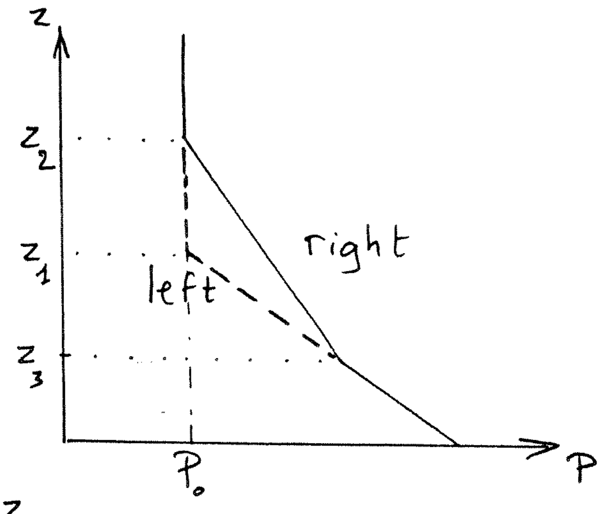
Crib for Engineering Tripos Part IA Paper 1 – Mechanical Engineering 2002

Section A: Thermofluid

Section A

1. a) Pressure distribution (sketch)



b) Levels z_1, z_2, z_3

Volume of oil : $L_2 \underset{\uparrow \text{depth}}{d} (z_2 - z_3) = 0.5\ m^3 \longrightarrow z_2 - z_3 = 0.5$

Volume of water : $L_2 d z_3 + L_1 d z_1 = 0.5\ m^3 \longrightarrow 0.5 z_1 + z_3 = 0.5$

Pressure at z_3 : hydrostatics on water side = hydrostatics on oil side

$$\hookrightarrow \cancel{P_0} + \rho_w \cancel{g}(z_1 - z_3) = \cancel{P_0} + \rho_o \cancel{g}(z_2 - z_3)$$

eliminating z_1, z_2 using the volume $\qquad 1 - 3z_3 = \frac{\rho_o}{\rho_w} 0.5$

$$\hookrightarrow z_3 = \frac{1}{3} - \frac{0.5}{3}\frac{\rho_o}{\rho_w} = \frac{1}{3} - \frac{0.5}{3}\frac{700}{1000} \simeq 0.217$$

hence $z_2 = 0.5 + z_3 \simeq 0.717$ and $z_1 = 1.2 z_3 \simeq 0.567$

c) Force on partition wall (see sketch in a))

F horizontal from right to left $= \left[\frac{1}{2}\rho_o g (z_2 - z_3)^2 - \frac{1}{2}\rho_w g (z_1 - z_3)^2\right] d$

$$\overleftarrow{F} \simeq \frac{700\ 9.81}{2} 0.5^2 - \frac{1000\ 9.81}{2} 0.35^2$$

d = depth = 1 m

$\simeq 258$ N

Comments: Well-answered question taken by 89 % of the candidates. Among possible errors, some students suggested that there should be the same mass of fluid on both sides. Others attempted to calculate a moment instead of the total force.

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2. Underwater chimney

a) Parallel streamlines $\longrightarrow$ uniform pressure across a jet

b) Bernoulli:
$$P_1 + \frac{1}{2}\rho_0 V_1^2 = P_{atm} + \rho_0 g H$$
$$P_1 = -\frac{1}{2}\rho_0 V_1^2 + P_{atm} + \rho_0 g H$$

c) S.F.E.E.: $\dot{m}\Delta h = \dot{Q} \longrightarrow \dot{m}(c_p T - c_p T_0) = \dot{Q}$

d) Bernoulli in the chimney: $P_1 = P_{atm} + \rho g H$

comparing with b): $\frac{1}{2}\rho_0 V_1^2 = (\rho_0 - \rho) g H = 0.4 (T - T_0) g H$

but $V_1 = \frac{\dot{m}}{\rho_0 A}$ and $T - T_0 = \frac{\dot{Q}}{\dot{m} c_p}$, so the relation above gives

