Part IA Engineering Tripos Part 2002

Paper 4 Mathematical Methods Answers

1 (a) (ii) $\dfrac{\underline{a}\,.\,(\underline{b}\times\underline{c})}{|\underline{a}\times\underline{b}+\underline{b}\times\underline{c}+\underline{c}\times\underline{a}|}$ (iii) $\underline{a}\,.\,(\underline{b}\times\underline{c})\dfrac{\underline{a}\times\underline{b}+\underline{b}\times\underline{c}+\underline{c}\times\underline{a}}{|\underline{a}\times\underline{b}+\underline{b}\times\underline{c}+\underline{c}\times\underline{a}|^2}$

(b) any multiple of $\underline{b}=3\,\underline{i}+\underline{j}+5\,\underline{k}$;

$$\frac{x-0}{3} = \frac{y-\frac{4}{3}}{1} = \frac{z-\frac{2}{3}}{5}$$ (N.B. not unique)

2 (a) -1 (c) (i) $r = 1.71$ and $\theta = 0.2145$ or 2.3089 or 4.4033

(c) (ii) $r = 1$ and $\theta = 0.675$

3 (a) (i) $x_{CF}(t) = Ae^{-t/2}\sin\left(\dfrac{\sqrt{3}}{2}t+\phi\right)$ (ii) $x_{PI}(t) = C\sin\left(\dfrac{\sqrt{3}}{2}t\right)+D\cos\left(\dfrac{\sqrt{3}}{2}t\right)$

(b) $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n-\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n$

4 (a) $\begin{bmatrix}\frac{13}{8} & \frac{3\sqrt{3}}{8}\\ \frac{3\sqrt{3}}{8} & \frac{7}{8}\end{bmatrix}$ (c) $\begin{bmatrix}-\frac{1}{2}\\ \frac{\sqrt{3}}{2}\end{bmatrix}$ (or any multiple) ; 2 and $\dfrac{1}{2}$ (d) $\dfrac{\sqrt{3}}{2}2^{100}\begin{bmatrix}\frac{\sqrt{3}}{2}\\ \frac{1}{2}\end{bmatrix}$

5 (a) $g(t) = 2e^{-2t}-e^{-t} \quad t>0;$ (b) $y(t) = \dfrac{e^{-t}}{1-\alpha}-\dfrac{2e^{-2t}}{2-\alpha}-\dfrac{\alpha e^{-\alpha t}}{(2-\alpha)(1-\alpha)}$

(c) $y(t) = e^{-t}-e^{-2t}$

6 (a) $\displaystyle\sum_{n=1}^{\infty}\frac{4}{n^3\pi}\left(1-(-1)^n\right)\sin nx$ (b) $-n^2a_n \quad -n^2b_n$

$$\sum_{n=1}^{\infty}\frac{4}{n^3\pi\left(k^2-n^2\right)}\left(1-(-1)^n\right)\sin nx$$

7 (b) (ii) .066 (iii) $.07\pm.01$

8 $x(t) = 2e^t-\cos 2t-\dfrac{1}{2}\sin 2t \qquad y(t) = e^t-\cos 2t$

9 (a) 58.3 (b) $(0\,,0)$ & $(-2\,,0)$ are saddle points ; $\left(-1\,,-\tfrac{1}{2}\right)$ is a maximum.

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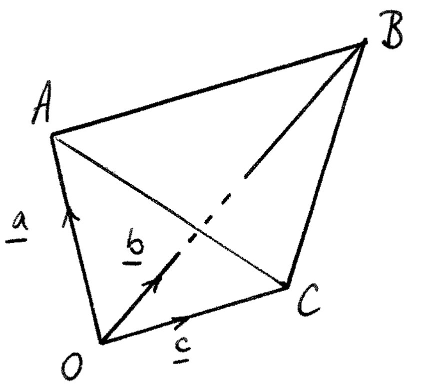
1A Paper 4 Mathematical Methods 2002

Section A

Q1. Vector equations of lines and planes

(a)



MARKS

(2) (i) To find normal, find vector product of any two non-parallel vectors in the plane e.g. $(\underline{b}-\underline{a})$ and $(\underline{c}-\underline{a})$

Consider a vector parallel to normal (not necessarily a unit vector!), $\underline{n}$

$$\underline{n} = (\underline{b}-\underline{a})\times(\underline{c}-\underline{a}) = \underline{b}\times\underline{c} - \underline{a}\times\underline{c} - \underline{b}\times\underline{a} + \underbrace{\underline{a}\times\underline{a}}_{\to 0}$$

(3) $$= \underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}$$

(ii) Plane goes through a point $\underline{a}$ and is of the form $\underline{r}.\hat{\underline{n}} = d$, where d is the perpendicular distance to O and $\hat{\underline{n}}$ is a unit normal

$$d = \underline{a}.\hat{\underline{n}} = \frac{\underline{a}.(\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a})}{|\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}|}$$

(3) $$= \frac{\underline{a}.(\underline{b}\times\underline{c})}{|\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}|}$$

Q1 (cont)

1(a)(iii) The nearest point to origin has position vector,

(2) $$\underline{r} = d\hat{\underline{n}} = \underline{a}.(\underline{b}\times\underline{c}) \frac{\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}}{|\underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}|^2}$$

1(b)(i) The direction of the line of intersection of 2 planes, $\underline{b}$, must be perpendicular to both normals (note $\underline{r}.\underline{n} = d$)

plane 1: $x + 2y - z = 2$ has normal parallel to $\underline{n}_1 = \begin{pmatrix}1\\2\\-1\end{pmatrix}$

plane 2: $3x + y - 2z = 0$ has normal parallel to $\underline{n}_2 = \begin{pmatrix}3\\1\\-2\end{pmatrix}$

$$\therefore \underline{b} = \underline{n}_1 \times \underline{n}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & -1 \\ 3 & 1 & -2 \end{vmatrix}$$

(5) $$= \underline{i}(-3) - \underline{j}(1) + \underline{k}(-5)$$

1(b)(ii) To find equation of line $\underline{r} = \underline{a} + \lambda\underline{b}$ we need to find a point $\underline{a}$ common to both planes such that

$$\underline{a}.\underline{n}_1 = 2 \quad \text{and} \quad \underline{a}.\underline{n}_2 = 0$$

$$a_x + 2a_y - a_z = 2$$
$$3a_x + a_y - 2a_z = 0$$

There is obviously no unique solution. Any point $\underline{a}$ satisfying above will yield the equation of the line of intersection

e.g. Try $a_x = 0 \Rightarrow a_y = 4/3 \quad a_z = 2/3$ from above equations

$\therefore (0, 4/3, 2/3)$ lies on line of intersection

Q1 (b) cont.

