Paper 4: Mathematical Methods

Solutions to 2005 Tripos Paper

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1. From the mathematics data book

$$\sin x = x - \frac{1}{6}x^3 + O(x^5)$$

Using the binomial theorem

$$\frac{1}{\sqrt{1+2x^2}} = (1+2x^2)^{-1/2} = 1 - x^2 + O(x^4)$$

Multiplying the two series together, we get

$$\frac{\sin x}{\sqrt{1+2x^2}} = \left(x - \frac{1}{6}x^3 + O(x^5)\right)\left(1 - x^2 + O(x^4)\right) = x - \frac{7}{6}x^3 + O(x^5)$$ [10]

Examiner's remarks: A straightforward short question on power series expansions which was answered correctly by most candidates. Those that made errors struggled with the binomial expansion of the denominator.

2. The auxiliary equation is

$$\lambda^2 + 2\lambda + 10 = 0 \Leftrightarrow \lambda = \frac{-2 \pm \sqrt{-36}}{2} = -1 \pm 3i$$

So the complementary function is

$$y = e^{-x}\left(A\cos 3x + B\sin 3x\right)$$

For a particular integral, try $y = Ce^{-2x}$. Substituting into the differential equation, we get

$$4Ce^{-2x} - 4Ce^{-2x} + 10Ce^{-2x} = e^{-2x} \Leftrightarrow C = 0.1$$

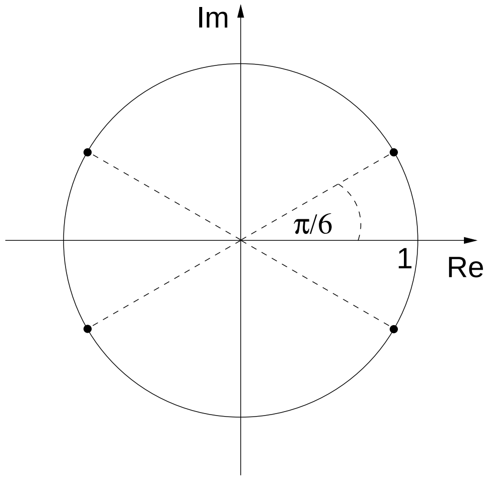
The general solution is therefore

$$y = e^{-x}\left(A\cos 3x + B\sin 3x\right) + 0.1e^{-2x}$$ [10]

Examiner's remarks: A straightforward short question on the solution of differential equations which was correctly answered by most candidates. The most common error was with the interpretation of the complex conjugate pair of roots as a damped oscillation.

3. This is a quadratic equation in z^2. Applying the usual formula for solving a quadratic equation, we get

$$\begin{aligned} z^2 &= \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{\pm i(\pi/3 + 2n\pi)} \\ \Leftrightarrow z &= e^{\pm i(\pi/6 + n\pi)} \end{aligned}$$



[10]

Examiner's remarks: A short question on complex numbers and complex roots of polynomial equations. Some candidates made simple algebraic errors in deriving the square root of a complex number.

4. (a) We need to find two vectors in the plane. One is clearly the direction vector of the line $[5\ 2\ 3]^T$. The point $(2, -3, 1/2)$ lies on the line, the vector joining this to $(1, 0, 2)$ gives us another vector in the plane, ie. $[1\ \ -3\ \ -3/2]^T$. Take the vector product to find the plane's normal:

$$\begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix} \times \frac{1}{2} \begin{bmatrix} 2 \\ -6 \\ -3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 12 \\ 21 \\ -34 \end{bmatrix}$$

The equation of the plane is therefore $12x + 21y - 34z = k$. Substitute either of the two points identified above to get $k = -56$. The plane is therefore $12x + 21y - 34z = -56$. [10]