$$\frac{1}{2}\frac{\dot{m}^2}{\rho_0 A^2} = 0.4 \frac{\dot{Q} g H}{\dot{m} c_p} \longrightarrow \dot{m}^3 = 0.8 \frac{\rho_0 A^2 \dot{Q} g H}{c_p}$$

$$\longrightarrow \dot{m} \simeq \left[0.8 \frac{10^3 \; 4 \; 10^{-6} \; 10^5 \; 9.81 \; 2}{4180}\right]^{1/3}$$

$$\dot{m} \simeq 1.15 \text{ kg.s}^{-1}$$

Comments: taken by 56 % of the candidates, this question was not too popular. The candidates could solve the steps individually, but very few appreciated the global picture: basically, hydrostatic pressure in the chimney (Bernoulli for the hot water, see (d)) leads to a lower pressure at its base than the corresponding cold water hydrostatic pressure. This difference drives the flow (Bernoulli for the cold water entering, see (b)). The heating sets the value of the product of mass flow rate and temperature difference (see (c)). A frequent mistake in the last part was to apply Bernoulli on a streamline starting on the cold side and ending on the hot side, which is not allowed because density is not constant. In (c), you might wonder whether the change in kinetic energy is significant in the SFEE: it is not actually, and you can check for that in retrospect, using $\dot{m}$, ρ_0 and ρ to evaluate the velocity before and after the heating element. Anyway, not only the difference between them, but also each kinetic energy (for instance $V_1^2/2$) is itself negligible compared to the thermal energy difference $c_p \Delta T$. This is because the fluid is a liquid and its density changes very little.

3. Cycle

a) $W_c = \int_1^2 P\,dV$ polytropic $PV^n = P_1 V_1^n \rightarrow P = P_1 \frac{V_1^n}{V^n}$

hence $W_c = \int_1^2 P_1 \frac{V_1^n}{V^n}\,dV = P_1 V_1^n \int_1^2 V^{-n}\,dV$

$$= P_1 V_1^n \frac{V_2^{-n+1} - V_1^{-n+1}}{-n+1} = \frac{P_1 V_1 - P_2 V_2}{n-1}$$

$$P_2 = P_1 \frac{V_1^n}{V_2^n} = 10^5\, r_v^n \simeq 10^5\, 8^{1.3} \simeq 1.5\ 10^6\ P_a$$

So $W_c \simeq \frac{10^5\ 0.4\ 10^{-3} - 1.5\ 10^6\ 0.05\ 10^{-3}}{0.3} \simeq -117\ J$

b) Temperature T_2 at the end of compression

$P_1 V_1 = m\ R\ T_1$

$P_2 V_2 = m\ R\ T_2$ $\longrightarrow$ ratio: $\frac{T_2}{T_1} = \frac{P_2 V_2}{P_1 V_1} \simeq \frac{1.5\ 10^6}{10^5} \times \frac{1}{8} \simeq \frac{15}{8}$

So $T_2 \simeq \frac{15}{8} T_1 \simeq 549\ K$ ($276\ °C$)

c) Combustion

Fixed amount of air : mass m heating $m\, c_v\, \Delta T = Q$

State eq. $m = \frac{P_1 V_1}{R T_1} \rightarrow \Delta T = T_3 - T_2 = \frac{Q}{c_V} \frac{R T_1}{P_1 V_1}$

$$T_3 - T_2 \simeq \frac{600\ \ 287\ \ 293}{718\ \ 10^5\ \ 0.4\ \ 10^{-3}} \simeq 1757\ K \rightarrow T_3 \simeq 2306\ K$$

c_v air databook page 13

d) Polytropic between 3 and 4 $\frac{P_4}{P_3} = \frac{V_3^{1.5}}{V_4^{1.5}} = \frac{1}{r_v^{1.5}} \longrightarrow P_4 \simeq 2.78\ 10^5\ P_a$

Eq. State $\frac{P_3}{P_1} = \frac{V_1}{V_3} \frac{T_3}{T_1} \simeq 8\ \frac{2306}{293} \longrightarrow P_3 \simeq 63\ 10^5\ P_a$

Expansion work (see a): $W_{exp} = \frac{P_3 V_3 - P_4 V_4}{0.5} \simeq 408\ J$

Net work $= W_{exp} + W_c \simeq 408 - 117 \simeq 291\ J$

Thermal efficiency: $\eta = \frac{W_{net}}{Q} \simeq \frac{291}{600} \simeq 0.484$

Comments: well-answered question taken by 95 % of the candidates. Just be aware of the fact that ‘polytropic’ does not mean ‘adiabatic’. It is precisely because it is not an adiabatic process (and also because of irreversibilities) that it is polytropic with exponent n $\neq \gamma$.

