt{a^2+b^2+c^2}}$

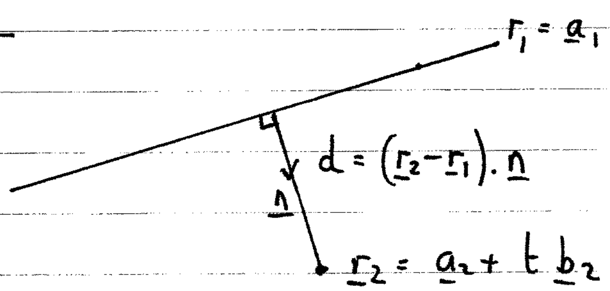


(6) (ii) distance to origin, $d' = \dfrac{d}{\sqrt{a^2+b^2+c^2}}$

(iii) Closest point is at $\dfrac{d\,(a, b, c)}{a^2+b^2+c^2}$.

Q1(c). lines

(B)



Perpendicular distance is $\underline{d = \left| (\underline{a}_2 - \underline{a}_1) \cdot \underline{n} \right|}$

where $\underline{n} = \dfrac{\underline{b}_1 \times \underline{b}_2}{|\underline{b}_1 \times \underline{b}_2|}$ is the perpendicular to both lines

Q2 (a) Series, approximations and limits

(i) L'hôpital's rule:

$$\lim_{x\to a} \frac{f(a)}{g(a)} = \frac{f'(a)}{g'(a)} \quad \text{if } f(a) = g(a) = 0$$

(4)

$$\lim_{x\to\frac{1}{3}} \left(\frac{1 - 2\cos(\pi x)}{1-3x} \right) = \lim_{x\to\frac{1}{3}} \left(\frac{2\pi \sin \pi x}{-3} \right)$$

$$= \frac{2\pi}{-3} \sin\frac{\pi}{3} = \frac{-\pi}{\sqrt{3}}$$

(ii) Using power series expansion for $\sin x = x - \frac{x^3}{3!} + O(x^5)$

(4)

$$\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right) = \lim_{x\to 0} \left[\frac{1}{\left(x - \frac{x^3}{3!} + O(x^5)\right)^2} - \frac{1}{x^2} \right]$$

$$= \lim_{x\to 0} \frac{1}{x^2} \left[\frac{1}{\left(1 - \frac{2x^2}{3!} + O(x^4)\right)} - 1 \right]$$

$$= \lim_{x\to 0} \frac{1}{x^2} \left[\frac{1 - \left(1 - \frac{2x^2}{3!} + O(x^4)\right)}{1 - \frac{2x^2}{3!} + O(x^4)} \right]$$

$$= \frac{2}{3!}$$

$$= \frac{1}{3}$$

(6) Complex numbers / hyperbolic fns

(i) $\cosh z = \cosh(x+iy)$

$= \cosh x \cos y + i \sinh x \sin y$ since $\begin{cases} \cosh iy = \cos y \\ \sinh iy = i \sin y \end{cases}$

If $\cosh z = i$;

$\therefore \cosh x \cos y = 0 \Rightarrow \cos y = 0 \quad y = (2n+1)\frac{\pi}{2} \quad n = 0, \ldots$

$\sinh x \sin y = 1 \Rightarrow (-1)^n \sinh x = 1$

$\sinh x = (-1)^n$

$\underline{x = (-1)^n \sinh^{-1}(+1)}$

$\left(x = \pm \sinh^{-1}(1) = \underline{\pm 0.88} \right)$

(6) $\boxed{\therefore \begin{array}{l} x = (-1)^n \sinh^{-1}(1) \\ y = (2n+1)\frac{\pi}{2} \end{array} \Bigg\} \; n = 0, \pm 1, \pm 2 \ldots}$

(ii) $z^6 - 2z^3 + 2 = 0$ (6 roots.)