Since the line can be expressed in form $\underline{r} = \underline{a} + \lambda \underline{b}$

or $$\frac{x - a_x}{b_x} = \frac{y - a_y}{b_y} = \frac{z - a_z}{c_z}$$

Substituting for $\underline{a}$ and $\underline{b}$:

(5) $$\underline{\frac{x - 0}{3} = \frac{y - 4/3}{1} = \frac{z - 2/3}{5}}$$

Other alternative solutions include

$$\underline{a}_2 = (-4, 0, 6) \Rightarrow \frac{x+4}{3} = \frac{y}{1} = \frac{z+6}{5}$$

$$\underline{a}_3 = \left(-\tfrac{2}{5}, \tfrac{6}{5}, 0\right) \Rightarrow \frac{x+2/5}{3} = \frac{y-6/5}{1} = \frac{z}{5}$$

$$\underline{a}_4 = (-1, 1, -1) \Rightarrow \frac{x+1}{3} = \frac{y}{1} = \frac{z+1}{5}$$

etc.

Q2. Limits, complex numbers and hyperbolic functions

2(a)

$$\lim_{x\to 0} \frac{\sin x - \tan x}{(1-\cos x)(1-e^{-x})} = \frac{\left[x - \frac{x^3}{3!} + O(x^5)\right] - \left[x + \frac{x^3}{3} + O(x^5)\right]}{\left[1 - \left(1 - \frac{x^2}{2} + O(x^4)\right)\right]\left[1 - \left(1 - x + O(x^2)\right)\right]}$$

$$= \frac{x^3\left(-\frac{1}{2}\right) + O(x^5)}{\left[\frac{x^2}{2} + O(x^4)\right]\left[x - O(x^2)\right]}$$

(4)

$$= \frac{-\frac{1}{2}x^3 + O(x^5)}{\frac{1}{2}x^3 + O(x^4)} = -1$$

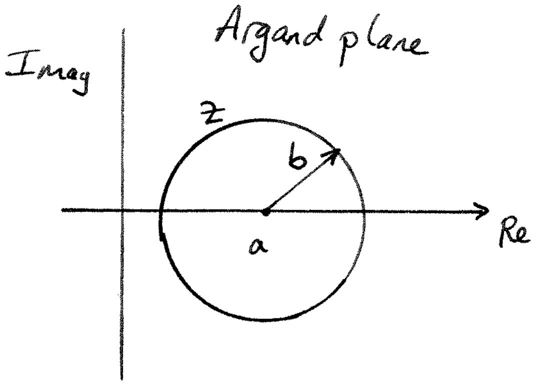
(note: power series preferred since l'hôpital's rule will require differentiation $\times 3$)

2(b) (i) $|z - a| = b$. Let $z = x + iy \Rightarrow |x + iy - a| = b$

$$\Rightarrow (x-a)^2 + y^2 = b^2$$

i.e. circle centre $(a, 0)$ radius b

(3)



(ii) $|z + 1| = |z|$. Let $z = x + iy$

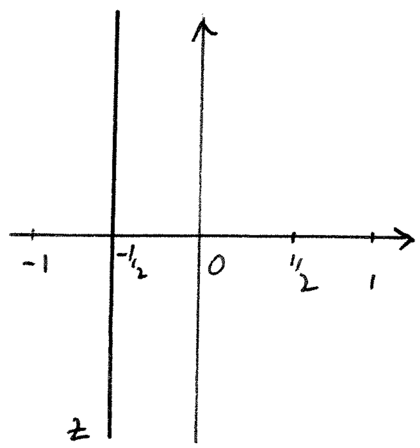
$$\Rightarrow (x+1)^2 + y^2 = x^2 + y^2$$

$$\therefore 2x + 1 = 0$$

$$x = -\tfrac{1}{2}$$

Vertical line through $x = -\frac{1}{2}$

(3)



Q2 contd

2(c)(i) $\quad z = (4+3i)^{1/3} = 5^{1/3} e^{i\left(\frac{\theta' + 2n\pi}{3}\right)}$ where $\tan\theta' = 3/4$

i.e. $\theta' = \tan^{-1} 3/4 = 0.6435$ rad and $n = 0, 1, 2$

(4) $\quad \therefore \; z = r e^{i\theta}$ where $r = 5^{1/3} = 1.71$

and $\theta = .2145$ or 2.3089 or 4.4033 rad

(Note: 3 possible values for θ)

(ii) $$z = \tanh\left(\frac{\pi}{6} + i\frac{\pi}{4}\right) = \frac{\sinh\left(\frac{\pi}{6} + i\pi/4\right)}{\cosh\left(\frac{\pi}{6} + i\pi/4\right)}$$

$$= \frac{\sinh\frac{\pi}{6}\cosh\left(\frac{i\pi}{4}\right) + \cosh\frac{\pi}{6}\sinh\left(\frac{i\pi}{4}\right)}{\cosh\frac{\pi}{6}\sinh\left(\frac{i\pi}{4}\right) + \sinh\frac{\pi}{6}\sinh\left(\frac{i\pi}{4}\right)}$$

[Note: sign is different to $\cos(A+B)$ expansion!]

$$= \frac{\sinh\frac{\pi}{6}\cos\frac{\pi}{4} + i\cosh\frac{\pi}{6}\sin\pi/4}{\cosh\frac{\pi}{6}\cos\frac{\pi}{4} + i\sinh\frac{\pi}{6}\sin\pi/4}$$

