ENGINEERING TRIPOS PART IA

Tuesday 12 June 2001 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than ***eight*** *questions, of which not more than* ***four*** *may be taken from section A and not more than* ***four*** *may be taken from section B.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

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SECTION A

Answer not more than ***four*** *questions from this section.*

1 (a) Consider a triangle ABC inscribed in a semi-circle with centre O, as shown in Fig. 1. Prove, by evaluating the scalar product $\overrightarrow{\mathrm{AB}} \cdot \overrightarrow{\mathrm{BC}}$, that the angle subtended at vertex B is always a right angle. [6]

(b) Consider the equation of a plane $ax + by + cz = d$. Find the point on the plane that is closest to the origin. [6]

(c) Determine the shortest distance between two lines $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ and $\mathbf{r} = \mathbf{c} + \mu\mathbf{d}$. [8]

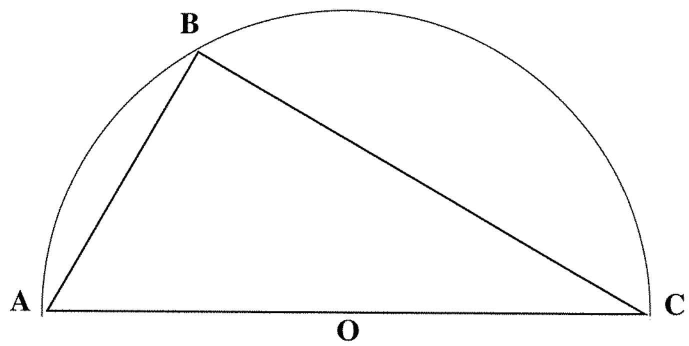

Fig. 1

2 (a) Using l'Hopital's rule, find:

$$\lim_{x \to 1/3} \left(\frac{1 - 2\cos(\pi x)}{1 - 3x} \right)$$

[4]

(b) Using a power series approximation, find:

$$\lim_{x \to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2} \right)$$

[4]

(c) Find all solutions of:

(i) $\cosh z = i$ [6]

(ii) $z^6 - 2z^3 + 2 = 0.$ [6]

(TURN OVER

3 (a) Find the solution of the differential equation:

$$4\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + x = 8\cos^2\left(\frac{t}{4}\right)$$

which satisfies the boundary conditions:

$$x = \frac{dx}{dt} = 0 \quad \text{when} \quad t = 0$$

[12]

(b) Find the solution of the linear difference equation:

$$4y_n + 4y_{n-1} + y_{n-2} = 0$$

given that $y_0 = 1$ and $y_1 = -1$. Verify that $y_2 = 0.75$. [8]

4 (a) Consider the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$

(i) Give a geometrical interpretation. [3]

(ii) For the special case in which the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are coplanar show that:

$$\mathbf{a}^t\mathbf{B}\mathbf{c} = \mathbf{c}^t\mathbf{A}\mathbf{b} = \mathbf{b}^t\mathbf{C}\mathbf{a} = 0$$

and identify the elements of the 3×3 matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$. [5]

(b) Find the value of s for which the following equations do not have a unique solution:

$$\begin{aligned} 2x + 3y + z &= 2 \\ x + 2z &= t \\ sx + 4y &= 2 \end{aligned}$$

For what value of t will the planes represented by these equations intersect in a line? [9]

(c) A rotation matrix $\mathbf{R}$ represents a rotation about an axis, $\mathbf{l}$, by an angle θ. Find one of the eigenvectors and the corresponding eigenvalue. [3]

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5 (a) A symmetric matrix $\mathbf{A}$ has distinct, real eigenvalues λ_1, λ_2 and λ_3. Show that the corresponding eigenvectors, $\mathbf{u}_1$, $\mathbf{u}_2$ and $\mathbf{u}_3$, are orthogonal. [6]