$z^3 = 1 \pm \sqrt{(1-2)} = \sqrt{2}\, e^{\pm \frac{i\pi}{4}} = 2^{\frac{1}{2}} e^{\pm i(\frac{\pi}{4} + 2n\pi)}$

$z = 2^{\frac{1}{6}} \left\{ e^{\pm \frac{i(\frac{\pi}{4} + 2n\pi)}{3}} \right\}$

$= \underline{2^{\frac{1}{6}} \left\{ e^{\frac{i\pi}{12}}, e^{i\frac{9\pi}{12}}, e^{i\frac{17\pi}{12}} \right\}}$ 6 roots

(6) and $\underline{2^{\frac{1}{6}} \left\{ e^{-\frac{i\pi}{12}}, e^{\frac{7\pi i}{12}}, e^{\frac{15\pi i}{12}} \right\}}$

Q3 Differential equations

(a) $4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 8\cos^2\left(\frac{t}{4}\right) \qquad x(0) = \left.\frac{dx}{dt}\right|_0 = 0$

(i) Complementary function $\quad x = e^{\lambda t}$

$$\therefore\ 4\lambda^2 + 4\lambda + 1 = 0$$
$$(2\lambda+1)^2 = 0$$
$$\lambda = -\tfrac{1}{2} \quad \text{(repeated root)}$$

$$\underline{x_{cF} = (At+B)\,e^{-\frac{t}{2}}}$$

(ii) Particular integral: $\quad \underline{RHS = 4\left(\cos\frac{t}{2} + 1\right)}$

Try:

$$x_{pI} = 4 + C\sin\tfrac{t}{2} + D\cos\tfrac{t}{2}$$

by substitution:

$$-C\sin\left(\tfrac{t}{2}\right) - D\cos\left(\tfrac{t}{2}\right) + 2C\cos\left(\tfrac{t}{2}\right) - 2D\sin\left(\tfrac{t}{2}\right) + C\sin\left(\tfrac{t}{2}\right) + D\cos\left(\tfrac{t}{2}\right) = 4\cos\left(\tfrac{t}{2}\right)$$

$$\therefore\ -C - 2D + C = 0 \qquad D = 0$$
$$-D + 2C + D = 4 \qquad C = 2$$

$$\underline{x_{pI} = 4 + 2\sin\left(\tfrac{t}{2}\right)}$$

(iii) General solution:

$$x(t) = (At+B)\,e^{-\frac{t}{2}} + 2\sin\left(\tfrac{t}{2}\right) + 4$$

(12)

$x = 0,\ t = 0 \ \rightarrow\ B = -4$

$\frac{dx}{dt} = 0,\ t = 0 \ \rightarrow\ A = -3$

$$\underline{\underline{x(t) = (-3t-4)\,e^{-\frac{t}{2}} + 2\sin\tfrac{t}{2} + 4}}$$

3(b) Linear difference equation.

$$4y_n + 4y_{n-1} + y_{n-2} = 0$$

$$y_n = \lambda^n \quad \therefore \; 4\lambda^2 + 4\lambda + 1 = 0$$

$$\lambda = -\tfrac{1}{2} \text{ repeated}$$

$$\therefore \; \underline{y_n = (An + B)\left(-\tfrac{1}{2}\right)^n}$$

$y_0 = 1 \quad n = 0 \qquad B = 1$

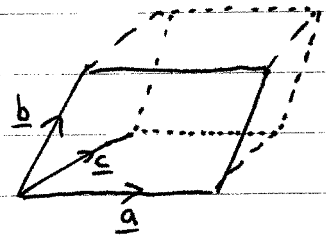
$y_1 = -1 \quad n = 1 \qquad (A+B)\left(-\tfrac{1}{2}\right) = -1 \qquad A = 1$

$$\therefore \; \underline{y_n = (n+1)\left(-\tfrac{1}{2}\right)^n}$$

(8) Check $y_2 = (3)\left(-\tfrac{1}{2}\right)^2 = 0.75$. QED.

Q4 (a) Determinants

(i) $\underline{a} \cdot (\underline{b} \times \underline{c})$ is volume of parallelpiped with sides $\underline{a}, \underline{b}, \underline{c}$.