(b) In general, a line will intersect a plane unless it happens to be parallel to it. For the line to be parallel to the plane, its direction vector must be perpendicular to the plane's normal. So take the scalar product of the direction vector $[1/4\ 2\ 3/2]^T$ and the plane's normal $[2\ \ -1\ 1]^T$ and see if it's zero:

$$2/4 - 2 + 3/2 = 0$$

So the line is indeed parallel to the plane. Consider the point $(2, -3, 1/2)$ on the line, and imagine heading off from this point along the plane's normal $[2\ -1\ 1]^T$ until we hit the plane. So we need to find the value of λ for which

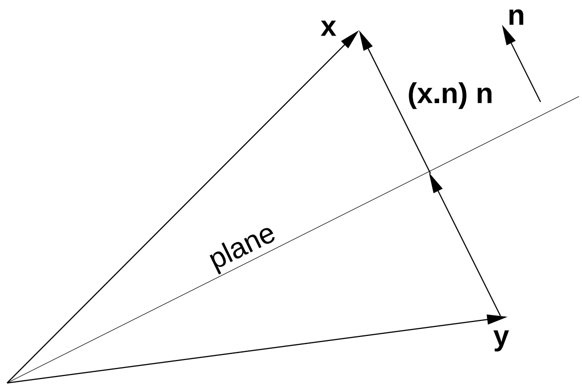
$$\begin{bmatrix} 2 \\ -3 \\ 1/2 \end{bmatrix} + \lambda \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

lies on the plane:

$$2(2 + 2\lambda) - (-3 - \lambda) + (1/2 + \lambda) = 3 \Leftrightarrow \lambda = -3/4$$

The distance between the line and the plane is thus $(3/4)\sqrt{2^2 + 1^2 + 1^2} = 1.84$. [10]

(c) Note that the plane passes through the origin.



From the diagram, it is clear that

$$\mathbf{y} = \mathbf{x} - 2(\mathbf{x}.\mathbf{n})\mathbf{n} = \mathbf{x} - 2\mathbf{n}(\mathbf{n}^T\mathbf{x}) = (\mathbf{I} - 2\mathbf{n}\mathbf{n}^T)\mathbf{x}$$

where $\mathbf{n}$ is the unit normal to the plane, $\frac{1}{\sqrt{6}}[2\ -1\ 1]^T$ in this case. Substitute to find $\mathbf{R}$:

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{6}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}[2\ -1\ 1] = \frac{1}{3}\begin{bmatrix} -1 & 2 & -2 \\ 2 & 2 & 1 \\ -2 & 1 & 2 \end{bmatrix}$$

Since this is a reflection, we'd expect the determinant to be -1. Checking our answer, we see that

$$\det \mathbf{R} = \frac{1}{27}[-1(4-1) - 2(4+2) - 2(2+4)] = -1$$ [10]

Examiner's remarks: A long question in three parts on vector equations of lines and planes and representation of geometric transformations with matrices. Part (a) required the candidates to derive the equation of a plane containing a line and point and was straightforward and answered correctly by most candidates. Many struggled with part (b) on computing the shortest distance between a line and a plane. Very few candidates successfully computed the matrix representing a reflection in part (c).

5. (a) The characteristic equation is

$$\lambda^2 - 8\lambda + 15 = 0 \Leftrightarrow (\lambda - 5)(\lambda - 3) = 0$$

The general solution is therefore $x_n = A5^n + B3^n$. Find A and B using the initial conditions:

$$\begin{aligned} A + B &= 0\,, \quad 5A + 3B = 1 \\ \Leftrightarrow A &= \frac{1}{2}\,, \quad B = -\frac{1}{2} \end{aligned}$$

The solution is therefore $x_n = \frac{1}{2}(5^n - 3^n)$. [7]

(b) (i) We can write the equations in matrix form as follows:

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{bmatrix} = \begin{bmatrix} 0.6 & 0 & 0 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \\ z_n \end{bmatrix}$$ [3]

