Mathematics IA (Paper 4) – 2004

Section A

Q1(a) Vectors / geometry

(i) Consider 2 vectors in plane ABC, $\vec{AB}$ and $\vec{AC}$

$$\underline{n} = (\vec{AB} \times \vec{AC}) = (\underline{b}-\underline{a}) \times (\underline{c}-\underline{a})$$

marks out of 2

$$\underline{n} = \underline{a} \times \underline{b} + \underline{b} \times \underline{c} + \underline{c} \times \underline{a} \qquad (4)$$

(ii) Shortest distance, d

$$d = \underline{a} \cdot \hat{\underline{n}} = \underline{a} \cdot \frac{\underline{n}}{|\underline{n}|} = \frac{\underline{a} \cdot (\underline{b} \times \underline{c})}{|\underline{a} \times \underline{b} + \underline{b} \times \underline{c} + \underline{c} \times \underline{a}|} \qquad (2)$$

(iii) Area of triangle ABC

$$= \frac{1}{2} |\vec{AB}| |\vec{AC}| \sin\theta = \frac{1}{2} |(\underline{b}-\underline{a}) \times (\underline{c}-\underline{a})|$$

$$= \frac{1}{2} |\underline{n}|$$

$$= \frac{1}{2} | \underline{a} \times \underline{b} + \underline{b} \times \underline{c} + \underline{c} \times \underline{a} | \qquad (2)$$

[note: area is a scalar]

(iv) Volume of tetrahedron, $V = \frac{1}{3} \times$ area of base, ABC $\times$ height

$$= \frac{1}{3} \frac{1}{2} \cancel{|\underline{n}|} \left(\frac{\underline{a} \cdot \underline{b} \times \underline{c}}{\cancel{|\underline{n}|}} \right)$$

$$V = \frac{1}{6} \underline{a} \cdot (\underline{b} \times \underline{c}) \qquad (2)$$

(v) If $\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$ then ABC is a plane through origin. (2

($\underline{a}$, $\underline{b}$ and $\underline{c}$ are coplanar).

Q1(b)

$\underline{l}$, direction of line of intersection $= \underline{n}_1 \times \underline{n}_2$ where $\underline{r}.\underline{n}_1 = d_1$ and $\underline{r}.\underline{n}_2 = d_2$

$$= \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= (-1)\underline{i} - \underline{j}(1) + \underline{k}(3)$$

$\underline{d}$, point on line of intersection $= \begin{pmatrix} 0 \\ 4 \\ -6 \end{pmatrix}$ or $\begin{pmatrix} -2 \\ 2 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix}$

$$\therefore \underline{r} = \begin{pmatrix} 0 \\ 4 \\ -6 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -3 \end{pmatrix}$$

(8).

Q2(a) limits

(i) $\lim_{x \to 0} \left[\frac{\tan x - x}{x - \sin x} \right] = \lim_{x \to 0} \left[\frac{x + \frac{x^3}{3} + O(x^5) - x}{x - \left(x - \frac{x^3}{3!} + O(x^5) \right)} \right]$

$= \frac{\frac{\cancel{x^3}}{3}}{\frac{\cancel{x^3}}{6}} = 2$ (3)

(ii) $\lim_{x \to \frac{1}{6}} \left[\frac{1 - 2\sin \pi x}{1 - 6x} \right] = \lim_{x \to \frac{1}{6}} \left[\frac{-2\pi \cos \pi x}{-6} \right]$ by L'hôpitals rule

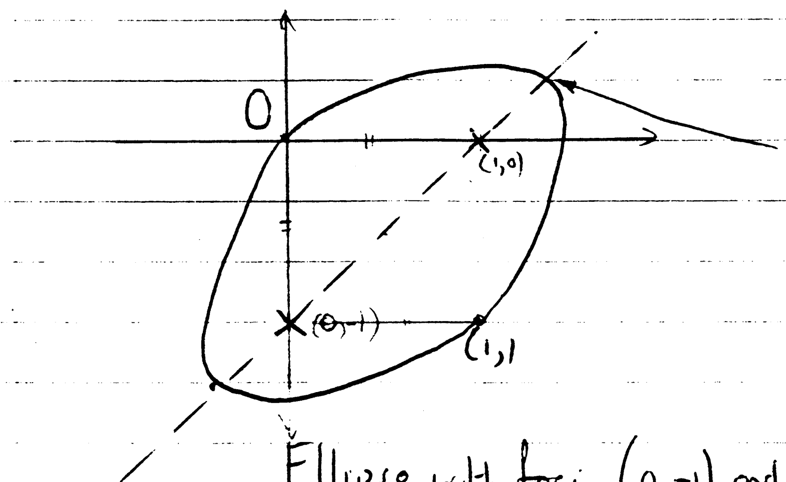
$= \frac{\cancel{2}\pi}{6} \cdot \frac{\sqrt{3}}{\cancel{2}}$

$= \frac{\pi}{2\sqrt{3}}$ (3)

(b) Complex numbers:

Sketch $|z+i| + |z-1| = 2$

In argand plane curve traced out so that distance to $(0,-1)$ and $(1,0)$ sum to 2. Passes through $(0,0)$ and $(1,1)$.
and sym[illegible] $y = x$.



