ENGINEERING TRIPOS PART IA

Tuesday 8 June 2004 1.30 to 4.30

Paper 4

MATHEMATICAL METHODS

Answer not more than **eight** *questions, of which not more than* **three** *may be taken from Section A, not more than* **four** *from Section B and not more than* **one** *from Section C.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

*Answer not more than **three** questions from this section.*

1 (a) Consider the tetrahedron, $OABC$, formed by the origin and the points A, B and C with position vectors $\underline{a}$, $\underline{b}$ and $\underline{c}$ respectively.

(i) Show that the normal to the plane ABC is given by

$$\underline{n} = (\underline{a} \times \underline{b}) + (\underline{b} \times \underline{c}) + (\underline{c} \times \underline{a})$$ [20%]

(ii) Find an expression for the shortest distance between the plane and the origin. [10%]

(iii) Express the area of the triangle ABC in terms of a vector product. [10%]

(iv) Hence find an expression for the volume of the tetrahedron $OABC$. [10%]

(v) Give a geometric interpretation for the case in which

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = 0$$ [10%]

(b) Find the equation of the line of intersection of the two planes:

$$\begin{aligned} 2x + y + z &= -2 \\ x + 2y + z &= 2 \end{aligned}$$

in the form $\underline{r} = \underline{d} + \lambda \underline{e}$. [40%]

2 (a) Find:

(i) $$\lim_{x \to 0} \frac{\tan x - x}{x - \sin x}$$

(ii) $$\lim_{x \to \frac{1}{6}} \frac{1 - 2\sin \pi x}{1 - 6x}$$ [30%]

(b) Sketch the curve on the Argand plane defined by the equation:

$$|z + \mathrm{i}| + |z - 1| = 2$$

where $z = x + iy$. [30%]

(c) Obtain expressions for:

(i) $\sin 4\theta$ in terms of $\sin\theta$ and $\cos\theta$

(ii) $\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)^{i}$ [40%]

(TURN OVER

3 (a) Find the solution of the following differential equation:

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = \cos 2t$$

which satisfies the boundary conditions:

$$x = \frac{dx}{dt} = 0 \quad \text{at} \quad t = 0$$ [50%]

(b) A rectangle, which is initially made up of 2 unit squares side by side, is made to grow in an iterative scheme by adding to its side a square whose dimension L_n is the sum of the dimensions of the last 2 squares such that L_n satisfies the difference equation

$$L_n = L_{n-1} + L_{n-2}$$

$$L_0 = L_1 = 1$$

The resulting shape is always a rectangle, as shown in Fig. 1.

(i) If the solution for L_n has the form:

$$L_n = A\lambda_1^n + B\lambda_2^n$$

find λ_1 and λ_2. [20%]

(ii) Find an expression for the area of the rectangle after adding a square of dimension L_n . [15%]

(iii) Hence find the ratio of successive areas of the rectangle for large n. [15%]

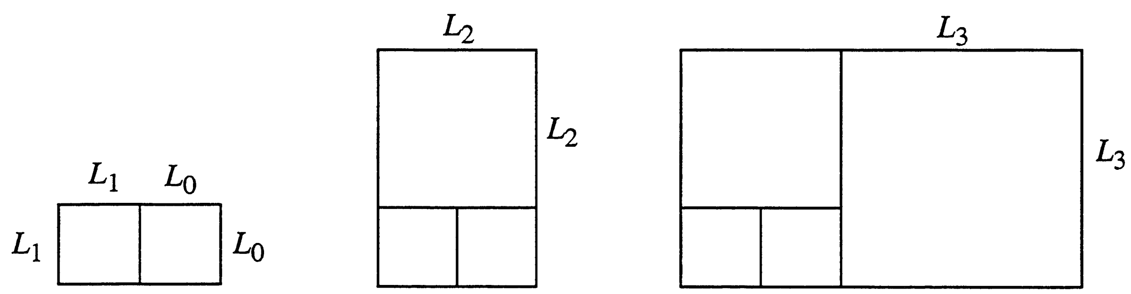

Fig. 1

4 (a) The 2×2 matrices A_i, $i = 1,2,3$, are given by:

$$A_1 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}; \quad A_2 = \begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix}; \quad A_3 = \begin{bmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix}$$

(i) For each of the matrices A_i, describe in geometrical terms the effects of multiplying an arbitrary vector $\underline{x}$ by A_i and sketch the transformation of points on a unit square. [20%]

(ii) In the case(s) for which the eigenvalues are real, determine the eigenvalues and the corresponding eigenvectors. [15%]

(iii) Hence estimate, for the case(s) for which the eigenvalues are real, the co-ordinates of a point (x_1, x_2) after applying the transformations 100 times when $\alpha > 1$. [15%]

(b) Consider the equation

$$\underline{y} = A\underline{x}$$

where $\underline{x}$ and $\underline{y}$ are vectors and A is a matrix with real elements.