[Note: substitution into calculator at this stage will give correct answer]

$$= \frac{\sinh\frac{\pi}{6} + i\cosh\frac{\pi}{6}}{\cosh\frac{\pi}{6} + i\sinh\pi/6}$$

$$= \frac{\sinh\frac{\pi}{6} + i\cosh\frac{\pi}{6}}{\cosh\frac{\pi}{6} + i\sinh\frac{\pi}{6}} \cdot \frac{\cosh\frac{\pi}{6} - i\sinh\frac{\pi}{6}}{\cosh\frac{\pi}{6} - i\sinh\frac{\pi}{6}}$$

$$= \frac{2\sinh\frac{\pi}{6}\cosh\frac{\pi}{6} + i\left(\cosh^2\frac{\pi}{6} - \sinh^2\frac{\pi}{6}\right)}{\cosh^2\frac{\pi}{6} + \sinh^2\frac{\pi}{6}}$$

Using databook formula, simplify to $z = \dfrac{\sinh\frac{\pi}{3} + i(1)}{\cosh\pi/3}$

$$\Rightarrow \quad z = \sqrt{\cancelto{1}{\frac{\sinh^2\frac{\pi}{3} + 1}{\cosh^2\pi/3}}} \; e^{i\tan^{-1}\frac{1}{\sinh\pi/3}}$$

(6) $$= 1 \times e^{i(0.675)}$$

Q3 Differential equations and difference equations

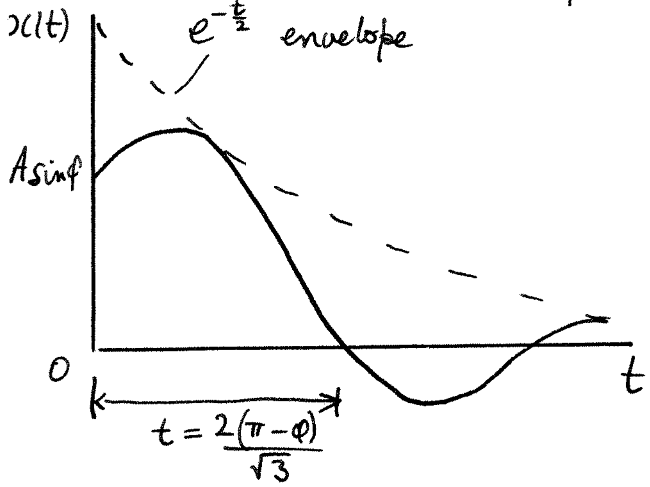
3 (a)(i) For complementary function find $x(t)$ satisfying

$$\ddot{x} + \dot{x} + x = 0$$

Try $x_{CF}(t) = e^{\lambda t} \Rightarrow \lambda^2 + \lambda + 1 = 0 \Rightarrow \lambda = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\Rightarrow x_{CF}(t) = A\, e^{-t/2} \sin\left(\frac{\sqrt{3}}{2}t + \varphi\right)$$

depend on boundary conditions



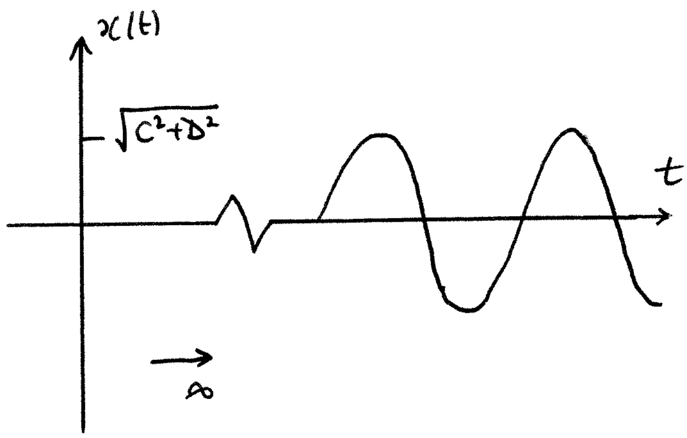
(6)

(ii) For particular integral, try $x_{PI}(t) = C\sin\frac{\sqrt{3}}{2}t + D\cos\frac{\sqrt{3}}{2}t$

Substitute to find C and D. The general solution is:

$$x(t) = x_{CF}(t) + x_{PI}(t)$$

$$\lim_{t\to\infty} x(t) = x_{PI}(t)$$



(4)

Q3 contd

3 (b) For Fibonacci series $a_{n+1} = a_n + a_{n-1}$

(i) For solution try $a_n = \lambda^n$

$$\therefore \lambda^{n+1} - \lambda^n - \lambda^{n-1} = 0$$

$$\Rightarrow \lambda^2 - \lambda - 1 = 0 \Rightarrow \lambda = \frac{1 \pm \sqrt{1+4}}{2}$$

$$= \underline{\frac{1 \pm \sqrt{5}}{2}}$$

General solution;

$$\underline{a_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n}$$

From boundary conditions $n=0$ $a_0 = 0$ $\therefore A + B = 0$

$n=1$ $a_1 = 1$ $\therefore A\frac{1+\sqrt{5}}{2} + B\frac{1-\sqrt{5}}{2} = 1$

$\therefore A = -B$ & $A = 1/\sqrt{5}$

(6) $$\therefore a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

[check: $a_2 = 1$ $a_3 = 2$]

(ii) As $n \to \infty$ $\left(\frac{1-\sqrt{5}}{2}\right)^n \ll 1$ and $\left(\frac{1+\sqrt{5}}{2}\right)^n \gg 1$

$$\text{As } n \to \infty \quad \frac{a_{n+1}}{a_n} \longrightarrow \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \cancel{\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}} \to 0}{\left(\frac{1+\sqrt{5}}{2}\right)^{n} - \cancel{\left(\frac{1-\sqrt{5}}{2}\right)^{n}} \to 0}$$

(4) $$\Rightarrow \underline{\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}} \quad \text{since } \lim_{n\to\infty}\left(\frac{1-\sqrt{5}}{2}\right)^n = 0$$

Q4 Matrices, transformations, determinants, e.vectors & e.values

4 (a) Consider 2×2 symmetric matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