(6)

Ellipse with foci $(0,-1)$ and $(1,0)$ going through origin.

(c)(i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$ by De Moivre (4)

$\therefore \sin 4\theta = \text{Im}\left[\cos\theta + i\sin\theta\right]^4 = 4\cos^3\theta\sin\theta - 4\cos\theta\sin^3\theta$

(ii) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^i = \left(e^{i\frac{\pi}{4}}\right)^i = e^{-\frac{\pi}{4}}$ (4)

Q3(a) Differential equations

(i). Solve characteristic equation $\lambda^2 + 4\lambda + 4 = 0$

$$(\lambda+2)^2 = 0$$

$$\lambda = -2, -2 \quad \text{(repeated root)}$$

Try complementary function, $\underline{x_{CF}(t) = (At+B)\,e^{-2t}}$

(ii) Try particular integral, $x(t) = C\sin 2t + D\cos 2t.$

$$\dot{x}(t) = 2C\cos 2t - 2D\sin 2t$$

$$\ddot{x}(t) = -4C\sin 2t - 4D\cos 2t$$

$\therefore$ $-4C - 8D + 4C = 0$

$-4D + 8C + 4D = 1$

$\therefore$ $C = \frac{1}{8}$, $D = 0$

$$\underline{x_{PI}(t) = \frac{1}{8}\sin 2t}$$

(iii) General solution

$$x(t) = \overbrace{(At+B)\,e^{-2t}}^{CF} + \overbrace{\frac{1}{8}\sin 2t}^{PI}$$

Apply boundary conditions $x = 0,\ t = 0$ $\quad B = 0$

$\dot{x} = 0,\ t = 0$ $\quad A + \frac{1}{4} = 0$

$$\therefore \quad \underline{\underline{x(t) = -\frac{t}{4}e^{-2t} + \frac{\sin 2t}{8}}}$$

(10)

Q3b (difference equations)

(b)(i) λ_1 and λ_2 satisfy:

$$\lambda^2 - \lambda - 1 = 0$$

$$\underline{\lambda = \frac{1}{2} \pm \frac{\sqrt{5}}{2}}$$

(4)

$$\underline{\lambda_1 = \frac{1}{2} + \frac{\sqrt{5}}{2}, \quad \lambda_2 = \frac{1}{2} - \frac{\sqrt{5}}{2}}$$

(3)

(ii) A_n, area after nth iteration $= L_n(L_n + L_{n-1})$
$= \underline{L_n(L_{n+1})}$ by difference equat

(iii) $$\lim_{n\to\infty} \frac{A_{n+1}}{A_n} = \frac{L_{n+2} \cdot \cancel{L_{n+1}}}{\cancel{L_{n+1}} \cdot L_n} = \frac{L_{n+2}}{L_n}$$

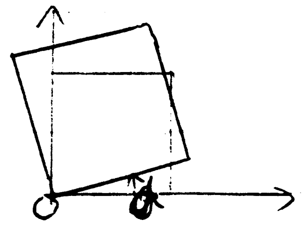
$$= \underline{\lambda_1^2} \quad \text{since } \lambda_2^n \to 0$$

$$= \underline{\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^2}$$

(3)

Q4 Matrices / transformations

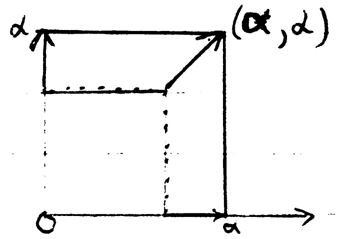
(a)(i) A_1



anti-clockwise rotation, area unchanged

$\underline{\det A_1 = 1}$

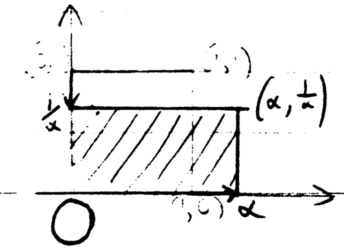
A_2



isotropic scaling, area changed by α^2.

$\underline{\det A_2 = \alpha^2}$

A_3



expansion of x-component by α, contraction of y-component by α preserving area.