(i) By finding an expression for $|\underline{y}|^2$, show that

$$\frac{|\underline{y}|^2}{|\underline{x}|^2} = \frac{\underline{x}^T A^T A \underline{x}}{\underline{x}^T \underline{x}}$$ [20%]

(ii) Show that $A^T A$ is a symmetric matrix. [10%]

(iii) What properties will the eigenvectors of $A^T A$ have ? [10%]

(iv) Find the value of $\underline{x}$ which minimises $\dfrac{|\underline{y}|}{|\underline{x}|}$. [10%]

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SECTION B

*Answer not more than **four** questions from this section.*

5 (a) Explain why the response of a linear system to a general input can be written in the form

$$y(t) = \int_0^t x(\tau)g(t-\tau)d\tau \quad \text{for } t \geq 0$$

where x is the input to the system, y is the output of the system and g is the impulse response. [20%]

(b) The input $x(t)$ and the output $y(t)$ of a linear system satisfy the differential equation

$$\frac{dy}{dt} + \alpha y = x$$

Find and sketch (i) the step response and (ii) the impulse response for this system. [30%]

(c) Using a convolution integral, find the response of the system to an input

$$x(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \end{cases}$$ [30%]

(d) Sketch the output $y(t)$ and explain how it is related to the step response. [20%]

6 (a) The function f, shown in Fig. 2(a), is represented over the range $-\pi \leq t \leq \pi$ by the Fourier series

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt \ .$$

Find the coefficients a_n and b_n . [45%]

(b) (i) Show that the function g, shown in Fig. 2(b), has a Fourier series representation of the form

$$a_0 + \sum_{n=1}^{\infty} 2a_n \cos nt$$

where the a_n are the same as those found in part (a). [15%]

(ii) Find a Fourier Series representation valid over this range for the function h, shown in Fig. 2(c). [15%]

(c) Approximations, g_1 and h_1, are obtained for g and h by truncating the Fourier series representation for each to the term up to and including $n = 10$. Explain carefully which approximation is likely to be the more accurate and which properties of g and h are responsible for this. Sketch g_1 and h_1 in comparison with g and h. [25%]

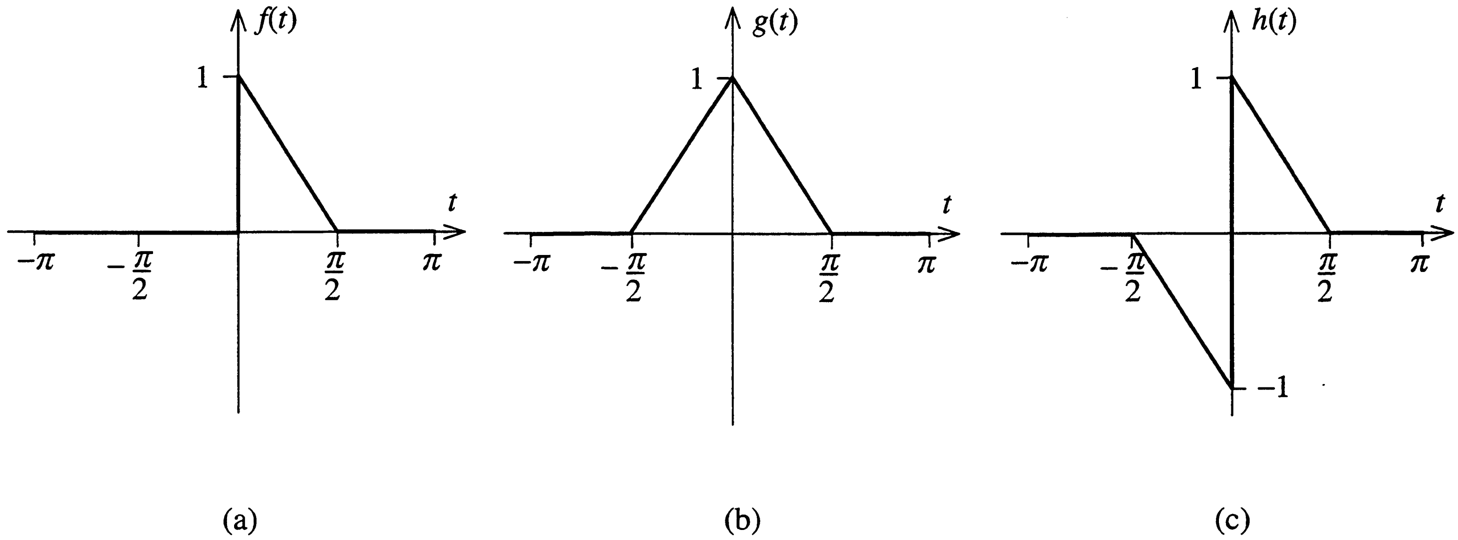

Fig. 2

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7 (a) (i) Show that the number of ways of choosing r items from n when the order does not matter is