From transformation of $B \to B'$, $\underline{x}'_B = A\underline{x}_B$

$$\begin{bmatrix} \frac{3\sqrt{3}}{8} \\ \frac{7}{8} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} b = \frac{3\sqrt{3}}{8} \\ d = \frac{7}{8} \end{array}$$

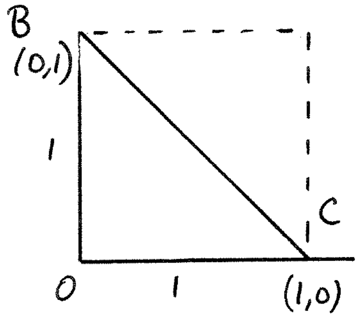
From transformation of $C \to C'$, $\underline{x}'_c = A\underline{x}_c$

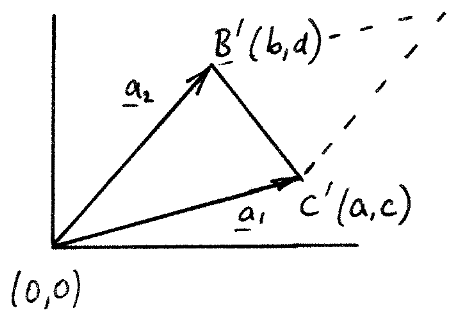
$$\begin{bmatrix} \frac{13}{8} \\ \frac{3\sqrt{3}}{8} \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{array}{l} c = \frac{3\sqrt{3}}{8} \\ d = 13/8 \end{array}$$

(4) $\therefore$ $A = \begin{bmatrix} 13/8 & \frac{3\sqrt{3}}{8} \\ \frac{3\sqrt{3}}{8} & \frac{7}{8} \end{bmatrix}$ and $A^T = A$ as required

Symmetric matrix

(b) Consider $A = [\underline{a}_1, \underline{a}_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$





Area before transformation $= \frac{1}{2} \times 1 \times 1 = \frac{1}{2}$

Area after transformation $= \frac{1}{2}|\underline{a}_1 \times \underline{a}_2| = \frac{1}{2}[ad - bc]$

$= \frac{1}{2} \times \det A$

Determinant of 2×2 matrix describes change in area

Q4 contd

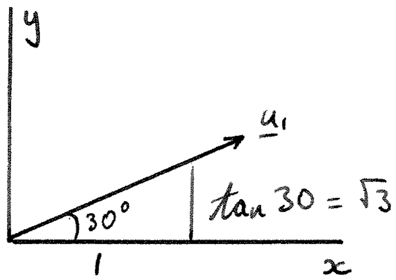
Since $\det A = ad - bc = \frac{13}{8} \times \frac{7}{8} - \frac{3\sqrt{3}}{8}\frac{3\sqrt{3}}{8} = \frac{91-27}{64} = 1$

(5) $\therefore$ Area is unchanged by the transformation

In general, $\det A = \lambda_1 \lambda_2$ = product of eigenvalues

If $\det A = 1 \Rightarrow \lambda_2 = \frac{1}{\lambda_1}$

(c) Eigenvector $\underline{u}_1 = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)^T$



Check that $A\underline{u}_1 = \lambda \underline{u}_1$

$$A\underline{u}_1 = \begin{bmatrix} \frac{13}{8} & \frac{3\sqrt{3}}{8} \\ \frac{3\sqrt{3}}{8} & \frac{7}{8} \end{bmatrix} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{13}{8}\frac{\sqrt{3}}{2} + \frac{3\sqrt{3}}{8}\frac{1}{2} \\ \frac{3\sqrt{3}}{8}\frac{\sqrt{3}}{2} + \frac{7}{8}\frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{16\sqrt{3}}{16} \\ \frac{16}{16} \end{bmatrix} = 2\begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}$$

$\therefore A\underline{u}_1 = 2\underline{u}_1$ and $\underline{u}_1$ is an eigenvector of eigenvalue 2

Other eigenvector must be orthogonal since A is real & symmetric and, from above, $\lambda_2 = \frac{1}{\lambda_1} = \frac{1}{2}$ $(\det A = \lambda_1 \lambda_2 = 1)$

(5) $\therefore \underline{u}_2 = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$ and $\lambda_2 = \frac{1}{2}$

(d) In general, $\underline{x} = \alpha \underline{u}_1 + \beta \underline{u}_2$

$$\Rightarrow A\underline{x} = \alpha \lambda_1 \underline{u}_1 + \beta \lambda_2 \underline{u}_2$$

$$\text{and } A^n \underline{x} = \alpha \lambda_1^n \underline{u}_1 + \beta \lambda_2^n \underline{u}_2$$

Consider $\underline{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$: $\alpha = \underline{u}_1 . \underline{x} = \cos 30° = \frac{\sqrt{3}}{2}$

Q 4 contd

After 100 applications of the transformation

$$\underline{x}' = A^{100}\underline{x} = \alpha\lambda_1^{100}\underline{u}_1 + \beta\lambda_2^{100}\underline{u}_2$$

$$\simeq \alpha 2^{100}\underline{u}_1 \quad \text{since } \lambda_2^{100} = \left(\tfrac{1}{2}\right)^{100} \simeq 0$$

(6) $$\therefore \underline{x}' = \frac{\sqrt{3}}{2} 2^{100} \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{bmatrix}.$$

i.e. <u>point moves along dominant eigenvector</u> (axis of expansion)

<u>Alternatively:</u>

$$\underline{x}' = A^{100}\underline{x} = \left[\underline{u}_1 \vdots \underline{u}_2\right] \begin{bmatrix} 2^{100} & 0 \\ 0 & \left(\frac{1}{2}\right)^{100} \end{bmatrix} \begin{bmatrix} \underline{u}_1^T \\ \cdots \\ \underline{u}_2^T \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

since $A^n = U\Lambda^n U^T$

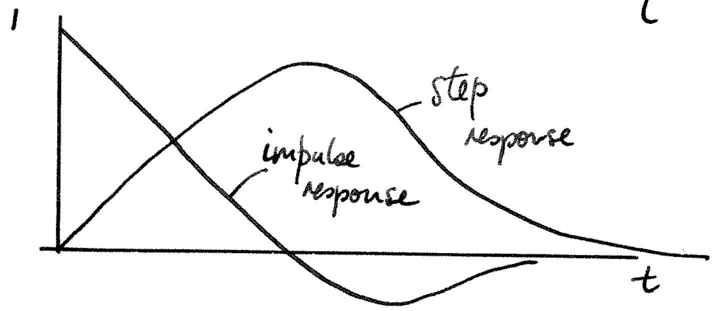
$$\therefore \underline{x}' \simeq 2^{100}\underline{u}_1 \frac{\sqrt{3}}{2}$$