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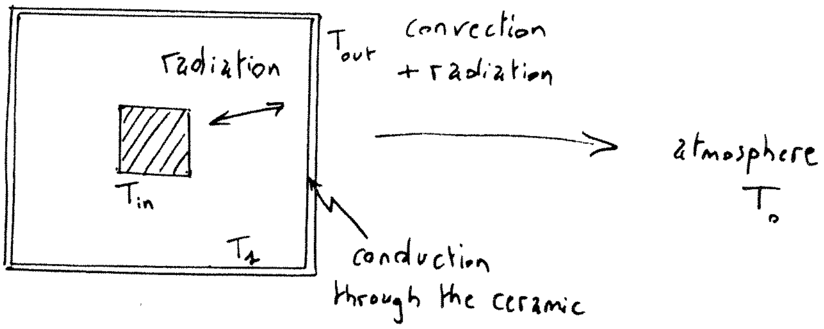
4. a)



b) At any point on the external surface, one can "see" only the atmosphere $\rightarrow$ $F = 1$

This is also true that any point on the central element emitting radiation will only "see" the internal wall of the outer element

$F_{inner-outer} = 1$

c) To the atmosphere

$$\dot{Q} = \underbrace{A\,h\,(T_{out} - T_0)}_{\text{convection}} + \underbrace{A F \sigma \left(T_{out}^4 - T_0^4\right)}_{\text{radiation}}$$

A surf. area $= 1\,m \times 4 \times 0.04\,m$

$$0.16 \times 15\,(500 - 293) + 0.16 \times 5.67 \times 10^{-8}\left(500^4 - 293^4\right) \simeq 997\ W$$

very close to $\dot{Q} = 1000\ W \longrightarrow T_{out} \simeq 500\ K$

d) Through the ceramic

Conduction $\dot{Q} = A\,\lambda\,\dfrac{\Delta T}{e} \longrightarrow \Delta T \simeq \dfrac{10^3 \times 10^{-3}}{0.16 \times 0.5} \simeq 12.5\ K$

$$T_1 = T_{out} + \Delta T \simeq 512.5\ K$$

e) Internal radiation

$$\dot{Q} = A_{in}\,F_{in-out}\,\sigma\left(T_{in}^4 - T_1^4\right) \longrightarrow T_{in}^4 = T_1^4 + \frac{\dot{Q}}{\sigma A_{in}}$$

A_{in} small element surf. area $= 1\,m \times 4 \times 0.01\,m$

$$T_{in}^4 \simeq 512.5^4 + \frac{10^3}{5.67 \times 10^{8} \times 0.04} \longrightarrow T_{in} \simeq 845\ K$$

Comments: this question was taken by 67 % of the candidates. This heat transfer question involves a specified heat flux $\dot{Q}$ driven by convection and radiation to the atmosphere. This heat flux also crosses a wall by conduction. Inside, it is transferred by radiation only. This simple question was not very popular and not well tackled compared to the rest of the section. Quite frequently, the heat flux was assumed wrongly to be transferred in parallel by conduction through the wall, convection in the atmosphere and radiation inside and outside.