(ii) The characteristic equation is

$$\begin{aligned} \begin{vmatrix} (0.6-\lambda) & 0 & 0 \\ 0.1 & (0.9-\lambda) & 0.2 \\ 0.3 & 0.1 & (0.8-\lambda) \end{vmatrix} &= 0 \Leftrightarrow (0.6-\lambda)\left[(0.9-\lambda)(0.8-\lambda) - 0.02\right] = 0 \\ \Leftrightarrow (0.6-\lambda)\left[\lambda^2 - 1.7\lambda + 0.7\right] &= 0 \Leftrightarrow \lambda_1 = 0.6\,, \quad \lambda_2 = 1.0\,, \quad \lambda_3 = 0.7 \end{aligned}$$

For the eigenvectors, consider each eigenvalue in turn.

$$\begin{aligned} \begin{bmatrix} 0.6 & 0 & 0 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} &= 0.6 \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \\ \Leftrightarrow 0.1u_x + 0.3u_y + 0.2u_z &= 0\,, \quad 0.3u_x + 0.1u_y + 0.2u_z = 0 \end{aligned}$$

Subtracting one equation from the other, we get

$$-0.2u_x + 0.2u_y = 0 \Leftrightarrow u_x = u_y$$

Substituting back into one of the original equations, we find that $u_z = -2u_x$. The eigenvector is therefore $[1\ 1\ -2]^T$.

$$\begin{aligned} \begin{bmatrix} 0.6 & 0 & 0 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} &= 1.0 \begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \\ \Leftrightarrow u_x &= 0\,, \quad -0.1u_y + 0.2u_z = 0 \end{aligned}$$

The eigenvector is therefore $[0\ 2\ 1]^T$.

$$\begin{aligned}\begin{bmatrix} 0.6 & 0 & 0 \\ 0.1 & 0.9 & 0.2 \\ 0.3 & 0.1 & 0.8 \end{bmatrix}\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} &= 0.7\begin{bmatrix} u_x \\ u_y \\ u_z \end{bmatrix} \\ \Leftrightarrow u_x &= 0\ ,\ \ 0.2u_y + 0.2u_z = 0\end{aligned}$$

The eigenvector is therefore $[0\ 1\ \ -1]^T$. A non-symmetric matrix may not have orthogonal eigenvectors, as is the case here. [12]

(iii) We need to find α, β and γ such that

$$\begin{bmatrix} N \\ 0 \\ 0 \end{bmatrix} = \alpha\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} + \beta\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} + \gamma\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

The top row tells us that $\alpha = N$. The other rows then give

$$\begin{aligned} 0 &= N + 2\beta + \gamma \\ 0 &= -2N + \beta - \gamma \end{aligned}$$

Sum the two equations to get

$$0 = -N + 3\beta$$

Hence $\beta = N/3$ and $\gamma = -5N/3$. Now,

$$\mathbf{p}_n = \mathbf{A}^n\mathbf{p}_0 = \mathbf{A}^n(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2 + \gamma\mathbf{u}_3) = \alpha\lambda_1^n\mathbf{u}_1 + \beta\lambda_2^n\mathbf{u}_2 + \gamma\lambda_3^n\mathbf{u}_3$$

Since $|\lambda_1| < 1$, $|\lambda_3| < 1$ and $\lambda_2 = 1$, as $n \to \infty$

$$\mathbf{p}_n \to \beta\mathbf{u}_2 = \frac{N}{3}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

[8]

Examiner's remarks: A long question in three parts on linear difference equations and eigenvectors and eigenvalues. The solution of a simple difference equation in part (a) was answered correctly by most candidates. The computation of the eigenvalues in part (b) was also answered correctly though many made simple algebraic errors in computing the eigenvalues. Few candidates attempted the last part of (b), which required them to estimate the components of a vector in the three non-orthogonal directions of the eigenvectors and to calculate the vector after repeated multiplication.