(b) Consider a general vector:

$$\mathbf{x} = \alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3$$

where $\mathbf{u}_i$ are the eigenvectors of $\mathbf{A}$. If all the eigenvalues of $\mathbf{A}$ are non-zero, evaluate:

(i) $\mathbf{A}^{-1}\mathbf{x}$ [2]

(ii) $\mathbf{A}^{N}\mathbf{x}$ where N is a positive integer [2]

(iii) $(\mathbf{A}^{-1})^{N}\mathbf{x}$ [2]

(c) A position vector is transformed by an orthogonal matrix $\mathbf{Q}$ such that:

$$\mathbf{y} = \mathbf{Q}\mathbf{x}$$

By first showing how the lengths and angles are changed by the transformation, show that the area of any triangle is unaffected. [8]

SECTION B

*Answer not more than **four** questions from this section.*

6 (a) Explain the role of *convolution* in Linear Systems. [4]

(b) A linear system has an impulse response $g(t)$. Show that the unit step response of such a system is given by:

$$y(t) = \int_0^t g(\tau)d\tau$$

[4]

(c) A finite duration integrator can be modelled by the impulse response given by:

$$g(t) = H(t) - H(t-a)$$

where $H(t)$ corresponds to a step function.

(i) If the input to this system is given by:

$$x(t) = A\exp(-bt)H(t)$$

find an expression for the output of this system. [8]

(ii) Demonstrate graphically that the result you obtain makes physical sense. [4]

(TURN OVER

7 (a) Find the Fourier series of:

$$f(x) = x^2$$

for $0 < x \leq 2$, by extending the function as an even function of x and making it periodic with period 4. [10]

(b) What would be the implications, concerning convergence, if you were to extend the function as an odd function of x? [3]

(c) Show that the complex Fourier series of:

$$f(x) = x$$

for $-2 < x \leq 2$ is given by:

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{2i(-1)^n}{n\pi} \exp(\frac{\pi i n x}{2})$$

[7]

This expression is correct for all values of n apart from n=0, in which case the Fourier coefficient is zero

8 (a) If $F(s)$ is the Laplace transform of $f(t)$, show that the first derivative of $F(s)$ with respect to s is given by:

$$\frac{dF(s)}{ds} = L[-tf(t)]$$

where $L[.]$ stands for the usual Laplace transform. [2]

(b) Show that if:

$$t\frac{d^2f(t)}{dt^2} + \frac{df(t)}{dt} + tf(t) = 0$$

with $f = 1$ and $\frac{df}{dt} = 0$ at $t = 0$, then:

$$(s^2+1)\frac{dF(s)}{ds} + sF(s) = 0$$

[6]

(c) Hence show that:

$$F(s) = \frac{C}{\sqrt{s^2+1}}$$

where C is a constant. [4]

(d) Show that $F(s)$ may be written:

$$F(s) = \frac{C}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}$$

and by expanding the last expression as a power series in $1/s$, show that:

$$f(t) = C\sum_{n=0}^{\infty}\frac{(-1)^n t^{2n}}{(2^n n!)^2}$$

[8]

(TURN OVER

9 (a) Find the positions and nature (i.e. maxima, minima or saddle points) of the stationary values of the function:

$$f(x, y) = 2x^3 + 6xy^2 - 3y^3 - 150x$$

[12]

(b) Sketch contour lines for this function. [8]

10 (a) Explain, briefly, the use of tree diagrams in the calculation of probabilities. [4]

(b) Assuming that the birth rate remains the same throughout the year and that people are born independently of each other, show that the probability that n people chosen at random will all have different birthdays is given by:

Note that if n is greater than d then the probability must, of course, be zero

$$p_n = \frac{d!}{(d-n)!d^n}$$

where d is the number of days in a year. [12]

(c) On the planet *Zog*, there are twenty days in a year. Show that in a group of 3 *Zogians* the probability that at least two have the same birthday is 0.145. [4]

END OF PAPER