$\underline{\det A_3 = 1}$

(4)

(ii) For A_2, eigenvalues are repeated $\lambda = \alpha, \alpha$
any vector is an eigenvector since $A\underline{x} = \lambda \underline{x}$
(choose $\underline{u}_1$ and $\underline{u}_2$ to be orthogonal, e.g. $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$)

For A_3, eigenvalues are $\lambda_1 = \alpha$ and $\lambda_2 = \frac{1}{\alpha}$ corresponding to $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$

since $A_3 \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}\alpha\\0\end{bmatrix}$ and $A_3 \begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\\frac{1}{\alpha}\end{bmatrix}$

(3)

(iii) $\underline{A_2^{100} \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\alpha^{100}x\\\alpha^{100}y\end{bmatrix}}$

$\underline{A_3^{100} \begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}\alpha^{100}x\\\alpha^{-100}y\end{bmatrix} \simeq \begin{bmatrix}\alpha^{100}x\\0\end{bmatrix}}$ for $\alpha > 1$

(3)

Q4(b)

(i) $\underline{y} = A\underline{x}$

$$|\underline{y}|^2 = \underline{y}\cdot\underline{y} = \underline{y}^T\underline{y} = (A\underline{x})^T(A\underline{x})$$
$$= \underline{\underline{x}^T A^T A^T \underline{x}}$$

$$|\underline{x}|^2 = \underline{x}\cdot\underline{x} = \underline{x}^T\underline{x}$$

$$\therefore \underline{\frac{|\underline{y}|^2}{|\underline{x}|^2} = \frac{\underline{x}^T A^T A\underline{x}}{\underline{x}^T\underline{x}}} \qquad (4)$$

(ii) $(A^TA)^T = A^TA \quad \therefore$ A^TA is a symmetric matrix (2)

(iii) Since A^TA has real elements and is symmetric, eigenvectors will be orthogonal and eigenvalues are real.

(2)

(iv) $\dfrac{|\underline{y}|^2}{|\underline{x}|^2} \geq \lambda_1$ where λ_1 is the smallest eigenvalue of A^TA.

(2)

Note: $\dfrac{|y|^2}{|x|^2} = \dfrac{x^TA^TAx}{x^Tx}$

$$= \frac{\alpha^2\lambda_1 + \beta^2\lambda_2 + \gamma^2\lambda_3}{\alpha^2+\beta^2+\gamma^2}$$

let $\underline{x} = \alpha\underline{u}_1 + \beta\underline{u}_2 + \gamma\underline{u}_3$

then $ATA\,\underline{x} = \alpha\lambda_1\underline{u}_1 + \beta\lambda_2\underline{u}_2 + \gamma\lambda_3\underline{u}_3$

$$\underline{\lambda_1 \leq \frac{|\underline{y}|^2}{|\underline{x}|^2} \leq \lambda_3 \qquad \text{if } \lambda_1 < \lambda_2 < \lambda_3}$$

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PART IA 2004
PAPER 4 <u>Section B</u>

5 a) $\delta(t) \longrightarrow$ Linear System $\longrightarrow g(t)$

If input is delayed by $\tau \Rightarrow$ output delayed by τ

$$\Rightarrow \quad \delta(t-\tau) \longrightarrow g(t-\tau)$$

Any arbitrary input $x(t)$ can be written as a sum of impulses

$$x(t) = \int_0^t x(\tau)\,\delta(t-\tau)\,d\tau \qquad (\text{assuming } x(t)=0 \quad t<0)$$

Since the system is linear

$$x(\tau)\,\delta(t-\tau) \longrightarrow \boxed{} \longrightarrow x(\tau)\,g(t-\tau)$$

and

$$\int_0^t x(\tau)\,\delta(t-\tau) \longrightarrow \boxed{} \longrightarrow \int_0^t x(\tau)\,g(t-\tau)\,d\tau$$

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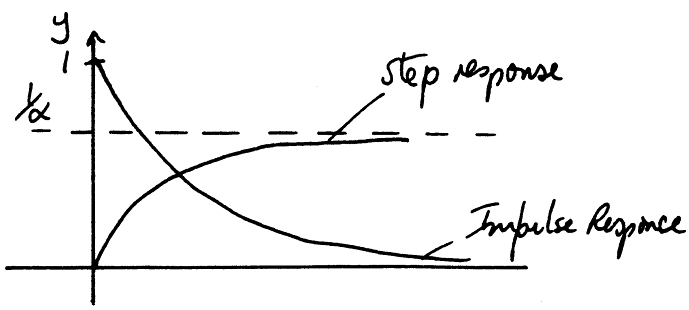
b) Step response: $\dfrac{dy}{dt} + \alpha y = 1 \quad \Rightarrow \quad y = \underset{\text{"P.I."}}{\dfrac{1}{\alpha}} + \underset{\text{"C.F."}}{Ae^{-\alpha t}}$

B.C. $y(0) = 0 \quad \Rightarrow \quad A = -\dfrac{1}{\alpha}$

$$\therefore \quad y(t) = \frac{1-e^{-\alpha t}}{\alpha} \qquad (t>0 \quad , y=0 \quad t<0)$$

Impulse response $= \dfrac{d}{dt}(\text{step response}) = e^{-\alpha t} \qquad (t>0 \ , y=0 \ t<0)$