5. a) mass flow rates in section 1

$$\dot m_1 = \underset{\underset{\underset{\underset{3.14\times 0.1^2\times 0.232\times 1417}{\shortparallel}}{\pi r^2 \rho_1 V_{1a}}}{\shortparallel}}{\dot m_{1a}} + \underset{\underset{\underset{\underset{3.14(0.3^2-0.1^2)\times 0.232\times 600}{\shortparallel}}{\pi (R^2-r^2)\rho_1 V_{1b}}}{\shortparallel}}{\dot m_{1b}}$$

state eq. perfect gas

$$\rho_1 = \frac{P_1}{R T_1} \simeq \frac{0.2\, 10^5}{287\times 300}$$

$$\to \rho_1 \simeq 0.232 \ kg\, m^{-3}$$

$$\dot m_1 \simeq 10.3 + 35.0 \simeq 45.3 \ kg.s^{-1}$$

b) S.F.M.E. between section 1 and section 2

$$\underset{\underset{0}{\shortparallel}}{\sum F_x} + \left(\sum p A\right)_x = \sum \dot m_{out} V_{out} - \sum \dot m_{in} V_{in}$$

$$P_1 A - P_2 A = \dot m V_2 - \dot m_{1a} V_{1a} - \dot m_{1b} V_{1b}$$

$$\text{So } V_2 = \frac{(P_1 - P_2) A + \dot m_{1a} V_{1a} + \dot m_{1b} V_{1b}}{\dot m}$$

$$\simeq \frac{(0.2 - 1) 10^5 \times 3.14 \times 0.3^2 + 10.3 \times 1417 + 35 \times 600}{45.3}$$

$$\simeq 287 \ m.s^{-1}$$

$$\text{by continuity} \quad \dot m = \pi R^2 \rho_2 V_2 \quad \to \rho_2 \simeq \frac{45.3}{3.14 \times 0.3^2 \times 287} \simeq 0.559 \ kg\, m^{-3}$$

$$\text{Eq state} \quad P_2 = \rho_2 R T_2 \quad \to \quad T_2 \simeq \frac{10^5}{0.559 \times 287} \simeq 623 \ K$$

c) S.F.F.E between section 1 and section 2

$$\dot Q - \underset{\underset{0}{\shortparallel}}{\dot W_x} = \dot m \left(h_2 + \frac{V_2^2}{2}\right) - \dot m_{1a}\left(h_1 + \frac{V_{1a}^2}{2}\right) - \dot m_{1b}\left(h_1 + \frac{V_{1b}^2}{2}\right)$$

$h = c_p T$ with $c_p \simeq 1005 \ J.K^{-1}.kg^{-1}$ [databook page 13]

$$\dot Q \simeq 45.3\left(1005 \times 623 + \frac{287^2}{2}\right) - 10.3\left(1005\times 300 + \frac{1417^2}{2}\right) - 35\left(1005\times 300 + \frac{600^2}{2}\right)$$

$$3.02 \times 10^7 \quad - \underset{1.34}{\cancel{1.03}\times 10^7} \quad - 1.69 \times 10^7$$

$$\dot Q \simeq 0$$

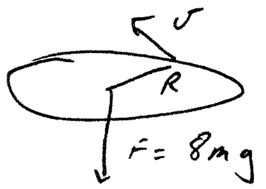
Comments: Very well answered question, taken by 88 % of the candidates, on a high speed air injector, where kinetic energy is comparable to thermal energy and where compressible effects are important. Most candidates took correctly into account the two distinct contributions for both momentum balance and energy balance.

Engineering Tripos 1A. Paper 1, Mechanical Engineering 2002.

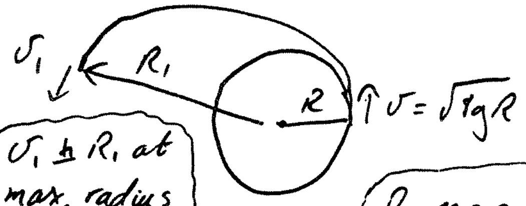
Solutions - Section B.

6. (a) When only acted on by a force towards (or parallel) to the axis.

