ENGINEERING TRIPOS PART IA

Tuesday 11 June 2002 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **three** *may be taken from Section A, not more than* **four** *from section B and not more than* **one** *from Section C.*

All questions carry the same number of marks.

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer not more than **three** questions from this section.*

1 (a) $\underline{a}$, $\underline{b}$ and $\underline{c}$ are the position vectors of points A, B and C which define a plane.

(i) Show that a vector normal to the plane is given by:

$$\underline{n} = \underline{a}\times\underline{b} + \underline{b}\times\underline{c} + \underline{c}\times\underline{a}$$ [5]

(ii) Find an expression in terms of $\underline{a}$, $\underline{b}$ and $\underline{c}$ for the distance of the plane to the origin. [3]

(iii) Find the position of the point on the plane nearest to the origin. [2]

(b) Find the direction of the line of intersection of the two planes given by:

$$x + 2y - z = 2$$
$$3x + y - 2z = 0$$ [5]

Hence find the equation of the line of intersection in the form:

$$\frac{x - a_1}{b_1} = \frac{y - a_2}{b_2} = \frac{z - a_3}{b_3}$$ [5]

2 (a) Find the limit as $x \to 0$ of:

$$\frac{\sin x - \tan x}{(1-\cos x)\left(1-e^{-x}\right)}$$ [4]

(b) Sketch the curves on the Argand plane defined by the equations:

(i) $|z-a| = b$ where a and b are positive real numbers [3]

(ii) $|z+1| = |z|$ [3]

(c) Express the following complex numbers in the form $r\,e^{i\theta}$:

(i) $z = (4+3i)^{1/3}$ [4]

(ii) $z = \tanh\left(\frac{\pi}{6} + i\,\frac{\pi}{4}\right)$ [6]

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3 (a) (i) Find and sketch the complementary function for the differential equation:

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} + x = \sin \omega_0 t$$

for $t > 0$. [6]

(ii) Find a particular integral for $\omega_0 = \frac{\sqrt{3}}{2}$ and sketch the general solution as $t \to \infty$. [4]

(b) The Fibonacci series is defined by the difference equation:

$$a_{n+1} = a_n + a_{n-1}$$

with $a_0 = 0$ and $a_1 = 1$.

(i) Find a_n. [6]

(ii) Show that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \frac{1+\sqrt{5}}{2}$$ [4]

4 Figure 1 shows the result of applying the linear transformation $\underline{x}' = \mathbf{A}\underline{x}$, described by the 2×2 symmetric matrix **A**, to the triangle *OBC*.

(a) Find the elements of the matrix **A**. [4]

(b) Show, by computing the determinant of **A**, that the area of the triangle is unchanged by the transformation. What does this imply about the eigenvalues of **A** ? [5]

(c) Show that the direction which is at 30° anti-clockwise to the positive *x*-axis is an eigenvector of **A** and find the other eigenvector and both eigenvalues of **A**. [5]

(d) Find the approximate position to which point *C* will move after 100 applications of the linear transformation. [6]

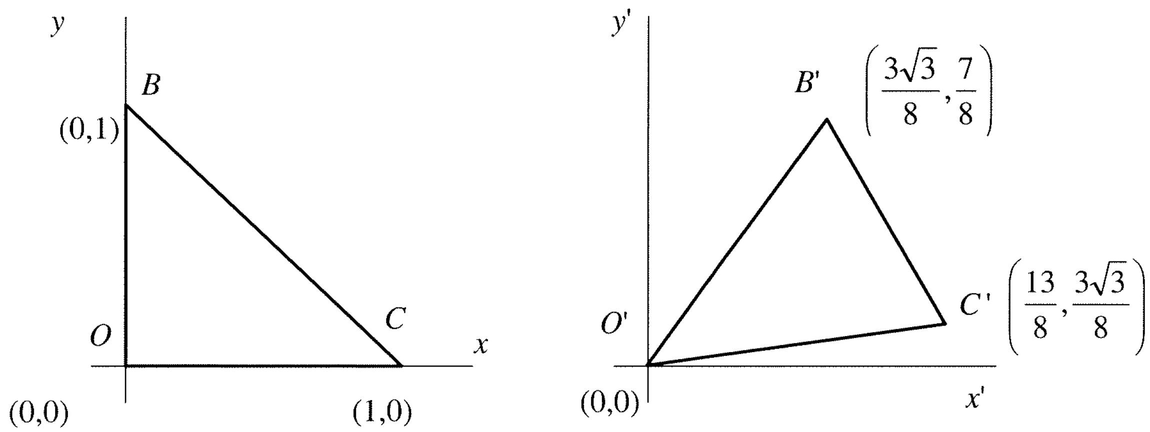

Fig. 1

SECTION B

*Answer not more than **four** questions from this section.*

5 (a) The step response of a linear system is given by:

$$y(t) = \begin{cases} 0 & t < 0 \\ e^{-t} - e^{-2t} & t \geq 0 \end{cases}$$

Find the impulse response of the system. [3]

Sketch both the step and impulse responses. [3]

(b) Using a convolution integral, find the response of the system to an input:

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-\alpha t} & t \geq 0 \end{cases}$$

where α is a non-negative constant. [8]

(c) Find the system response for the limiting case $\alpha \rightarrow 0$ and interpret this result. [6]

6 (a) Find the half-range sine wave Fourier Series representation of the function:

$$f(x) = x(\pi - x) \qquad 0 \le x \le \pi$$

which is valid in the range $0 \le x \le \pi$. [6]