6. Taking the Laplace Transform of both sides of the equation, we get

$$sX - 1 + X = \frac{1}{s+2} \Leftrightarrow X = \frac{1}{s+1} + \frac{1}{(s+1)(s+2)} = \frac{2}{s+1} - \frac{1}{s+2}$$

Looking up the inverse Laplace Transform, we get $x = 2e^{-t} - e^{-2t}$. [10]

Examiner's remarks: This question concerned the application of Laplace transforms to solve a differential equation. This was universally well answered with only a few students making errors. The most popular error was to transform $x(t)$ into $1/s^2$ rather than $X(s)$.

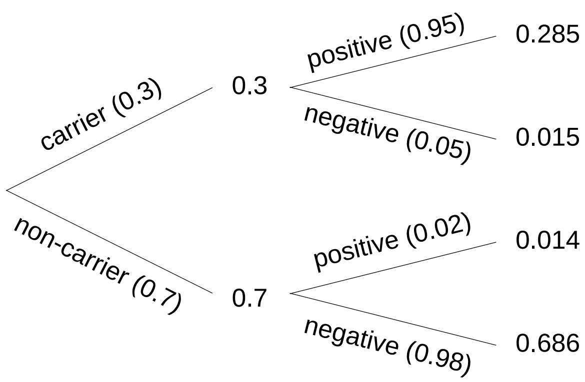
7. The direction of steepest descent is $-\nabla h$, where

$$\nabla h = \begin{bmatrix} \frac{\partial h}{\partial x} \\ \frac{\partial h}{\partial y} \end{bmatrix} = \begin{bmatrix} 2xe^{-2x} - 2(x^2+y)e^{-2x} \\ e^{-2x} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ at the origin}$$

The unit vector in the direction of steepest descent is therefore $[0\ \ -1]^T$. The steepest gradient at this point is given by $|\nabla h| = 1$. [10]

Examiner's remarks: This question concerned the calculation of a gradient in a two dimensional height field. Almost all students could calculate the partial derivatives correctly, although some failed to understand that 'descent' implied reversing the direction of the grad. Other than this, the most popular error was to misinterpret the direction of steepest descent in the x-y plane into a vertical plane, leading to many claims of infinite steepness.

8. The problem is best tackled using a tree diagram.



So the probability of a positive test is $0.285 + 0.014 = 0.299$, and the probability that I carry MRSA given a positive test is $0.285/0.299 = 0.953$. [10]

Examiner's remarks: This question concerned probability. It was correctly answered by a very large number of candidates. Those who failed to answer it correctly mostly did so because they did not appear to understand how to transform the information given to them in the question

9. (a) The principle of superposition applies to linear systems as follows. If the system's response to input x_1 is y_1, and the response to input x_2 is y_1, then the response to input $Ax_1 + Bx_2$ is $Ay_1 + By_2$. [4]

(b) $x(t)$ is the superposition of a positive step, magnitude 1, at time $t=0$, and a negative step, magnitude 2, at time $t=T$. The response is therefore

$$y(t) = \begin{cases} 0 & t < 0\,, \\ 1 - e^{-t} & 0 \le t \le T\,, \\ (1-e^{-t}) - 2(1-e^{-(t-T)}) = e^{-t}(2e^{T}-1) - 1 & t > T\,. \end{cases}$$

In the limit $T \to 0$, the input becomes a negative step, magnitude 1, and $y(t)$ simplifies to

$$y(t) = \begin{cases} 0 & t < 0\,, \\ e^{-t}(2e^{0}-1) - 1 = e^{-t} - 1 & t > 0\,. \end{cases}$$

This is, as expected, the negative of the step response. [8]

(c) First we obtain the impulse response $g(t)$ by differentiating the step response:

$$g(t) = \frac{d}{dt}(1-e^{-t}) = e^{-t}$$

Now use convolution to find the response to the input $x(t)$. Since $x(t)$ is discontinuous, we need to consider two distinct ranges of t. First, for $t \le T$, we have

$$\begin{aligned} y(t) &= \int_0^t x(\tau) g(t-\tau)\, d\tau = \int_0^t (1-e^{-\tau}) e^{-(t-\tau)}\, d\tau \\ &= e^{-t} \int_0^t (1-e^{-\tau}) e^{\tau}\, d\tau = e^{-t} \int_0^t (e^{\tau}-1)\, d\tau \\ &= e^{-t} \left[e^{\tau} - \tau\right]_0^t = e^{-t}(e^{t} - t - 1) \\ &= 1 - te^{-t} - e^{-t} \end{aligned}$$