(3)

(ii) If $\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar $\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$

$$\Rightarrow \underline{a}^T B \underline{c} = 0 \quad \text{where } B\underline{c} \equiv \underline{b} \times \underline{c}$$

$$\therefore B = \begin{bmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{bmatrix}$$

and similarly for A and C since:

(5) $$\underline{a} \cdot (\underline{b} \times \underline{c}) = \underline{c} \cdot (\underline{a} \times \underline{b}) = \underline{b} \cdot (\underline{c} \times \underline{a}) = 0$$

(b)
$$\begin{aligned} 2x + 3y + z &= 0 \\ x \quad + 2z &= t \\ sx + 4y \quad &= 2 \end{aligned}$$

For no unique solution $\begin{vmatrix} 2 & 3 & 1 \\ 1 & 0 & 2 \\ s & 4 & 0 \end{vmatrix} = 0$

(4) $2(-8) - 3(-2s) + 1(4) = 0 \Rightarrow s = 2.$

Q4(b)(cont).

For planes to intersect in a line

$$2x + 3y + z = 2$$
$$x + 2z = t$$
$$2x + 4y = 2$$

$$\left.\begin{aligned} 3x + 6y &= 4 - t \\ 2x + 4y &= 2. \end{aligned}\right\} \text{must be same equation.}$$

(4) $\therefore \dfrac{4-t}{3} = 1 \Rightarrow t = 1.$

(4) (c) $R\underline{L} = \underline{L}$ $\therefore \underline{L}$ is eigenvector with eigenvalue $\lambda = 1$.

Q5 Matrices, eigenvectors and eigenvalues

(a) $A\underline{u}_i = \lambda_i \underline{u}_i$ and $A\underline{u}_j = \lambda_j \underline{u}_j$ $\quad \underline{u}_i \neq \underline{u}_j \quad \lambda_i \neq \lambda_j$

Now consider $u_j^T(Au_i) = u_j^T A u_i = \lambda_i u_j^T u_i$

$$u_i^T(Au_j) = u_i^T A u_j = u_j^T A^T u_i = \lambda_j u_i^T u_j$$

Since $A = A^T$ and $u_i^T u_j = u_j^T u_i$

$$(\lambda_i - \lambda_j)\ \underline{u}_j . \underline{u}_i = 0$$

(6) hence $\underline{u}_i . \underline{u}_j = 0$ if $i \neq j$ and eigenvectors are perpendicular.

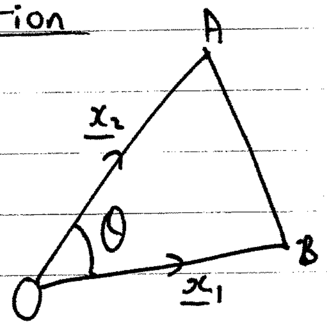
(b) $A\underline{x} = \alpha\lambda_1\underline{u}_1 + \beta\lambda_2\underline{u}_2 + \gamma\lambda_3\underline{u}_3$

(2) (i) $A^{-1}A\underline{x} = \underline{x}$ $\quad \therefore\ A^{-1}\underline{x} = \dfrac{\alpha\underline{u}_1}{\lambda_1} + \dfrac{\beta\underline{u}_2}{\lambda_2} + \dfrac{\gamma\underline{u}_3}{\lambda_3}$

(2) (ii) $A^N\underline{x} = \alpha\lambda_1^N\underline{u}_1 + \beta\lambda_2^N\underline{u}_2 + \gamma\lambda_3^N\underline{u}_3$

(2) (iii) $(A^{-1})^N = \dfrac{\alpha\underline{u}_1}{\lambda_1^n} + \dfrac{\beta\underline{u}_2}{\lambda_2^n} + \dfrac{\gamma\underline{u}_3}{\lambda_3^N}$

Q5(c) Rotation



$$\underline{y}_i = Q\underline{x}_i$$

Check effect on lengths and angles:

$$\underline{y}_i \cdot \underline{y}_i = |\underline{y}_i|^2 = (Q\underline{x}_i)^T Q\underline{x}_i$$
$$= \underline{x}_i^T Q^T Q \underline{x}_i$$
$$= \underline{x}_i^T \underline{x}_i$$
$$= |\underline{x}_i|^2$$

$\therefore$ lengths are unchanged

Check angles θ

$$\frac{\underline{y}_2 \cdot \underline{y}_1}{|\underline{y}_2||\underline{y}_1|} = \cos\theta' = \frac{\underline{x}_2^T Q^T Q \underline{x}_1}{|Q\underline{x}_1|\,|Q\underline{x}_2|} = \frac{\underline{x}_2 \cdot \underline{x}_1}{|\underline{x}_1||\underline{x}_2|} = \cos\theta$$

$\therefore$ angles are unchanged

(8) The area of triangle is given by $\frac{1}{2}|\underline{a}\times\underline{b}| = \frac{1}{2}|\underline{a}||\underline{b}|\sin\theta$

$\therefore$ area is unchanged.