Sketch the function $F(x)$, which is equal to the sum of the series, for the range $-4\pi \le x \le 4\pi$ and comment on the behaviour of the coefficients as $n \to \infty$. [4]

(b) Explain how the coefficients in a Fourier Series representation of a function $y(x)$ are related to those in the representation of $\dfrac{d^2y}{dx^2}$. [4]

Hence, or otherwise, find the Fourier Series representation of the function $y(x)$ which satisfies the differential equation:

$$\frac{d^2y}{dx^2} + k^2 y = x(\pi - x) \qquad 0 < x < \pi$$

where k is a non-integer positive constant, subject to the boundary conditions $y(0) = y(\pi) = 0$. [6]

7 (a) Explain briefly the circumstances under which probabilities can be:

(i) added;

(ii) multiplied. [4]

(b) A robot is used to insert a memory chip during the production-line assembly of a printed circuit board. 1.000 % of memory chips are supplied with faulty connectors. A new robot fits 99.999 % of correctly-manufactured chips correctly and 50.000 % of the faulty ones correctly, despite the connector imperfections. When a robot's manipulator bearing becomes worn, these figures become 99.900 % and 12.500 % respectively.

The proportion of circuit boards with incorrectly fitted memory chips produced on a large number of production lines is observed to be 0.532 %.

(i) Sketch a tree diagram which applies to the process of fitting a chip. [5]

(ii) Hence calculate the probability that a given robot has a worn bearing. [7]

(iii) If the recorded data only indicates that the proportion of circuit boards with incorrectly fitted chips is between 0.53 % and 0.54 %, estimate the uncertainty in your answer to (ii). [4]

8 Solve, using Laplace transforms, the simultaneous differential equations:

$$\dot{x} + 4x - 5y = 5e^{t}$$
$$\dot{y} - \dot{x} + y = 0$$

for $t > 0$, when $x(0) = 1$, $y(0) = 0$ [14]

Verify that the solutions you obtain for x and y satisfy the differential equations and boundary conditions. [6]

9 (a) Explain how the rate of change of a function $\phi(x, y)$ with distance along the direction given by the unit vector $\underline{n}$ is related to $\nabla\phi$ [3]

Find the rate of change with distance of the function:

$$\phi = x^3 y + 2xy^2 - \frac{x}{y}$$

along the curve $y = x^2$ at the point (2,4) in the sense of x increasing. [6]

(b) Sketch contour lines for the function:

$$f(x, y) = x^2 y + 2xy - y^2$$ [5]

Find the position and nature (i.e. maximum, minimum or saddle point) of each stationary point of f. [6]

SECTION C

*Answer not more than **one** question from this section.*

10 (a) (i) Convert 3145728.0 into IEEE standard single precision floating point format. [5]

(ii) Consider the following C++ code segment:

```
float x    = 1.5707963;
double y   = 1.5707963;

cout << "tan(x) is " << tan(x) << endl;
cout << "tan(y) is " << tan(y) << endl;
```

When the code is executed, the console displays:

```
tan(x) is 1.32454e+07
tan(y) is 3.73205e+07
```

Explain the difference between the two results. [4]

(b) Figure 2 lists the C++ code for a function called `FindRoot()` which is used to find the square root of a positive single precision floating point number without using the Math library function.

(i) Describe briefly the algorithm being used to find the root and identify the role of the key variables. [5]

(ii) What is the accuracy of the solution and how many times will the while-loop execute ? [3]

(iii) In what range must the root lie for the `FindRoot()` function to return a solution ? When will the algorithm fail to find the correct solution ? [3]

(cont.

```
//FindRoot function definition
//
float FindRoot(float square)
{
  float precision = 0.00001;
  float low = 0.1, high = 10.0, mid;
  float f_low, f_mid;
  bool converged = false;

  while (!converged)
    {
      mid = 0.5*(low+high);
      f_low = square - low*low;
      f_mid = square - mid*mid;
      if( f_mid==0.0 ) return mid;

      if( f_low*f_mid<0.0 ) high = mid;
      else low = mid;

      converged = ((high-low)<precision);
    }
  return mid;
}
```

Fig. 2

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11 (a) The C++ function definition in Fig. 3 attempts to use the method of least-squares to fit a straight line $y = a + b\,x$ to a discrete set of points:

$$(x_1,y_1)\ (x_2,y_2)\ \ldots\ (x_n,y_n)$$

using the formulae in the Mathematics Data Book. The definition contains bugs.

(i) Identify each of the bugs and explain their consequences. State how to correct them. [8]

(ii) Write a suitable C++ definition for the data structure `ModelData` which is passed to the function. [4]

(b) What is meant by algorithmic complexity ? [3]

(i) What is the algorithmic complexity of the least-squares algorithm ? [2]

(ii) Describe an algorithm for sorting the pairs (x_1,y_1) (x_2,y_2) ... (x_n,y_n) in part (a) in the order of increasing x-value and state the complexity of the algorithm. [3]

(cont.

```
void LeastSquares( ModelData data);
{
  float sum_x, sum_xx, sum_xy, sum_y;
  for( int i=1; i<data.n; i++ );
    {
      sum_x = sum_x + data.x[i];
      sum_y = sum_y + data.y[i];
      sum_xx = sum_xx + data.x[i]*data.x[i];
      sum_xy = sum_xy + data.x[i]*data.y[i];

      data.a = (sum_xx*sum_y - sum_x*sum_xy)/
               (sum_xx*data.n - sum_x*sum_x);
      data.b = (sum_y - data.n*a)/sum_x;
}
```

Fig. 3

END OF PAPER