(b) (i) 'F = ma' : $a = v^2/R = \frac{8mg}{m} \Rightarrow \underline{v = \sqrt{8gR}}$



(ii)



$v_1 \perp R_1$ at max. radius

B moves up $R_1 - R$

Conservation of momentum

$\Rightarrow R_1 v_1 = R v$

$\frac{v}{v_1} = \rho$, where $\rho = R_1/R$

Conservation of energy of masses at A and B after removal of 7m

$$\underset{\text{gain in PE of B}}{\cancel{m}g(R_1 - R)} + \underset{\text{gain in KE of B (stationary)}}{0} + \underset{\text{gain in KE of A. N.B. no radial vel.}}{\frac{1}{2}\cancel{m}(v_1^2 - v^2)} = 0$$

$\Rightarrow (\rho - 1) + \frac{1}{2g}\frac{v^2}{R}\left(\frac{1}{\rho^2} - 1\right) = 0 \Rightarrow \rho^3 - 5\rho^2 + 4 = 0$ (using $\frac{v^2}{gR} = 8$)

$(\rho - 1)(\rho^2 - 4\rho - 4) = 0 \Rightarrow \rho = 2 + 2\sqrt{2}$, neglecting -ve root, as required.

Known root

(Alternatively substitute in given solution.)

(iii) As for (b)(i) $\frac{v_1^2}{R_1} = \frac{Mg}{m}$ ← Final mass at B, ← mass at A $\Rightarrow M = \frac{mv^2}{gR} \cdot \frac{v_1^2}{v^2} \cdot \frac{R}{R_1} = \frac{8m}{\rho^3}$

Change in mass $= m - M = m(1 - 8/\rho^3) = 0.93m$

(iv) ~~Gain~~ Loss in KE of A = Gain in PE of B $= mg(R_1 - R) = mgR(1 + 2\sqrt{2})$
$= 3.83\,mgR$

This energy has gone to lifting B.

[A sketch seemed to help understand what is going on in b(ii). Easy to lose a sign in energy conservation.]

7.