14/2/01

Solutions to IA Maths Exam 2001

Section B

Q 6

a) If the impulse response of a linear system is $g(t)$, then the output to any input $f(t)$ is

$$y(t) = \int_{-\infty}^{t} g(t-\tau) f(\tau)\, d\tau .$$

Results follow from approximating $f(t)$ by rectangular slices at sampled times, approximating each slice by a delta function, summing the relevant impulse responses, then taking the limit of sample interval $\to 0$

b) Step response $= \int_0^t g(\tau)\, d\tau \qquad t > 0.$

direct from the expression in (a).

c) Impulse response $g(t) = H(t) - H(t-a)$

Input $x(t) = A e^{-bt} H(t)$.

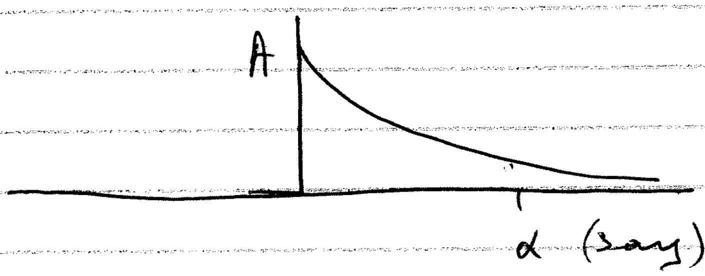
Therefore, the output is:

1) $t < a$, $y(t) = \int_0^t A e^{-bT}\, dT = \frac{A}{b}\left(1 - e^{-tb}\right)$

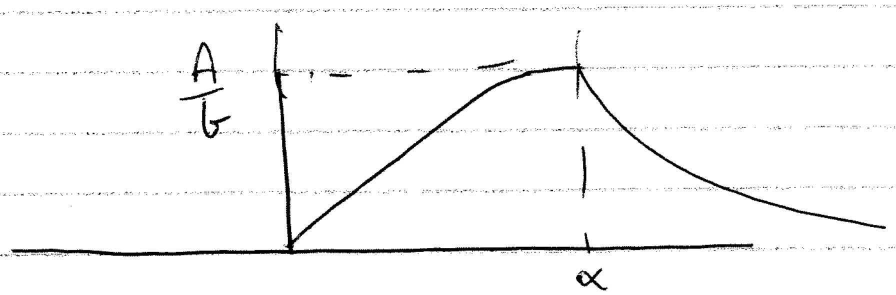
2) $t > a$

$$y(t) = \int_{t-a}^{t} A e^{-bT}\, dT = \frac{A}{b} e^{-bt}\left(e^{ab} - 1\right)$$

d) Input



Output



Q7 a) $f(x) = x^2 \quad -2 \leq x \leq 2$ is even

$\therefore$ cosine series

$$f(x) = \frac{a_0}{2} + \sum a_n \cos \frac{2n\pi x}{4}$$

$$= \frac{a_0}{2} + \sum a_n \cos \frac{n\pi x}{2}$$

$$a_0 = \frac{1}{2}\int_{-2}^{2} x^2\,dx = \frac{8}{3}$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx = \frac{1}{2}\int_{-2}^{2} x^2 \cos\frac{n\pi x}{2}\,dx$$

