ENGINEERING TRIPOS PART IA

Tuesday 10 June 2003 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than ***eight*** *questions, of which not more than* ***three*** *may be taken from Section A, not more than* ***four*** *from section B and not more than* ***one*** *from Section C.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer not more than **three** questions from this section.*

1 A regular octahedron has vertices at the points $(a, \pm a, 0)$, $(-a, \pm a, 0)$ and $(0, 0, \pm h)$ in a right-handed co-ordinate system with origin O.

(a) Sketch the octahedron and find the necessary relation between h, the height of the octahedron, and a. [25%]

(b) Determine:

(i) the equation of the plane corresponding to the octahedron face that lies entirely in the positive xz quadrant, i.e. $(x \geq 0, z \geq 0)$;

(ii) the unit normal to this plane, and hence find the shortest distance from the origin to this face;

(iii) the angle between the plane in (i) and an adjacent face in the positive z region. [60%]

(c) Write down the matrix $Q(\theta)$ that describes a rotation of a vector by an angle θ about the z axis. For a 45° rotation, using $Q(\theta)$ find the new vertices of the octahedron in terms of a. [15%]

2 (a) Find the following limits using only one application of l'Hopital's rule and one application of power series approximation:

(i) $$\lim_{x\to 0} \frac{x^3}{x - \sin x \cos x}$$

(ii) $$\lim_{x\to 1} \frac{1+\cos \pi x}{\tan^2 \pi x}$$ [50%]

(b) Find all solutions of the following equations and plot them on one Argand diagram:

(i) $$z^4 + 4 = 0$$

(ii) $$z^6 - z^3 + 1 = 0$$ [50%]

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3 (a) Find the solution of the following differential equation:

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{-x}$$

which satisfies the boundary conditions:

$$\frac{dy}{dx} = y = 0 \quad \text{at} \quad x = 0$$ [40%]

(b) Find the solution of the following differential equation:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x$$

which satisfies the boundary conditions:

$$y = 1 \ \text{at} \ x = 0 \qquad \text{and} \qquad \frac{dy}{dx} = 2 \ \text{at} \ x = 0$$ [60%]

4 A symmetric 3×3 matrix A is given as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix}$$

(a) Give brief definitions of orthogonal and orthonormal vectors as well as an orthogonal matrix. Show that the matrix A above is not orthogonal. [20%]

(b) Determine the eigenvalues and corresponding eigenvectors of A. Given that Λ is a diagonal matrix whose elements are the eigenvalues of A, state a relation between A and Λ that enables the original matrix A to be calculated from its eigenvalues and eigenvectors. Verify the relation you stated by calculating A. [60%]

(c) Calculate A^{∞}. [20%]

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SECTION B

*Answer not more than **four** questions from this section.*

5 (a) Describe the relationship between the impulse and step responses of a linear system. [20%]

(b) The input $x(t)$ and the output $y(t)$ of a linear system satisfy the differential equation

$$\frac{dy}{dt} + 3y = x$$

Show that the impulse response of this system is given by

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-3t} & t \geq 0 \end{cases}$$ [20%]

(c) Using a convolution integral, find the response of the system to inputs

(i)

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases}$$ [20%]

(ii)

$$x(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & 0 \leq t \leq T \\ 0 & t \geq T \end{cases}$$ [20%]

(d) For the case $T = \frac{1}{2}$, sketch the two responses found for part (c). [20%]

6 (a) Sketch the function

$$f(t) = \begin{cases} 1 + \dfrac{2t}{\pi} & -\pi \le t < 0 \\ 1 - \dfrac{2t}{\pi} & 0 \le t \le \pi \end{cases}$$ [15%]

(b) The function is represented over the range $-\pi \le t \le \pi$ by the Fourier Series

$$f(t) = \sum_{n=1}^{\infty} a_n \cos nt$$

Using your sketch, explain why there are no sine terms in this representation and also why there is no constant term. [15%]

(c) Find the coefficients a_n in the Fourier Series representation of f. [30%]

(d) Comment on the behaviour of a_n as $n \to \infty$ and explain which properties of f are responsible for this. [20%]

(e) Identify the values of n for which the a_n are zero and explain, with the aid of sketches, why this is so. [20%]

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7 (a) Explain the meaning of the terms *exclusive* and *independent* when applied to events and the consequences for probabilities of those events. [20%]

(b) Players A and B play a game in which they roll alternately a six-sided die, with faces marked from 1 to 6, until one of them wins the game by throwing a 6. Show that for a game consisting of two rolls each, with A going first, the probabilities of each player winning are given by

$$\mathrm{P}(A \text{ wins}) = \frac{61}{216} \quad \text{and} \quad \mathrm{P}(B \text{ wins}) = \frac{305}{1296}$$ [20%]

(c) The game is extended (with A still going first) so that they continue playing until one player wins.