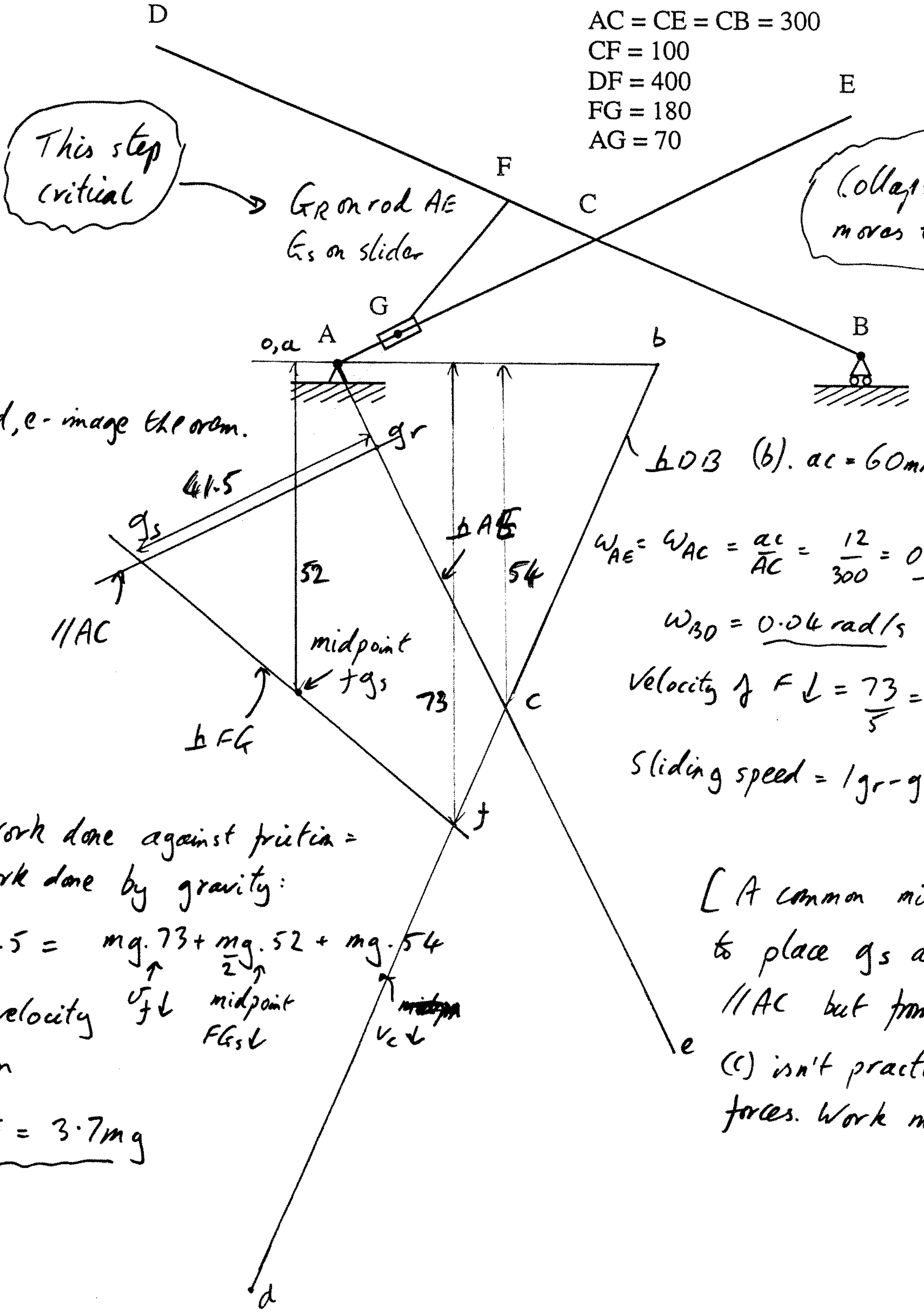
ENGINEERING TRIPOS PART IA

Paper 1, Mechanical Engineering

Monday 10 June 2002 9 to 12

Loose-leaf copy of Figure 7, drawn to scale.

Dimensions in mm
AC = CE = CB = 300
CF = 100
DF = 400
FG = 180
AG = 70



This step critical → G_R on rod AE, G_s on slider

Collapsing ⇒ B moves to right.

gr, f, d, e - image theorem.

(b). ac = 60mm → $\frac{60}{50} \times 10 = 12$ mm/s

$\omega_{AE} = \omega_{AC} = \frac{ac}{AC} = \frac{12}{300} = 0.04$ rad/s ↺

$\omega_{BD} = 0.04$ rad/s ↻ by symmetry

Velocity of F ↓ $= \frac{73}{5} = 15$ mm/s

Sliding speed $= |g_r - g_s| = \frac{41.5}{5} = 8.3$ mm/s

(c) Work done against friction = work done by gravity:

$$F.41.5 = mg.73 + \frac{mg}{2}.52 + mg.54$$

41.5: mm on velocity diagram; 73: v_f ↓; 52: midpoint FG_s ↓; 54: ~~midpt~~ v_c ↓

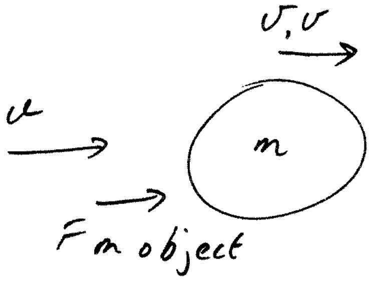
⇒ F = 3.7mg

[A common mistake was to place gs along a line //AC but from A, not gr. (c) isn't practical resolving forces. Work much easier.]

8. (a) Force $= \dot{m}\Delta v = \underline{\rho A u^2}$

(b) Relative velocity $= u - v$. Mass δm sticking in time $\delta t = (u-v)\,\delta t.\,\rho A \Rightarrow \frac{dm}{dt} = \rho A (u-v)$ (1)

(c)



$F = \dot{m}\Delta v = \dot{m}(u-v)$ (2) [or use conservation of momentum for system]

$F = ma = m\frac{dv}{dt}$ (3)

$(2)+(3) \Rightarrow \frac{dm}{dt}(u-v) = m\frac{dv}{dt}$

Need to eliminate v using (1) [$\frac{d^2m}{dt^2} = -\rho A \frac{dv}{dt}$ from (1)]

$$\Rightarrow \frac{dm}{dt}\cdot\frac{1}{\rho A}\frac{dm}{dt} = m\cdot\frac{-1}{\rho A}\cdot\frac{d^2m}{dt^2} \Rightarrow -m\frac{d^2m}{dt^2} = \left(\frac{dm}{dt}\right)^2$$

(d) $m^2 = \alpha t + \beta \Rightarrow \frac{dm}{dt}2m = \alpha \Rightarrow 2\frac{dm}{dt}\cdot\frac{dm}{dt} + 2m\frac{d^2m}{dt^2} = 0$

$\Rightarrow -m\frac{d^2m}{dt^2} = \left(\frac{dm}{dt}\right)^2$ as required.