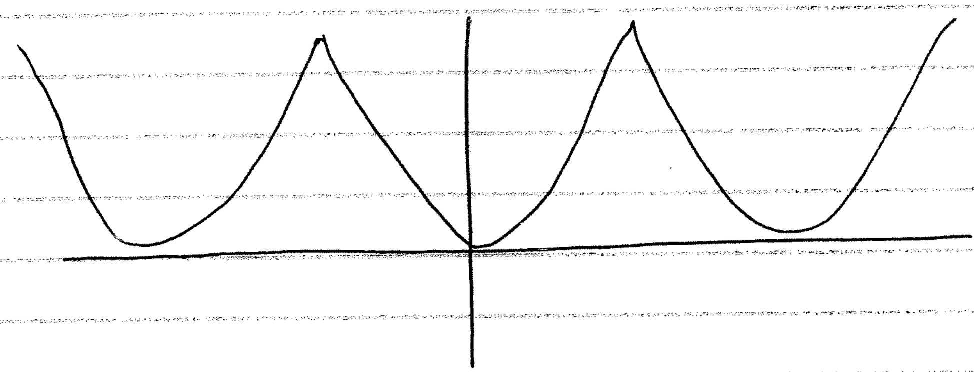
$$= \frac{1}{2}\left[x^2 \frac{\sin n\pi x/2}{n\pi/2}\right]_{-2}^{2} - \int_{-2}^{2} 2x \frac{2}{n\pi} \sin\frac{n\pi x}{2}\,dx.$$

$$= -\frac{2}{n\pi}\int_{-2}^{2} x \sin\frac{n\pi x}{2}\,dx$$

$$= -\frac{2}{n\pi}\left\{\left[x\left(\frac{-2}{n\pi}\right)\cos\frac{n\pi x}{2}\right]_{-2}^{2} + \frac{2}{n\pi}\int_{-2}^{2}\cos\frac{n\pi x}{2}\,dx\right\}.$$

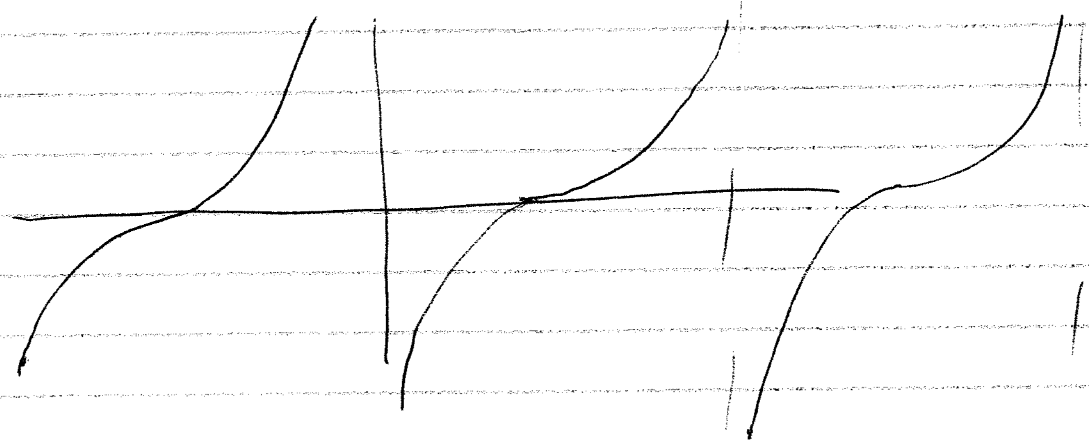
$$= \left(\frac{2}{n\pi}\right)^2 4\cos n\pi + 0 = \underline{\underline{\frac{16(-1)^n}{n^2\pi^2}}}$$

b)



Extended function has jumps in f'
so convergence goes as $\frac{1}{n^2}$ as found.

If extended as an odd $f^=$, f has jumps and we have $1/n$ convergence.



c) $$f(x) = x$$

If $f(x) = \sum c_n e^{in\pi x/2}$

$$\therefore c_n = \frac{1}{4}\int_{-2}^{2} f(x)\, e^{-in\pi x/2}\, dx$$

e cont)

$$\therefore c_n = \frac{1}{4}\int_{-2}^{2} x\, e^{-in\pi x/2}\, dx$$

$$= \frac{1}{4}\left\{\left[x\left(\frac{-2}{in\pi}\right)\right]_{-2}^{2} + \frac{2}{in\pi}\int_{-2}^{2} e^{-in\pi x/2}\, dx\right\}.$$

$$= -\frac{1}{in\pi}\left(e^{-in\pi} + e^{in\pi}\right) + \frac{1}{2in\pi}\left(\frac{-2}{in\pi}\right)\left[e^{-in\pi x/2}\right]_{-2}^{2}$$