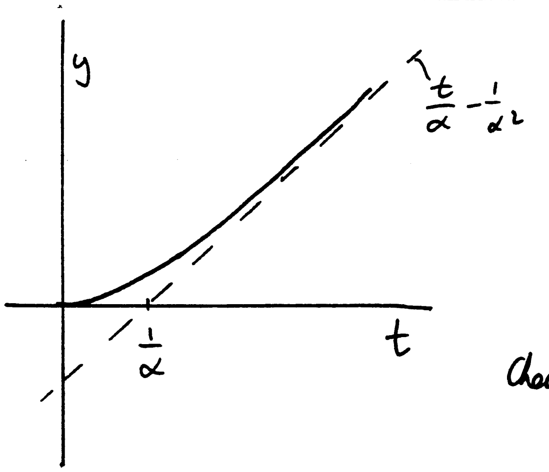
(c) $$y(t) = \int_0^t x(\tau) g(t-\tau)\,d\tau = \int_0^t \tau e^{-\alpha(t-\tau)}\,d\tau$$

$$= e^{-\alpha t}\left\{\left[\frac{\tau e^{\alpha\tau}}{\alpha}\right]_0^t - \int_0^t \frac{e^{\alpha\tau}}{\alpha}\,d\tau\right\}$$

$$= e^{-\alpha t}\left\{\frac{t e^{\alpha t}}{\alpha} - \left[\frac{e^{\alpha\tau}}{\alpha^2}\right]_0^t\right\}$$

$$= \frac{t}{\alpha} - \frac{1}{\alpha^2} + \frac{e^{-\alpha t}}{\alpha^2}$$

d)



The input $x(t) = t$ is the integral of a unit step

$\Rightarrow$ $y(t)$ = integral of step response.

Check: $\frac{dy}{dt} = \frac{1}{\alpha} - \frac{e^{-\alpha t}}{\alpha}$ = step response ✓

Examiner's Note

A straight-forward question - well done by most candidates.

Common errors:

1) Step response means <u>unit</u> step response, there should be no arbitrary constants in either the step response or impulse response.

2) Some candidates confused x and t

N.B. $$\underset{\text{fn}(t)}{\frac{dy}{dt}} + \underset{\text{fn}(t)}{\alpha y} = \underset{\text{fn}(t)}{x}$$

$y = ax + b$ is <u>not</u> a P.I. for this equation

You can not find a P.I. until you know how x depends on t.

6 a) $f(t) = 1 - \frac{2t}{\pi} \quad 0 < t < \frac{\pi}{2} \qquad f = 0$ otherwise.

$$\therefore a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\int_0^{\pi/2}\left(1 - \frac{2t}{\pi}\right)\cos nt\,dt$$

$$= \frac{1}{\pi}\left[\left(1-\frac{2t}{\pi}\right)\frac{\sin nt}{n}\right]_0^{\pi/2} - \frac{1}{\pi}\int_0^{\pi/2} -\frac{2}{\pi}\frac{\sin nt}{n}\,dt$$

$$= \frac{2}{n\pi^2}\left[-\frac{\cos nt}{n}\right]_0^{\pi/2} = \underline{\frac{2}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)}$$

$$\underline{a_0 = \frac{1}{\pi}\left\{\frac{\pi}{2}\cdot\frac{1}{2}\right\} = \frac{1}{4}} \qquad \left(\frac{a_0}{2} = \text{average value} = \text{area under } \Delta \div 2\pi\right)$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\int_0^{\pi/2}\left(1-\frac{2t}{\pi}\right)\sin nt\,dt$$

$$= \frac{1}{\pi}\left[\left(1-\frac{2t}{\pi}\right)\frac{-\cos nt}{n}\right]_0^{\pi/2} + \frac{1}{\pi}\int_0^{\pi/2} -\frac{2}{\pi}\frac{\cos nt}{n}\,dt$$

$$= \frac{1}{n\pi} - \frac{2}{n^2\pi^2}\left[\sin nt\right]_0^{\pi/2} = \underline{\frac{1}{n\pi} - \frac{2}{n^2\pi^2}\sin\frac{n\pi}{2}}$$

b) (i) $g(t)$ is an <u>even</u> fn $\therefore$ only cosine terms

and for g "a_n" $= \frac{1}{\pi}\int_{-\pi}^{\pi} g(t)\cos nt\,dt = 2\cdot\frac{1}{\pi}\int_0^{\pi/2} g(t)\cos nt\,dt$