Paper 4 Section B 2002

5a)

Impulse response $g(t) = \frac{d}{dt}$ (step response)

$$= \begin{cases} -e^{-t} + 2e^{-2t} & t > 0 \\ 0 & t < 0 \end{cases}$$



b)

$$y(t) = \int_0^t x(T)\,g(t-T)\,dT = \int_0^t e^{-\alpha T}\left(2e^{-2(t-T)} - e^{-(t-T)}\right)dT$$

$$= 2e^{-2t}\int_0^t e^{(2-\alpha)T}\,dT - e^{-t}\int_0^t e^{(1-\alpha)T}\,dT$$

$$= 2e^{-2t}\left[\frac{e^{(2-\alpha)T}}{2-\alpha}\right]_0^t - e^{-t}\left[\frac{e^{(1-\alpha)T}}{1-\alpha}\right]_0^t$$

$$= \frac{2}{2-\alpha}\left(e^{-\alpha t} - e^{-2t}\right) - \frac{1}{1-\alpha}\left(e^{-\alpha t} - e^{-t}\right)$$

$$= \frac{e^{-t}}{1-\alpha} - \frac{2e^{-2t}}{2-\alpha} - \frac{\alpha\, e^{-\alpha t}}{(2-\alpha)(1-\alpha)}$$

(and $y(t) = 0 \quad t < 0$)

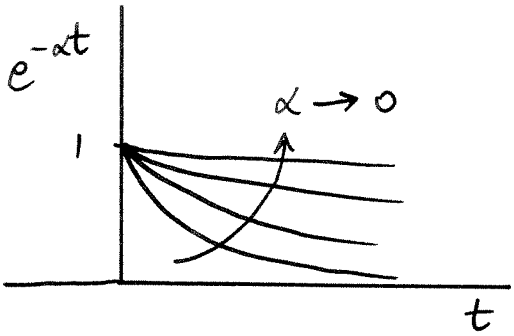
c) As $\alpha \to 0$ $\quad y(t) \simeq \begin{cases} e^{-t} - e^{-2t} & t > 0 \\ 0 & t < 0 \end{cases}$

This is the step response.

5 (continued)

This happens because the input $\begin{cases} e^{-\alpha t} & t>0 \\ 0 & t<0 \end{cases}$

approaches the unit step as $\alpha \to 0$.



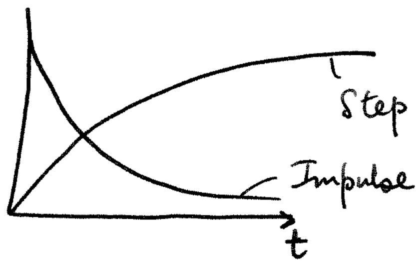
Examiner's Note:

Average Mark 13.9/20

The question was well done in the main.

Commonest errors:

- Curve sketching was abysmal. Not all linear systems have

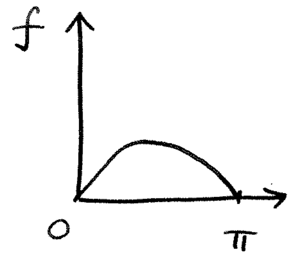


as their step & impulse responses.

- Some candidates had trouble with exponentials. e.g. $\int_0^t e^{(2-\alpha)T}\,dT = e^{(2-\alpha)}\int_0^t e^{T}\,dT$

or $\int_0^t e^{(2-\alpha)T}\,dT = \left[(2-\alpha)\,e^{(2-\alpha)T}\right]_0^t$

6 a) $f(x) = \sum_{1}^{\infty} b_n \sin nx$



where $b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx \, dx$

$$\Rightarrow \quad b_n = \underbrace{\frac{2}{\pi}\left[-x(\pi-x)\frac{\cos nx}{n}\right]_0^{\pi}}_{0} + \frac{2}{n\pi}\int_0^{\pi}(\pi - 2x)\cos nx \, dx$$

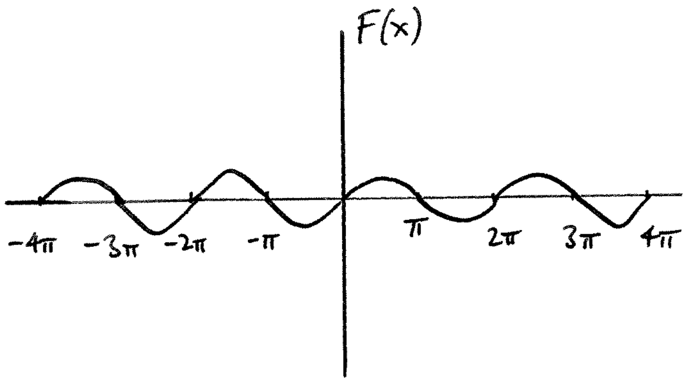
$$= \frac{2}{n\pi}\underbrace{\left[(\pi-2x)\frac{\sin nx}{n}\right]_0^{\pi}}_{0} - \frac{2}{n\pi}\int_0^{\pi}(-2)\frac{\sin nx}{n}\, dx$$

(a)

$$= \frac{4}{n^2\pi}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = \frac{4}{n^3\pi}\left(1 - \cos n\pi\right)$$

$$= \frac{4}{n^3\pi}\left(1 - (-1)^n\right) = \begin{cases} \frac{8}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

(b) (c)



F is an odd function & periodic 2π.