Next, for $t > T$, we have

$$\begin{aligned} y(t) &= \int_0^T x(\tau) g(t-\tau)\, d\tau + \int_T^t x(\tau) g(t-\tau)\, d\tau \\ &= \int_0^T (1-e^{-\tau}) e^{-(t-\tau)}\, d\tau + \int_T^t (1-e^{-T}) e^{-(t-\tau)}\, d\tau \\ &= e^{-t}\left[e^{\tau} - \tau\right]_0^T + (1-e^{-T}) \left[e^{-(t-\tau)}\right]_T^t \\ &= e^{-t}(e^{T} - T - 1) + (1-e^{-T})(1-e^{-(t-T)}) \\ &= e^{-t}e^{T} - Te^{-t} - e^{-t} + 1 - e^{-t}e^{T} - e^{-T} + e^{-t} \\ &= 1 - Te^{-t} - e^{-T} \end{aligned}$$

The system's response is therefore

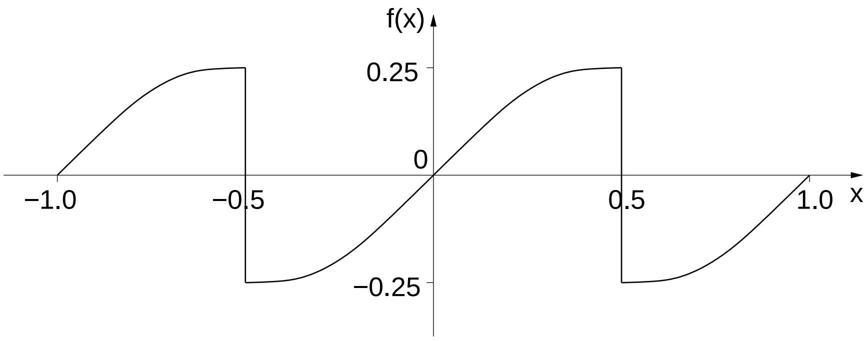
$$y(t) = \begin{cases} 0 & t < 0\,, \\ 1 - te^{-t} - e^{-t} & 0 \le t \le T\,, \\ 1 - Te^{-t} - e^{-T} & t > T\,. \end{cases}$$

At $t = T$, both expressions evaluate to $1 - Te^{-T} - e^{-T}$. The solution is therefore continuous at $t = T$. [18]

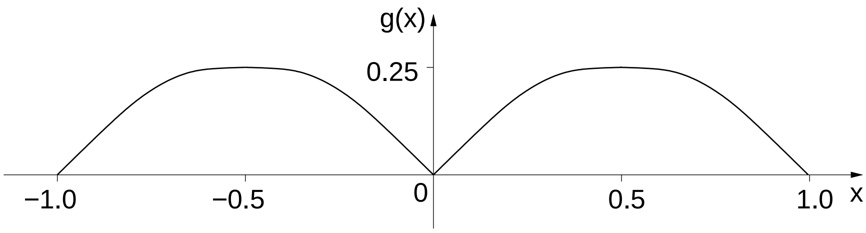
Examiner's remarks: This question concerned impulse and step responses, requiring the students to perform superposition, calculate an impulse response and convolve this with an input signal. This question was problematic and very few students were able to answer it correctly (or even begin to answer it correctly). There were a large number of kinds of error made, the most popular being not recognising that the step response given in the question only applied for $t > 0$, mixing up t, T and τ when computing convolutions, not splitting integration ranges correctly, and failing to differentiate the step response to obtain an impulse response.