$= 2\cdot\frac{1}{\pi}\int_0^{\pi/2} f(t)\cos nt\,dt = 2 \times a_n$ for f

(since over the range $0 < t < \pi/2 \quad g = f$)

$$\Rightarrow \quad \underline{g(t) = a_0 + \sum_1^\infty 2a_n\cos nt}$$

(ii) $h(t)$ is <u>odd</u> $\Rightarrow$ no cosine terms

& exactly as above b_n for $g = 2 \times b_n$ for f

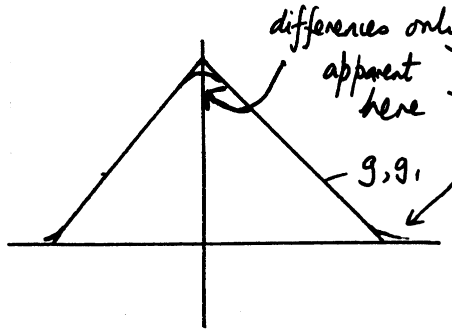
$$\underline{\text{i.e. } g(t) = \sum_1^\infty 2b_n \sin nt}$$

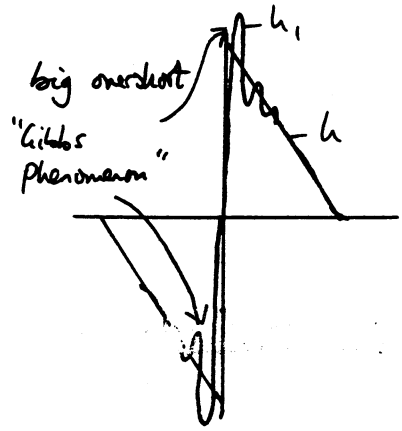
c) The coefficients for g are proportional to $\frac{1}{n^2}$ at high n

while those for h " " " $\frac{1}{n}$

$\therefore$ Expect g, to be more accurate

The reason that this would be expected is because g is a continuous fn while h is <u>discontinuous</u>; the smoother a function is (i.e. the more continuous derivatives it has) the faster the series will converge.





<u>Examiner's Note:</u>

1) Candidates made mincemeat of the explanation for the rate of convergence, with nearly the whole year getting it right.

2) Some candidates answered part (b) by saying, correctly, that

$$g(t) = f(t) + f(-t) = \frac{a_0}{2} + \sum_1^\infty a_n \cos nt + \sum_1^\infty b_n \sin nt$$
$$+ \frac{a_0}{2} + \sum_1^\infty a_n \cos n(-t) + \sum_1^\infty b_n \sin n(-t)$$
$$= a_0 + \sum_1^\infty 2a_n \cos nt \quad \text{since cosine even, sine odd}$$

& similarly

$$h(t) = f(t) - f(-t) = \sum_1^\infty 2b_n \sin nt$$

This was a very neat way of doing it.

3) Integration by parts needs practice; it comes up every year.

4) $\cos\frac{n\pi}{2} \neq (-1)^n$ as many candidates seemed to think
$\sin\frac{n\pi}{2} \neq 0$ .. ..
Perfectly ok to leave it as $\cos n\pi/2$, $\sin\frac{n\pi}{2}$

5) $f(t)$ only non-zero between 0 & $\pi/2$ does <u>NOT</u> mean that f is periodic of period $\pi/2$ & $\therefore$ contains only harmonics $4n$.

7 a) (i)

n

No of ways of picking 1st $= n$
" " " " " 2nd $= n-1$
⋮
" " " " " r^{th} $= n-r+1$

} assuming order matters

$\therefore$ No of ways of picking r from n in which order matters

$$= n(n-1)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)(n-r)\cdots 1}{(n-r)\cdots 1}$$

$$= \frac{n!}{(n-r)!}$$

The r that are chosen can be rearranged in $r!$ ways
(r choices for 1st, $r-1$ for second, etc)

$\therefore$ No of ways of choosing r from n in which order doesn't matter $= \dfrac{n!}{(n-r)!\,r!}$

(ii) $$(1-x)^{-(n+1)} = 1 - (n+1)(-x) + \frac{-(n+1)(-n-2)}{2!}(-x)^2 + \cdots \quad \text{Binomial}$$

$$= 1 + (n+1)x + \frac{(n+1)(n+2)}{2!}x^2 + \cdots$$

and $\binom{n}{n} = 1$ $\quad \binom{n+1}{n} = n+1$ $\quad \binom{n+2}{n} = \dfrac{(n+2)!}{n!\,2!} = \dfrac{(n+2)(n+1)}{2!}$

etc

$$\therefore (1-x)^{-(n+1)} = \binom{n}{n} + \binom{n+1}{n}x + \binom{n+2}{n}x^2 + \cdots$$

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b) (i) $$P_0 = 1 - (P_1 + P_2 + \cdots + \cdots) = 1 - (\alpha p^1 + \alpha p^2 + \alpha p^3 + \cdots)$$

$$= 1 - \frac{\alpha p}{1-p} \quad \text{(series is GP)}$$

(ii) $P(k$ boys in family of $n)$ = No ways of choosing k from n $\times$ Prob there are k boys $\times$ Prob rest girls

$$= \binom{n}{k}\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{n-k} = \binom{n}{k}\left(\frac{1}{2}\right)^n$$