Find P(A wins) and P(B wins). [25%]

(d) B proposes to use a die which has been loaded to show a 6 with probability p. If A throws first and uses the unloaded die, B uses the loaded one and the game continues until one player wins, find P(A wins) and P(B wins). [25%]

(e) Estimate the value of p which ensures a fair game (i.e. that each player has an equal probability of winning). [10%]

8 (a) Solve using Laplace Transforms

$$\frac{dy}{dt} + 2y = e^{-t}\cos t \qquad \text{for } t > 0$$

where $y(0) = 1$. [70%]

(b) Verify that your solution satisfies the differential equation and the boundary condition. [30%]

9 (a) Find the function $\phi(x, y)$ for which

$$\nabla\phi = \left(\frac{2x}{y} + 2y^2 + 3 \,,\, -\frac{x^2}{y^2} + 4xy + \cos y \right)$$

and $\phi(\,0\,,1\,) = 1$. [30%]

(b) (i) Sketch contours of the function f given by

$$f(x, y) = xy\,(x - y + 2)$$ [20%]

(ii) Find and classify the stationary points of f and mark them on your sketch. [50%]

SECTION C

*Answer not more than **one** question from this section.*

10 The C++ function, `Partition`, in Fig. 1 partitions an array `list[]` into two halves such that:

```
list[lo]..list[k-1] < list[k] < list[k+1]...list[hi]
```

where the central item, `list[k]`, is called the key. All values in the lower sub-array are below the key value and all the values in the higher sub-array are above the key value. `Partition` is a sub-function that is commonly used within the QuickSort algorithm.

(a) Write a commented `QuickSort` function that calls the function `Partition` in Fig. 1 and uses a recursive algorithm. Briefly explain the algorithm and write down a sample function call to `QuickSort` to sort an array `list` with `n` elements. [30%]

(b) The `QuickSort` function is used to sort a list of 100 structural sections by cross-sectional area for a new bridge design. Write down the data type definition for `Section` that includes separate fields for `section_label`, `index`, `cross_sectional_area` and `length`, and then declare a variable `section_list` of type `Section` that defines an array to store the data for 100 sections. [25%]

(c) What changes would need to be made to the functions `Partition` and `QuickSort` to be able to sort the array `section_list` by `cross_sectional_area`? Why could the algorithm fail? [30%]

(d) What is the algorithmic complexity of the recursive QuickSort algorithm? Would it be less time efficient to sort `section_list` by bubble sort or exchange sort rather than QuickSort and why? [15%]

(cont.

```
int Partition (item list[], int lo, int hi)
// Partition function for QuickSort
{
    item x, tmp;

    x = list[(lo+hi)/2.0]; //Key value

    while (lo < hi)
    {
        while ((lo < hi) && (x < list[hi]))
        hi--;
        while ((lo < hi) && (x > list[lo]))
        lo++;

        // Swap list[hi] and list[lo]
        tmp = list[hi];
        list[hi] = list[lo];
        list[lo] = tmp;
    }
    return lo;
}
```

Fig. 1

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11 (a) Define the machine accuracy, ε_m, of a floating point representation. [10%]

(b) The value of π can be computed using the following equation:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Figure 2 shows a C++ program that computes an estimate for π based on the first n terms in the series where n is entered by the user from the keyboard. The program contains a number of errors. Identify five errors and suggest corrections. [50%]

(c) The value of the machine precision for floats is $\varepsilon_m = 6 \times 10^{-8}$. When the corrected program is run, it is found that if the user enters 10000 they get exactly the same estimate for π as when they enter 9000. Why does this happen and at approximately what number of terms does the estimate for π stop increasing? [30%]

(d) What could be done to fix the problem identified in part (c)? [10%]

(cont.

```
#include<iostream>
#include<cmath>
using namespace std;

int main(){
  float term, total, estimate;
  int i;

  cin >> num_terms;

  for(i=0; i<=num_terms; i++){
    term = 1/(i*i);
    total = total + term;
    }

  estimate = sqrt(total/6);

  cout.setprecision(8);
  cout <<"estimate is"<< estimate << endl;
}
```

Fig. 2

END OF PAPER