10. (a) $x - x^2$ passes through the origin and $(0.5, 0.25)$. It has a gradient of 1 at the origin and 0 at $(0.5, 0.25)$.

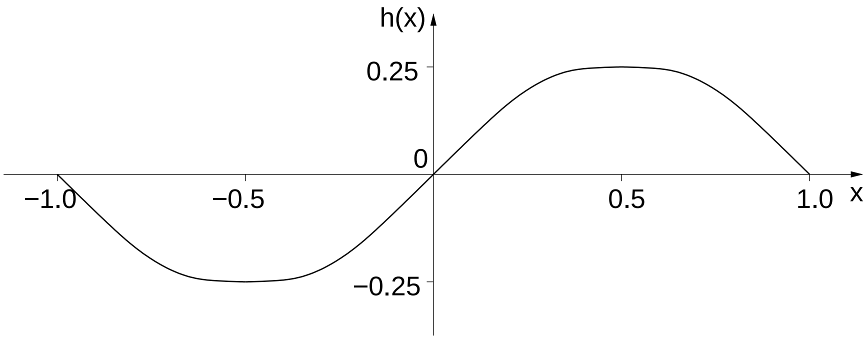
(i) The lowest frequency term in the Fourier series ($n = 1$) is $\sin 2\pi x$, which has a period of 1. $f(x)$ is therefore an odd function with period 1.



(ii) The lowest frequency term in the Fourier series ($n = 1$) is $\cos 2\pi x$, which has a period of 1. $g(x)$ is therefore an even function with period 1.



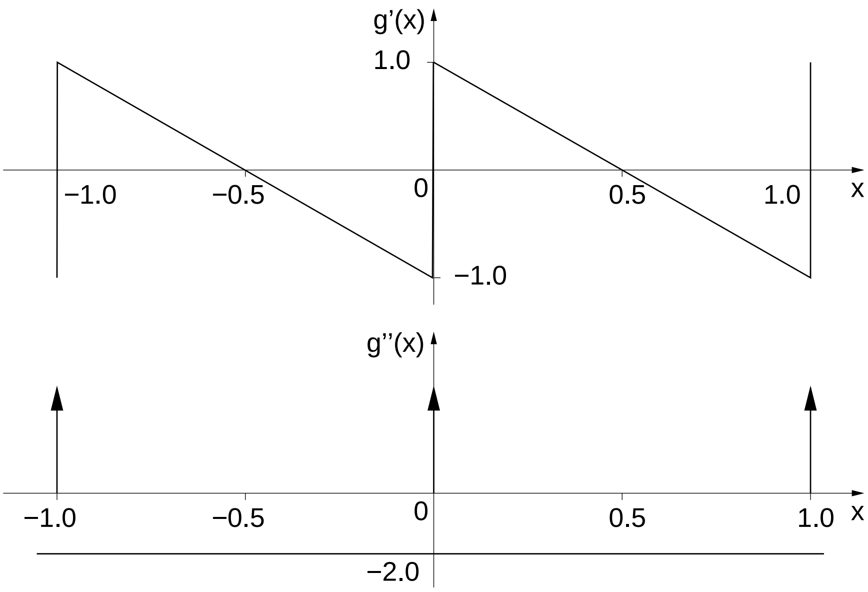
(iii) The lowest frequency term in the Fourier series ($n = 1$) is $\sin \pi x$, which has a period of 2. $h(x)$ is therefore an odd function with period 2. All the constituent terms ($\sin \pi x$, $\sin 3\pi x$, $\sin 5\pi x$ ...) are symmetric around $x = 1/2$, so $h(x)$ must also be symmetric around $x = 1/2$.



[11]

(b) $f(x)$ has a discontinuity in value, so it converges as $1/n$. $g(x)$ has a discontinuity in gradient, so it converges as $1/n^2$. $h(x)$ has a discontinuity in its second derivative, so it converges as $1/n^3$. Hence $f(x)$ converges the slowest, $h(x)$ converges the fastest. [4]

(c) To avoid tricky integration by parts, first differentiate $g(x)$ twice.