Aliter: Prob of child being male $= \frac{1}{2}$ = prob of being female

No of possibilities in family of n $= 2^n$

No of ways in which k can be boys from n $= \binom{n}{k} \Rightarrow P = \binom{n}{k}\left(\frac{1}{2}\right)^n$

(iii) Prob k boys $=$ P(k children) $\times$ P(k boys in k)
+ P($k+1$ children) $\times$ P(k boys in $k+1$)
+ P($k+2$ children) $\times$ P(k boys in $k+2$)
+ . . . .

$$= \alpha p^k \binom{k}{k}\left(\tfrac{1}{2}\right)^k + \alpha p^{k+1}\binom{k+1}{k}\left(\tfrac{1}{2}\right)^{k+1} + \alpha p^{k+2}\binom{k+2}{k}\left(\tfrac{1}{2}\right)^{k+2} + \cdots$$

$$= \alpha\left(\frac{p}{2}\right)^k\left[\binom{k}{k} + \binom{k+1}{k}\frac{p}{2} + \binom{k+2}{k}\left(\frac{p}{2}\right)^2 + \cdots\right]$$

$$= \alpha\left(\frac{p}{2}\right)^k \frac{1}{(1-p/2)^{k+1}} \quad \text{using a (ii)}$$

$$= \frac{2\alpha p^k}{(2-p)^{k+1}}$$

<u>Examiner's Note:</u>

1) I have never seen so many attempts to fiddle answers in all my borne days! Outrageous fiddling does not leave the examiner with the impression that you know what you are doing.

2) Very few candidates who attempted this question (about 40% of the year tried it) could prove the formula for ${}_nC_r$.

8(a) Taking L.T.

$$s^2 Y - s y_0 - \dot{y}_0 + 2(sY - y_0) - 3Y = \frac{1}{s^2+1}$$

$$\therefore \quad (s^2 + 2s - 3)Y = 1 + \frac{1}{s^2+1}$$

$$\therefore \quad Y = \frac{1}{(s+3)(s-1)} + \frac{1}{(s+3)(s-1)(s^2+1)} = \frac{s^2+2}{(s+3)(s-1)(s^2+1)}$$

$$= \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

$$\Rightarrow \quad A(s+3)(s^2+1) + B(s-1)(s^2+1) + (Cs+D)(s^2+2s-3) = s^2+2$$

$$\Rightarrow \quad s^3: \quad A + B + C = 0 \quad (1)$$

$$s^2: \quad 3A - B + 2C + D = 1 \quad (2)$$

$$s^1: \quad A + B - 3C + 2D = 0 \quad (3)$$

$$s^0: \quad 3A - B - 3D = 2 \quad (4)$$

$$(1) - (3) \Rightarrow \quad 4C - 2D = 0 \quad \Rightarrow \quad D = 2C$$

$$(2) - (4) \Rightarrow \quad 2C + 4D = -1 \quad \Rightarrow \quad C = -\frac{1}{10} \quad D = -\frac{2}{10}$$

Then
$$\left.\begin{array}{l} A + B = \frac{1}{10} \\ 3A - B = \frac{7}{5} \end{array}\right\} \Rightarrow \quad A = \frac{3}{8} \quad B = -\frac{11}{40}$$

$$\therefore \quad Y = \frac{3}{8}\frac{1}{s-1} - \frac{11}{40}\frac{1}{s+3} - \frac{1}{10}\frac{s+2}{s^2+1}$$

$$\Rightarrow \quad y(t) = \frac{3}{8}e^{t} - \frac{11}{40}e^{-3t} - \frac{1}{10}(\cos t + 2\sin t)$$

(b)

$$\dot{y}(t) = \frac{3}{8}e^{t} + \frac{33}{40}e^{-3t} - \frac{1}{10}(-\sin t + 2\cos t)$$

$$\ddot{y}(t) = \frac{3e^{t}}{8} - \frac{99}{40}e^{-3t} - \frac{1}{10}(-\cos t - 2\sin t)$$

$$\therefore\ y(0) = \frac{3}{8} - \frac{11}{40} - \frac{1}{10} = \frac{15-11-4}{40} = 0 \quad ✓$$

$$\dot{y}(0) = \frac{3}{8} + \frac{33}{40} - \frac{2}{10} = \frac{15+33-8}{40} = \frac{40}{40} = 1 \quad ✓$$

$$\ddot{y} + 2\dot{y} - 3y = \frac{3e^{-t}}{8}\left[1+2-3\right] - \frac{11e^{-3t}}{40}\left[9-6-3\right]$$

$$- \frac{1}{10}\left[-\cos t - 2\sin t - 2\sin t + 4\cos t - 3\cos t - 6\sin t\right]$$

$$= \sin t \quad ✓$$

Examiner's Note:

1) The whole year had a very good go at this. Even those who couldn't get the right answer could get an answer which was not far wrong and so gained 14 marks (out of 20).