$b_n = O\left(\frac{1}{n^3}\right)$ since F is continuous & F' is continuous F'' is discontinuous

---

6 contd)

(b) If $y(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nx + \sum_1^\infty b_n \sin nx$

$\Rightarrow \quad y''(x) = -\sum_1^\infty n^2 a_n \cos nx - \sum_1^\infty n^2 b_n \sin nx$

$\Rightarrow$ coefficients in representation of y''

$= - n^2 \times$ coefficients in that for y.

Put/try $y = \sum_1^\infty b_n \sin nx \quad \Rightarrow \quad y'' = \sum_1^\infty -n^2 b_n \sin nx$

$$\sum_1^\infty (-n^2 b_n + k^2 b_n) \sin nx = \sum_{\substack{1 \\ n \text{ odd}}}^\infty \frac{8}{n^3\pi} \sin nx$$

Since Fourier series unique $\Rightarrow$ can equate coeffs

$$\Rightarrow \quad (-n^2+k^2) b_n = \begin{cases} \dfrac{8}{n^3\pi} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\text{i.e.} \quad b_n = \begin{cases} \dfrac{8}{n^3\pi(k^2-n^2)} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

Examiner's Note:

Average mark 10.5/20.

Not many attempted the last part — so more marks given to earlier parts.

- Answers ⓐ, ⓑ or ⓒ all acceptable for part (a)
- Integration by parts needs practice!
- As does $\sin n\pi = 0$, $\cos n\pi = (-1)^n$, $\int \sin nx = -\frac{\cos nx}{n}$ etc.

7(a) (i) The probabilities of events A and B can be added when A & B are <u>exclusive</u> (i.e. can not both happen)

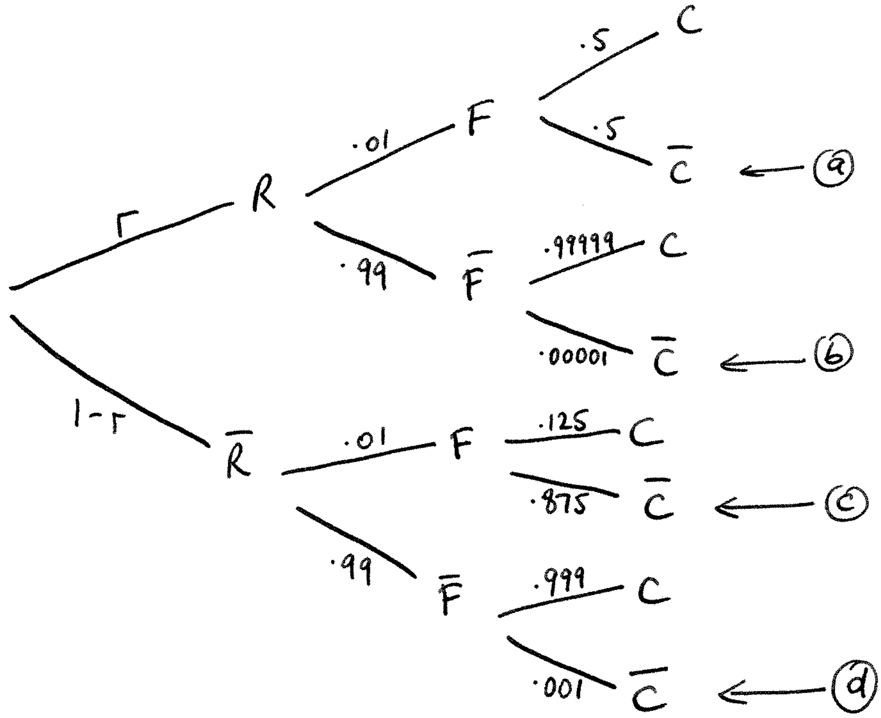
$$p(A \cup B) = p(A) + p(B)$$

(ii) The probabilities can be multiplied when A and B are <u>independent</u>

$$P(A \cap B) = p(A)\,p(B)$$

(b) Let R = Robot ok F = chip faulty
C = correctly fitted.

(i)



(ii) Let prob robot ok $= r$

$$P(\bar{C}) = P\text{ⓐ} + P\text{ⓑ} + P\text{ⓒ} + P\text{ⓓ}$$

$$= r \cdot .01 \times .5 + r \cdot .99 \times .00001 + (1-r) \cdot .01 \times .875 + (1-r) \cdot .99 \times .001$$

7 contd.) $P(\bar{c}) = .00532$

$\Rightarrow \quad .00532 = .00974 - .00473r$

$\Rightarrow \quad r = .934 \quad \Rightarrow \underline{P(\bar{r}) = 6.56\%}$ or $\underline{.066}$

(iii) $P(\bar{c}) = .0053 \Rightarrow P(\bar{r}) = .061$

$P(\bar{c}) = .0054 \Rightarrow P(\bar{r}) = .082$

$\therefore \quad \underline{P(\bar{r}) = .07 \pm .01}$

Examiner's Note:

Average Mark 12.7/20

N.B. • The terms exclusive and independent have a strict technical meaning

• When multiplying probabilities along "branches" you have to use probabilities i.e. $0 \leq$ number ≤ 1. Using percentages 99.9, 12.5 leads to answers which examiners find distinctly unimpressive.

8. $L(\dot{x}) = sX - x(0) = sX - 1$

$L(\dot{y}) = sY - y(0) = sY$

Taking L.T. of equations

$$\Rightarrow \left.\begin{aligned} sX - 1 + 4X - 5Y &= \frac{5}{s-1} \\ sY - (sX - 1) + Y &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} (s+4)X - 5Y &= \frac{s+4}{s-1} \\ sX - (s+1)Y &= 1 \end{aligned}$$

Eliminating Y

$$\Rightarrow \left[(s+4)(s+1) - 5s\right]X = \frac{(s+4)(s+1)}{s-1} - 5$$

i.e. $$(s^2+4)X = \frac{(s+4)(s+1)}{s-1} - 5 = \frac{s^2+9}{s-1}$$

$$\therefore \quad X = \frac{s^2+9}{(s^2+4)(s-1)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}$$