To find α and β use initial conditions at $t=0, v=0, m=m_0$

$\Rightarrow \beta = m_0^2$.

At $t=0$ $\frac{dm}{dt} = \rho A u \Rightarrow \alpha = 2m_0\rho A u$.

[Short but mathematically tricky question.]

9. (a)(i) $T\dot{y}_1 + y_1 = x_1 \Rightarrow T\ddot{y}_1 + \dot{y}_1 = \dot{x}_1 \Rightarrow T\frac{d}{dt}(\dot{y}_1) + \dot{y}_1 = \dot{x}_1$

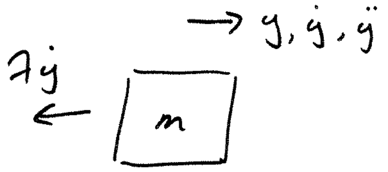
i.e. $\dot{x}_1$ and $\dot{y}_1$ also satisfy the governing equation.

(ii) Since step input = $\int$ impulse input

step response = $\int$ impulse response using (i)

(b) (i) Change in momentum = impulse : $mv = Q \Rightarrow v_{0+} = Q/m$

(ii) After impulse 'F=ma'



$-\lambda\dot{y} = m\ddot{y}$

$T\ddot{y} + \dot{y} = 0$ with $T = m/\lambda$

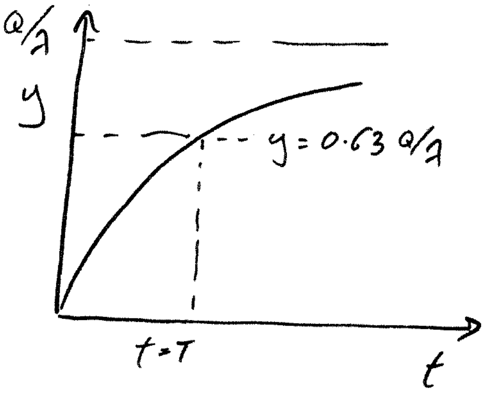
$\dot{y} = Ae^{-t/T}$

At $t = 0$ $\dot{y} = Q/m \Rightarrow A = Q/m$

Integrate:

$\Rightarrow y = B - \frac{QT}{m}e^{-t/T}$: B.C. $y = 0, t = 0$

$\Rightarrow y = \frac{QT}{m}(1 - e^{-t/T}) = \frac{Q}{\lambda}(1 - e^{-t/T})$

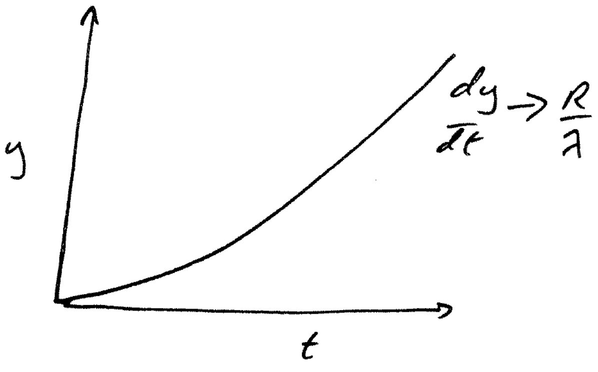


(c) Use result that step response is integral of impulse response [or by direct ~~integr~~ solution again]

$\Rightarrow \dot{y} = \frac{R}{\lambda}(1 - e^{-t/T})$ taking answer to b(ii) replacing Q by R

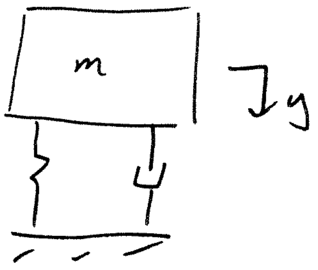
$\Rightarrow y = \frac{Rt}{\lambda} + \frac{RT}{\lambda}(e^{-t/T} - 1)$ (constant of integration using $y = 0$ @ $t = 0$)