2) If the answer isn't right don't start the question again – go back and check the algebra – starting again wastes a collosal amount of time and trees for at most 6 more marks.

3) Many candidates reduced the algebra considerably by using a combination of the cover-up rule and selected values for s

e.g. $$\frac{s^2+2}{(s+3)(s-1)(s^2+1)} = \frac{A}{s-1} + \frac{B}{s+3} + \frac{Cs+D}{s^2+1}$$

"Cover up" with $s=1 \Rightarrow A = \frac{3}{4.2} = 3/8$

"Cover up" with $s=-3 \Rightarrow B = \frac{9+2}{(-4)(10)} = -\frac{11}{40}$

$$s=0 \Rightarrow \frac{2}{3(-1)(1)} = \frac{A}{-1} + \frac{B}{3} + \frac{D}{2} \Rightarrow D$$

Coefficient of $s^3 = 0$ (after cross-multiplying) gives C ($A+B+C=0$).
(or $s \to \infty$ L.H.S. $\to \frac{1}{s^2}$ R.H.S. $\to \frac{A}{s} + \frac{B}{s} + \frac{C}{s} \Rightarrow A+B+C=0$)

9(a) (i) $P\,dx + Q\,dy$ is an exact differential if and only if $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

(ii) $\dfrac{\partial P}{\partial y} = \alpha x - \dfrac{3y^2}{x^2}$ $\qquad \dfrac{\partial Q}{\partial x} = 12x - \dfrac{3y^\beta}{x^2}$

$\therefore \ \alpha = 12, \ \beta = 2$

$$\frac{\partial f}{\partial x} = P = 12xy - \frac{y^3}{x^2} - \sin x$$

$$\Rightarrow f = 6x^2y + \frac{y^3}{x} + \cos x + g(y) \quad \text{for some } g(y) \quad (1)$$

$$\frac{\partial f}{\partial y} = Q = 6x^2 + \frac{3y^2}{x}$$

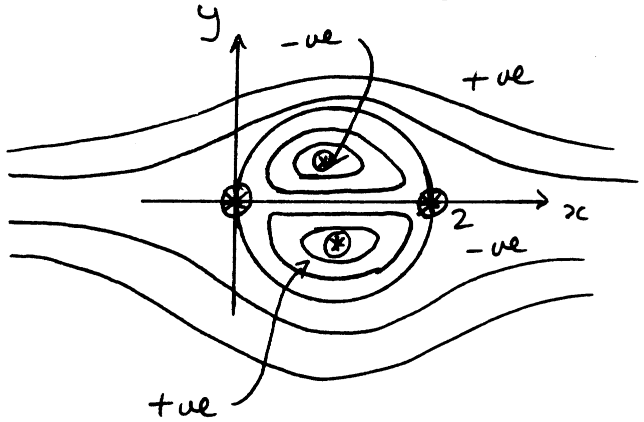
$$\Rightarrow f = 6x^2y + \frac{y^3}{x} + h(x) \quad \text{for some } h(x) \quad (2)$$

(1) & (2) can only be the same if $g(y) = \text{const}$ $\quad h(x) = \cos x$ (+ const)

$$\therefore f = 6x^2y + \frac{y^3}{x} + \cos x \ (+\text{ const})$$

b) (i) $g = y\{(x-1)^2 + y^2 - 1\} = 0$

$\Rightarrow y = 0$ or $(x-1)^2 + y^2 = 1$ ← circle



$\boxed{\text{(0,0) and (2,0) are saddles}}$

By symmetry / signs of g expect min at $(1, ?)$ max at $(1, -?)$

At a stationary point $\dfrac{\partial g}{\partial x} = \dfrac{\partial g}{\partial y} = 0$

if $g = x^2y - 2xy + y^3$

$\therefore \dfrac{\partial g}{\partial x} = 2xy - 2y = 0$

and $\frac{\partial g}{\partial y} = x^3 - 2x + 3y^2 = 0$

i.e. $2y(x-1) = 0$ AND $x^2 - 2x + 3y^2 = 0$

$\therefore$ $y = 0$ OR $x = 1$ AND $x^2 - 2x + 3y^2 = 0$

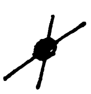
i.e. $y = 0 \Rightarrow x^2 - 2x = 0$ i.e. $x = 0$ or $x = 2$ (as expected)

or $x = 1 \Rightarrow 1 - 2 + 3y^2 = 0 \Rightarrow y^2 = \frac{1}{3} \Rightarrow y = \pm \frac{1}{\sqrt{3}}$

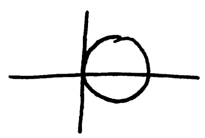
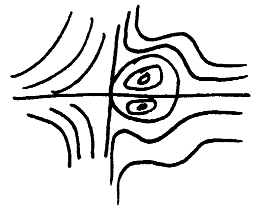
i.e. $(1, \frac{1}{\sqrt{3}})$ is a minimum

$(1, -\frac{1}{\sqrt{3}})$ is a maximum.