A, B & C found from "cover-up" rule or equating coeff

$$\Rightarrow As^2 + (B-A)s - B + Cs^2 + 4C = s^2 + 9$$

$\Rightarrow \quad A + C = 1 \qquad B - A = 0 \qquad -B + 4C = 9$

i.e. $B = A$

$$\begin{array}{r} -A + 4C = 9 \\ \hline 5C = 10 \end{array} \quad \Rightarrow \quad C = 2 \quad A = -1 \quad B = -1$$

$$\therefore X(s) = -\frac{s+1}{s^2+4} + \frac{2}{s-1} = -\frac{s}{s^2+4} - \frac{1}{2}\frac{2}{s^2+2^2} + \frac{2}{s-1}$$

Inverting $\underline{x(t) = -\cos 2t - \frac{1}{2}\sin 2t + 2e^t}$

From above $$Y(s) = \frac{s+4}{5}\left[X - \frac{1}{s-1}\right] = \frac{s+4}{5}\left[\frac{1}{s-1} - \frac{(s+1)}{s^2+4}\right]$$

8 contd).

$$\Rightarrow \quad Y(s) = \frac{(s+4)5}{5(s-1)(s^2+4)} = \frac{s+4}{(s-1)(s^2+4)} = \frac{As+B}{s^2+4} + \frac{C}{s-1}$$

As above

$$As^2 + (B-A)s - B + Cs^2 + 4C = s+4$$

$$\Rightarrow \quad A + C = 0 \qquad B - A = 1 \qquad -B + 4C = 4$$

$$\Downarrow$$

$$C = -A \qquad B + 4A = -4$$

$$5A = -5$$

$$\Rightarrow \quad A = -1 \qquad C = 1 \qquad B = 0$$

i.e. $$Y(s) = -\frac{s}{s+4} + \frac{1}{s-1} \quad \Rightarrow \quad \underline{y = -\cos 2t + e^t}$$

(b) Substituting

$$\dot{x} + 4x - 5y = 2\sin 2t - \cos 2t + 2e^t$$
$$+ 4(-\cos 2t - \tfrac{1}{2}\sin 2t + 2e^t)$$
$$- 5(-\cos 2t + e^t)$$
$$= 5e^t \quad \checkmark$$

$$\dot{y} - \dot{x} + y = 2\sin 2t + e^t - (2\sin 2t - \cos 2t + 2e^t)$$
$$+ (e^t - \cos 2t)$$
$$= 0 \quad \checkmark$$

B.C. $\underline{x(0) = 1 \quad y(0) = 0}$ ✓

<u>Examiner's Note</u> Average Mark 12.4/20

With hindsight algebra too tough! Algebraic errors thus treated "sympathetically".

Apart from algebra, commonest errors: $L(e^t) = \frac{1}{s+1}$

$\frac{1}{s^2+4} = \frac{1}{(s-2)(s+2)}$; candidates handwriting s & 5 confused.

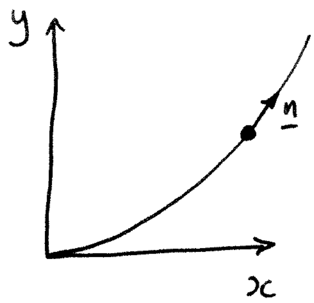
9 (a)

In general
$$d\varphi = d\underline{x}\cdot\nabla\varphi$$

Taking $d\underline{x} = \underline{n}\,ds$ ($\underline{n}$ unit, ds = distance)

$$\Rightarrow \quad d\varphi = ds\,\underline{n}\cdot\nabla\varphi$$

i.e. $$\frac{d\varphi}{ds} = \underline{n}\cdot\nabla\varphi$$



On curve $y = x^2$

$$\underline{n} \;//\; (dx, dy) = (dx, 2x\,dx)$$

$$\Rightarrow \underline{n} = \frac{(1, 2x)}{\sqrt{1+4x^2}} = \frac{1, 4}{\sqrt{17}} \quad \text{at } (2,4).$$

$$\nabla\varphi = \left(3x^2y + 2y^2 - \frac{1}{y},\; x^3 + 4xy + \frac{x}{y^2}\right)$$

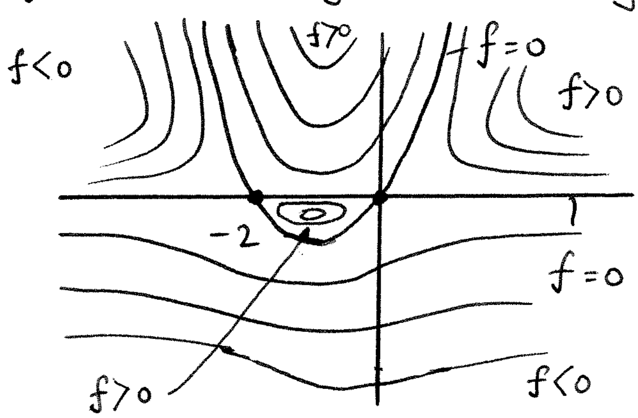
$$= \left(48 + 32 - \frac{1}{4},\; 8 + 32 + \frac{1}{8}\right) \quad \text{at } (2,4)$$

$$\therefore \quad \underline{n}\cdot\nabla\varphi = \frac{d\varphi}{ds} = \frac{79.75 + 4\times 40.125}{\sqrt{17}} = \underline{58.3}$$

(b) $$f = x^2y + 2xy - y^2 = y(x^2 + 2x - y)$$

$$f = 0 \Rightarrow \quad y = 0 \quad \text{or} \quad y = x^2 + 2x$$

parabola passing through $(0,0)$ & $(-2,0)$



9 contd) From the sketch it is clear that there are 3 stationary points and that 2 of these are saddle points at $(0,0)$ & $(-2,0)$. We expect the third to be a <u>maximum</u> and lie in the closed region between the parabola and $y=0$.

Stationary when $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$$\Rightarrow \quad 2xy + y = 0 \quad \text{AND} \quad x^2 + 2x - 2y = 0$$

$$\Rightarrow \quad y = 0 \text{ or } x = -1 \quad \text{AND} \quad 2y = x^2 + 2x$$

- $y = 0 \quad \Rightarrow \quad x^2 + 2x = 0 \quad$ ie. $\quad x = 0$ or -2
- $x = -1 \quad \Rightarrow \quad y = \frac{1-2}{2} = -\frac{1}{2}$.