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!}$$ [10%]

(ii) Show, using the binomial expansion, that

$$\frac{1}{(1-x)^{n+1}} = \binom{n}{n} + \binom{n+1}{n}x + \binom{n+2}{n}x^2 + \ldots$$ [10%]

(b) The probability P_n that a family has n children is αp^n where $n \geq 1$.

(i) Show that the probability that a family is childless is given by

$$P_0 = 1 - \frac{\alpha p}{1-p}$$ [20%]

(ii) Assuming that children of either sex are equally likely (irrespective of family size), show that the probability that a family of n children contains precisely k boys is

$$\binom{n}{k}\left(\frac{1}{2}\right)^n$$ [30%]

(iii) Show, using part (a) (ii), that for $k \geq 1$, the probability that a family will have precisely k boys is

$$\frac{2\alpha p^k}{(2-p)^{k+1}}$$ [30%]

8 (a) Solve using Laplace Transforms

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t \qquad \text{for } t > 0$$

where $\frac{dy}{dt}(0) = 1$ and $y(0) = 0$. [70%]

(b) Verify that your solution satisfies the differential equation and the boundary conditions. [30%]

9 (a) (i) State the conditions under which

$$P(x,y)\,dx + Q(x,y)\,dy$$

is an exact differential. [15%]

(ii) For

$$P(x,y) = \alpha x y - \frac{y^3}{x^2} - \sin x$$

$$Q(x,y) = 6x^2 + \frac{3y^\beta}{x}$$

determine the values of α and β which make $df = P(x,y)\,dx + Q(x,y)\,dy$ an exact differential and find the corresponding function $f(x,y)$. [25%]

(b) (i) Sketch contours of the function g given by

$$g(x,y) = y\left((x-1)^2 + y^2 - 1\right)$$ [20%]

(ii) Find and classify the stationary points of g and mark them on your sketch. [40%]

(TURN OVER

SECTION C

*Answer not more than **one** question from this section.*

10 (a) Figure 3 shows a C++ program which attempts to estimate the value of π.

(i) What is the algorithm that is being implemented?

(ii) Identify each of the bugs in the program, explain their consequences and state how to correct them.

[60%]

(b) Modify the program to estimate the value of π by exploiting the power series expansion

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + \ldots$$

Explain how the choice of the value of x will affect convergence. [40%]

(cont.

```
// Program with bugs to estimate the value of pi.

int main()
{
  float sum, estimatePi;

  while(N<1 & N>1000)
    {
      cout << "Please enter the number terms (1<N<1000): ";
      cin >> N;
    }

  for(int i = 1; i <= N; i++);
    {
      sum = sum + 1/(i^2);
      estimatePi = sqrt(6.0*sum);
    }
  cout << "Estimate of PI = " << estimatePi << endl;
  return estimatePi;
}
```

Fig. 3

(TURN OVER

11 Figure 4 shows a C++ computer program.

(a) The program contains a loop in `main()`. How many times will the code inside this loop be executed ? [10%]

(b) Calculate the values that the variables `i`, `hi`, `lo`, `mid` and `val` will have on the first four iterations after the function `f` is called. Show your answer as a table with each variable corresponding to a column and each row to an iteration. [40%]

(c) What (approximately) will be the output of the program ? How accurate will the result be ? [20%]

(d) What change must be made if a 100 times more precision is desired ? How many times longer will the program take to run ? [20%]

(e) If the number of iterations of the loop is made very large, at some point the values of `hi` and `lo` will cease to change. Why is this ? How many iterations will take place before this occurs ? (You may assume that the machine precision is $\varepsilon_m = 6 \times 10^{-8}$). [10%]

(cont.

```
#include<iostream>
using namespace std;

float f(float x);

int main()
{
  int i;
  float lo=0;
  float hi=8;
  float mid;

  for(i=1; i<14; i++) {
    mid = (hi+lo)/2;
    float val = f(mid);
    if( val > 0 ) {
      hi = mid;
    } else {
      lo = mid;
    }
  }
  mid = (hi+lo)/2;
  cout << mid << endl;
  return 0;
}

float f(float x)
{
  return x*x-3*x-2;
}
```

Fig. 4

END OF PAPER