Examiner's Note:

1) The key to the second part of the question is getting the sketch correct.  ← This is a saddle, 2 contours appearing to cross / run into each other.

 This is either a max or a min.

2) The commonest error, at least for those who attempted the sketch, was for candidates to draw the axes, then the circle  and then draw other contours  . N.B. the y-axis is not necessarily a contour.

This type of error occurs in many years. Suggest you draw the axes in pencil & the lines $g = 0$ in pen so as to avoid this.

3) Getting the sketch correct is much easier than working out the stationary points "blind" and classifying them with 2nd derivs.

C1

Mathematics I A (Paper 4) - 2004 Section C

Q10 (C++ programming and laboratory course)

(a)(i) Algorithm being implemented is to find approximation of π from Euler's formula

$$\lim_{N \to \infty} \sum_{i=1}^{N} \frac{1}{i^2} = \frac{\pi^2}{6}$$ (4)

(b)

Bugs	Solution
1 N not declared as integer variable N not initialised to 0	int N = 0;
2 Logical AND symbol is not & but && need logical OR ||	while(N<1 || N>1000)
3. for loop never execute for( );	for( )
4. sum never initialised. Initialise on declaration	sum = 0.0; float sum = 0.0;
5. Integer arithmetic will give incorrect answer i^2 does not give square. Incorrect syntax	$\frac{1.0}{i*i}$
6. Estimate of PI calculated each time. Not needed but not a bug.	
7 Missing headers - iostream - cmath - using namespace std;	
8 return estimate?.	return 0;

(8)

10(b)(i) Can use $\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} \dots \quad |x| \le 1$

Typically use $x = 1$ for $\frac{\pi}{4}$
$x = \frac{1}{\sqrt{3}}$ for $\frac{\pi}{6}$

Convergence faster for $x = \frac{1}{\sqrt{3}}$

eg. $x=1$ $$\sum_{i=1}^{N} (-1)^{i+1} \frac{1}{(2i-1)} = \frac{\pi}{4}$$

eg $x = \frac{1}{\sqrt{3}}$ $$\sum_{i=1}^{N} (-1)^{i+1} \frac{1}{(2i-1)(\sqrt{3})^{(2i-1)}} = \frac{\pi}{6}$$

$$\sum_{i=1}^{N} (-1)^{i+1} \frac{\sqrt{3}}{(2i-1)\, 3^{i}} = \frac{\pi}{6}$$

$$\therefore \frac{\pi}{6} = \sqrt{3}\left(\frac{1}{3} - \frac{1}{3 \times 3^2} + \frac{1}{5 \times 3^3} \dots \right)$$

```
#include <iostream>
#include <cmath>

using namespace std;

int main(){
  int N=1000;
  float iterm =-1;
  float sum=0;
  for(int i=1; i<=N; i++)
  {
    iterm = -1*iterm/3;
    sum = sum + iterm/(2*i-1);
  }
  float estimatePi = 6 * sqrt(3) * sum;
  cout << estimatePi << endl;
}
```

Q 11(a) 13 times

(b)

i	hi	lo	mid	val
1	8	0	4	2
2	4	0	2	−4
3	4	2	3	−2
4	4	3	3.5	−0.25

(c). Roots of $x^2 - 3x - 2 = 0$

$$x = \frac{3 \pm \sqrt{17}}{2}$$

Program will converge to upper root $\frac{3+\sqrt{17}}{2} \simeq 3.5616$

At end of i^{th} iteration search range is $8 \times \left(\frac{1}{2}\right)^i$.
error is $4 \times \left(\frac{1}{2}\right)^i$

For $i = 13$, accuracy $= 4 \times \left(\frac{1}{2}\right)^{13} = 4.9 \times 10^{-4}$

(d) For 100 times more precision:

$$4 \times \left(\tfrac{1}{2}\right)^i = 4 \times \left(\tfrac{1}{2}\right)^{13} \times \frac{1}{100}$$

$i < 20$

$\cancel{\ln 4} + i \ln \frac{1}{2} = \cancel{\ln 4} + 13 \ln$ $+ \ln 100$

$i = \dfrac{13 \ln \frac{1}{2} + \ln 100}{\ln \frac{1}{2}}$

$= 13 + \ln 100 / \ln \frac{1}{2}$

$= 19.6$

Will take longer by factor of $\frac{20}{13}$
(excluding fixed overheads outside loop)

(e) Finite precision (23 bit mantissa) in float format. Solve $\left(\frac{3+\sqrt{17}}{2}\right) \times \varepsilon_n = 4 \times \left(\frac{1}{2}\right)^i$
$i = 24.157$. So would expect no change on 25th iteration.

Rr Ala