$$= \frac{-2}{in\pi}(-1)^n + 0$$

$$\therefore c_n = \frac{2i}{n\pi}(-1)^n .$$

Q 8

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\therefore \frac{dF(s)}{ds} = \int_0^\infty -t\,e^{-st} f(t)\,dt$$

$$= L\{-t\,f(t)\}$$

b)

$$t\frac{d^2 f(t)}{dt^2} + \frac{df(t)}{dt} + t\,f(t) = 0$$

$$f(0) = 1\ , \quad \frac{df(0)}{dt} = 0 .$$

Taking the Laplace transform of the above equation

$$L\left(tf'' + f' + tf\right) = 0$$

$$\therefore\ -\frac{d}{ds}L(f'') + L(f') - \frac{d}{ds}L(f) = 0$$

using the result from part (a).

$$L(f') = s\,F(s) - 1$$

$$L(f'') = s^2 F(s) - s\,f(0) - f'(0)$$

$$= s^2 F(s) - s .$$

$\therefore$
$$-\frac{d}{ds}\left(s^2F - s\right) + sF - 1 - \frac{dF}{ds} = 0.$$

$$\therefore (s^2+1)\frac{dF}{ds} + sF = 0.$$

$$\therefore \frac{dF}{F} = -\frac{s\,ds}{s^2+1}$$

$$\therefore \ln F = -\frac{1}{2}\ln(s^2+1) + c.$$

$$\therefore F = C\,(s^2+1)^{-1/2}.$$

$$\therefore F = \frac{C}{s}\left(1 - \frac{1}{2s^2} - \frac{\frac{1}{2}(-\frac{3}{2})}{2!}\frac{1}{s^4} + \cdots\right).$$

$$= C\left(\frac{1}{s} - \frac{1}{2}\frac{1}{s^3} + \frac{1.3}{2^2\,2!}\frac{1}{s^5} - \frac{1.3.5}{2^3\,3!}\frac{1}{s^7}\cdots\right)$$

taking inverse Laplace transform,

$$f(t) = C\sum_{n=0}^{\infty}\frac{(-1)^n\,t^{2n}}{(2^n\,n)^2}$$

Question 9.