Check using 2nd derivs & databook formulae

$$\frac{\partial^2 f}{\partial x^2} = 2y \qquad \frac{\partial^2 f}{\partial x \partial y} = 2x + 2 \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

At $(0,0)$ $\quad \frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial x \partial y} = 2 \quad \frac{\partial^2 f}{\partial y^2} = -2 \Rightarrow \Delta = f_{xx}f_{yy} - f_{xy}^2 < 0$

$(-2,0)$ $\quad \frac{\partial^2 f}{\partial x^2} = 0 \quad \frac{\partial^2 f}{\partial x \partial y} = -2 \quad \frac{\partial^2 f}{\partial y^2} = -2 \Rightarrow \Delta < 0$

$(-1,-\frac{1}{2})$ $\quad \frac{\partial^2 f}{\partial x^2} = -1 \quad \frac{\partial^2 f}{\partial x \partial y} = 0 \quad \frac{\partial^2 f}{\partial y^2} = -2$ & $\Delta > 0$

$\therefore$ <u>$(0,0)$, $(-2,0)$ saddles $(-1,-\frac{1}{2})$ maximum.</u>

<u>Examiner's Note:</u> Average Mark 11.3/20

- <u>Either</u> the sketch <u>or</u> the formulae are sufficient to classify the stationary points
- The <u>key</u> to this type of question is the sketch.

Engineering Tripos Part IA 2002
Paper 4 (Mathematical Methods) Section C
Model Answers

Note that the text of the questions has been abbreviated. The official full text is available separately.

Question 10

C10 a i.

Convert 3145728.0 to IEEE float. [5 marks]

This is an integer. Converting it to binary yields 11,0000,0000,0000,0000,0000, which is $1.1 \cdot 2^{21}$. The IEEE exponent is 21+127=148 (bias to allow for negative exponents). Including one bit for the sign, and dropping the first 1 of the normalized mantissa, we write it all down as 0 10010100 10000000000000000000000.

C10 a ii.

Consider this C++ code segment. Why do strange things happen with `tan(x)` and `tan(y)`? [4 marks]

The number 1.5707963 can't be represented exactly, neither as float nor as double, although the double will be a more accurate representation since it has twice as many bits. Same goes for the tangent.

Having said that, that number is almost $\pi/2$, whose tangent is infinite. So this is an ill-conditioned numerical problem. This is why the two results are so wildly different.

C 10 b i.

Describe the algorithm used in the supplied `FindRoot()` function and identify the role of the main variables. [5 marks]

The bisection method is used to find the root.

The solution is always between `low` and `high`. At each iteration, the search interval is halved, and we check whether the root is in the lower or upper half. We exit when the size of the interval is smaller than the required `precision` or when (if ever) we land on the exact root.

The condition inside the `if` statement checks whether the signs of `f_low` and `f_mid` are same or opposite. Meaning: if opposite signs, then the root is between `low` and `mid`.

The main variables are as follows:

- `low` and `high` define the interval for the root.
- `mid` is the midpoint of that interval.
- `f_low` and `f_mid` are the images of `low` and `mid` via the function $f(x) = \texttt{square} - x^2$.
- `precision` is the upper bound for the error we accept on the solution.
- `square` is the number of which we wish to find the root.

C 10 b ii.

What is the accuracy of the solution and how many times will the `while`-loop execute? [3 marks]

Assuming the input is within range, the maximum distance between the solution and the actual root will be half the size of the final interval, i.e. 0.000005.

The initial interval is from 0.1 to 10. This is halved at every iteration, and the program stops when the size of the interval goes below 0.00001. This happens in $\lceil \log_2 \frac{10-0.1}{0.00001} \rceil = 20$ iterations. (It may stop earlier if it hits on the solution exactly, but this is statistically unlikely.)

Regardless of the input value, and even if the input is out of range, the program will still stop after at most 20 iterations—it won't loop forever.

C 10 b iii.

In what range must the root lie to get a solution? When will the algorithm fail to find the correct solution? [3 marks]

A solution will always be produced no matter what. However, for a *correct* one, the root r (not the input!) must be such that $0.1 < r < 10$.

Inputs (not roots!) outside the range 0.01 to 100, including of course negative ones, will fail to produce a correct solution.

Question 11

C 11 a i.

Identify bugs in the supplied `LeastSquares()` function and correct them. [8 marks]

- The variables `sum_...` are never initialized. They should be initialized to 0.
- The `for` loop has an off-by-one error: it should count from 0, not from 1, or it will miss the first element of the array.
- The semicolon at the end of the header (1st line) shouldn't be there.
- The semicolon at the end of the `for` line shouldn't be there, or the bit that follows will be taken to be outside the body of the loop.
- A close curly bracket is missing before the `data.a =...` section, to conclude the body of the loop.
- The `a` in the last expression is a typo for `data.a`.
- The `data` variable is passed by value, but should be passed by reference (`ModelData& data` in the header) otherwise no result will be returned to the calling program.

C 11 a ii.

Write a C++ definition for `ModelData`. [4 marks]

```
#define SIZE 100
struct ModelData {
    int n;
    float x[SIZE];
    float y[SIZE];
    float a;
    float b;
};
```

C 11 b.

What is meant by algorithmic complexity? [3 marks]
An estimate of the running time of the algorithm as a function of the size of the input when the size of the input tends to infinity. Only the dominant term is taken into account, ignoring smaller ones and multiplicative constants.

C 11 b i.

What is the complexity of the least-squares algorithm? [2 marks]
A constant amount of computation is performed for each of the n points, so $O(n)$.

C 11 b ii.

Describe an algorithm for sorting the pairs and state its complexity. [3 marks]
The exchange-sort algorithm sorts a list of n elements using n passes. In the first pass we look at the whole list, find its smallest element, and put it in the first position, exchanging it with whatever was there. At each successive pass we look at the rest of the list, find the minimum of that and put it in the first position of the sublist. The loop invariant is that, at the end of pass i, the first i elements are sorted.

The complexity is $O(n^2)$ because the operation of finding the smallest element of a sublist is $O(n)$ and it has to be repeated once for each of the n passes.