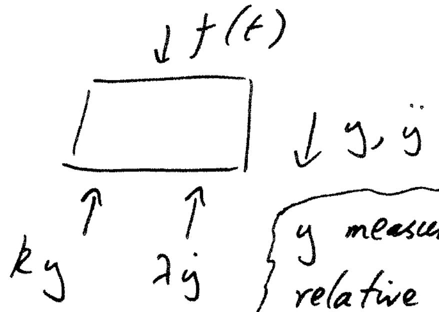
Easiest to sketch noting that it is the integral of (b)



[Direct solution in (c) rather than using the 'hint', part (a) tended to make the question long.]

10.





y measured relative to equilibrium position

(a) 'F = ma'

$-ky - \lambda\dot{y} + f(t) = m\ddot{y}$ Here $f(t) = \Delta mg\, H(0)$ where $H(0)$ is step fn.

$\Rightarrow \ddot{y}\frac{m}{k} + \frac{\lambda}{k}\dot{y} + y = f(t)/k$.

$= \frac{\lambda}{2\sqrt{k/m}}$

Databook case 4.4 with $\omega_n = \sqrt{k/m}$, $\zeta = \frac{\lambda}{2k}\omega_n$, $X = \frac{\Delta mg}{k}$

(b) $\omega_n = \sqrt{4000/15} = 16.3$ rad/s $X = \frac{\Delta mg}{k} = 4.9 \times 10^{-5}$ m $= 49\mu m$

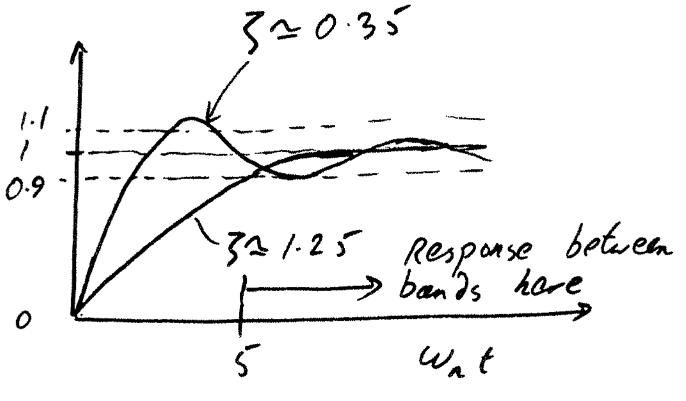
We need variation of y about new equilibrium position $< 5\mu m$ after non dimensional time $\omega_n t = 16.3.0.3 = 4.9 \simeq 5$

i.e $y/X = 1 \pm \frac{5}{49\mu m}$ $\Rightarrow 0.9 < y/X < 1.1$ approx.

From curves, case 4.4, curves for $0.35 < \zeta < 1.25$ approx satisfy this.

i.e. $610 > \lambda > 170$ Ns/m

using $\lambda = \frac{2k}{\omega_n}\zeta$



(c) Now we need much lower damping (assume $\zeta \ll 1$)

Number of cycles $n = 60 \times 16.3/2\pi = 156$ in 60 seconds.

Logarithmic decrement (in databook) $\Rightarrow \ln\left(\frac{y_1}{y_2}\right) = 2\pi\zeta \Rightarrow \ln\left(\frac{49}{5}\right) = 2\pi\zeta n$

(or use $\frac{y}{x} = 1 - e^{-\zeta\omega_n t}\cos\omega_n t$, putting $\cos = \pm 1$) $\Rightarrow \zeta = 2.3 \times 10^{-3}$, $\underline{\lambda > 1.1 \text{ Ns/m}}$

(d) Probably impractical to reduce m. Hence increase ω_n by increasing k and λ accordingly. Keep λ up to critical damping. [But this may cause problems with harmonic response.]

[Main problems were in not understanding the form of solution (as given in the databook) and not using the databook to solve the problem.]