$$f(x,y) = 2x^3 + 6xy^2 - 3y^3 - 150x.$$

$$\frac{\partial f}{\partial x} = 6x^2 + 6y^2 - 150$$

$$\frac{\partial f}{\partial y} = 12xy - 9y^2$$

For stationary points, these partial derivatives are zero and

$$\therefore \quad x^2 + y^2 = 25$$

$$y(4x - 3y) = 0$$

From the second equation, either

$$y = 0 \qquad \text{or} \qquad 4x = 3y.$$

Putting $y=0$ into the first eq$^{=}$ gives

$$x = \pm 5$$

$\therefore$ the points $(5,0)$ and $(-5,0)$ are solutions

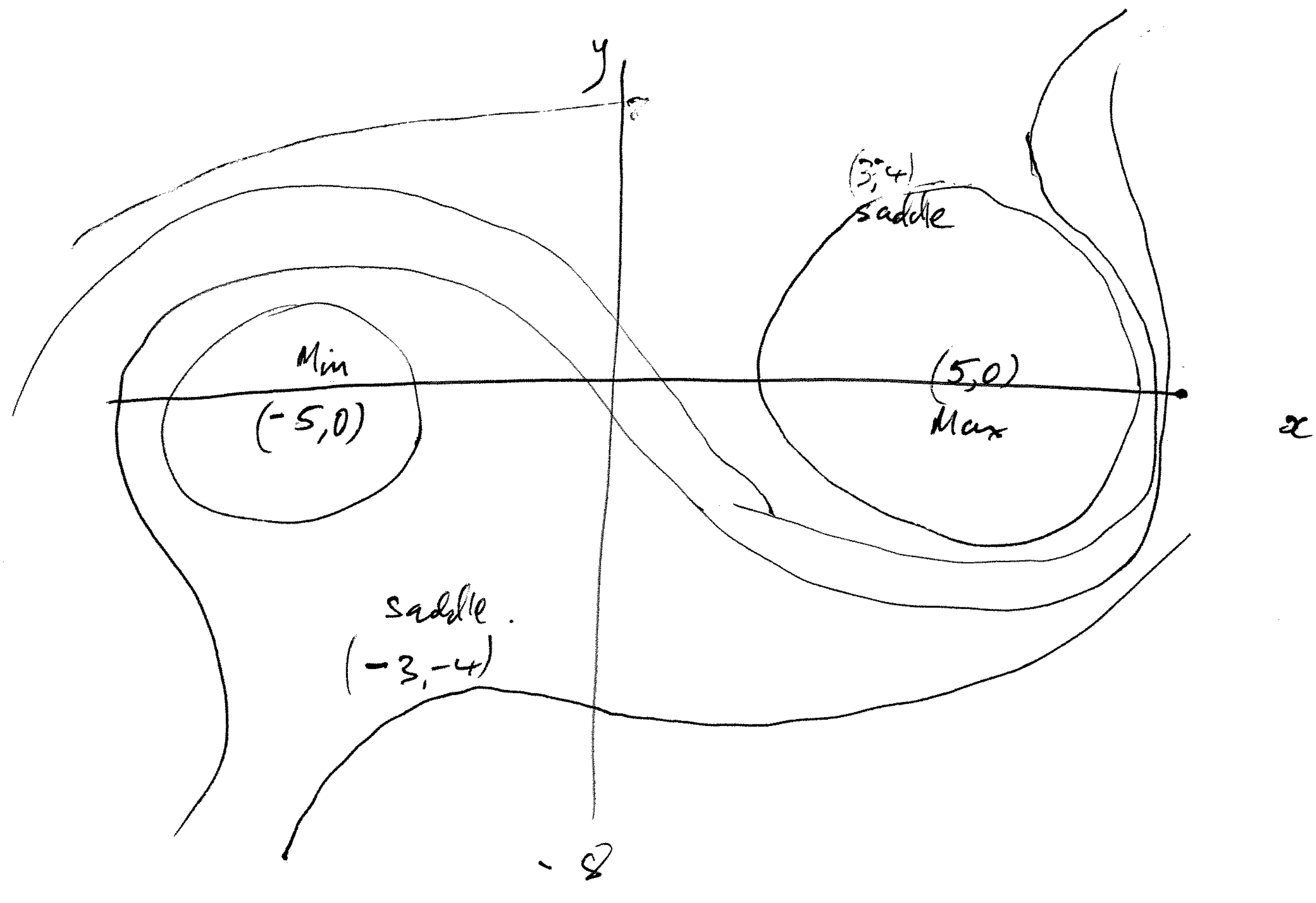
Putting $x = 3/4\, y$ into the first eq$^{=}$ gives $y = \pm 4$ $\therefore$ $(3,4)$ and $(-3,-4)$ are solutions

9 cont.

We need to classify these points by looking at

$$\frac{\partial^2 f}{\partial x^2} = 12x, \quad \frac{\partial^2 f}{\partial y^2} = 12x - 8y, \quad \frac{\partial^2 f}{\partial x \partial y} = 12y.$$

	$\partial^2 f/\partial x^2$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x \partial y}$	$\left[\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right]$	nature	value
5,0	60	60	0	+	min	-500
-5,0	-60	-60	0	+	Max	500
3,4	36	-36	48	−	saddle pt.	-300
-3,-4	-36	36	-48	−	saddle pt.	300.



Q. 10

a) Book work

b) 1^{st} person has any birthday they like, out of d

2^{nd} person is different with probability

$$\left(\frac{d-1}{d}\right)$$

3^{rd} person is different from both

with probability $\left(\frac{d-2}{d}\right)$

So prob that n people are all different is

$$P = \left(\frac{d-1}{d}\right)\left(\frac{d-2}{d}\right)\cdots\left(\frac{d-n+1}{d}\right)$$

$$= \frac{(d-1)!}{(d-n)!\,d^{n-1}} = \frac{d!}{(d-n)!\,d^{n}}$$

But $\binom{d}{n} = \dfrac{d!}{n!(d-n)!}$, $\therefore P = \binom{d}{n}\dfrac{n!}{d^{n}}$

$d=20$, $n=3$, $\therefore P = \cdot 855$

$\therefore 1-P = \underline{\underline{\cdot 0145}}$.