The arrows are impulse functions of magnitude 2: there is also a constant of -2. The Fourier series of the impulse train $I(x)$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

where

$$a_n = 2\int_{-0.5}^{0.5} I(x)\cos 2n\pi x\, dx = 2 \times 2\cos 0 = 4$$

Summing the Fourier series for the impulse train with the constant, we get

$$g''(x) = -2 + 2 + \sum_{n=1}^{\infty} 4\cos 2n\pi x$$

Integrating twice, we find:

$$g(x) = -\frac{4}{4\pi^2}\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2} + Ax + B$$

Since $g(x)$ is periodic, the constant of integration A is clearly zero. The other constant of integration B is the average value of $g(x)$:

$$B = \frac{1}{0.5}\int_0^{0.5}(x - x^2)\,dx = 2\left[\frac{1}{2}x^2 - \frac{1}{3}x^3\right]_0^{0.5} = \frac{1}{6}$$

The coefficients a_n are therefore

$$a_n = \begin{cases} \dfrac{1}{3} & \text{for } n = 0 \\ -\dfrac{1}{n^2\pi^2} & \text{for } n \geq 1 \end{cases}$$ [15]

Examiner's remarks: This question concerned the modelling of a given function with three different possible Fourier series. This question was also problematic, with a surprisingly large number of candidates unable to draw a graph of $x - x^2$ in the range 0–0.5, or unable to correctly determine the implied structure of the three Fourier series. The final part of the question was well attempted by most candidates but rarely answered correctly, with many failing to set integration ranges correctly, or failing to perform the (slightly lengthy) integration by parts correctly. Only one candidate spotted the trick used in the crib — but was unable to perform the differentiation correctly to carry this out.

11. The `factorial` function works out the factorial of its first parameter, placing the answer in the second (reference) parameter. It returns `true` if the answer is valid, and `false` if it was asked to find the factorial of a negative number (note that zero is allowed — the factorial of zero is one). So the program prints "The factorial of 5 is 120" on the screen. If, however, the first parameter were passed by reference, `b` would be decremented by the `factorial` loop and the program would print "The factorial of 1 is 120". Calculating the factorial of n requires approximately n iterations of the `factorial` loop: the algorithmic complexity is therefore $\mathcal{O}(n)$. [10]

    Examiner's remarks: A bimodal distribution of marks, candidates generally either knew this material well or didn't know it at all. 156 candidates scored 7+, 67 scored 3−. So it is pleasing to see that more than half the candidates have a solid grasp of parameter passing and algorithmic complexity. Of the 55 candidates scoring middling marks, the most common pattern was failing to understand the effect of the call-by-reference, but getting the algorithmic complexity correct.

12. IEEE standard single precision floating point format has a sign bit s, followed by an 8-bit exponent e and a 23-bit mantissa m. The decimal equivalent is

$$(-1)^s \times 1.m \times 2^{e-127}$$

where the mantissa $1.m$ and the exponent e are in binary, and therefore need converting into decimal first. For this example, we have $e = 10010111_2 = 151_{10}$, $1.m = 1.1_2 = 1.5_{10}$, number $= (-1)^1 \times 1.5 \times 2^{151-127} = -1.5 \times 2^{24} = -25165824$.

The number 0.5 is 1.0×2^{-1}. The difference between the exponents of `a` and `b` is therefore 25. To perform the addition, the CPU would first match the exponents by shifting the mantissa of `b` 25 places to the right. This would, unfortunately, reduce `b` to zero. The result of the addition would therefore be -25165824. [10]

Examiner's remarks: Another bimodal distribution of marks. 170 candidates scored 7+, 55 scored 3−. So it is pleasing to see that nearly 60% of the candidates have a solid grasp of floating point representation and its consequences for machine addition. Of the 51 candidates scoring middling marks, the most common pattern was forgetting how the IEEE floating point representation works, but saying sensible things about rounding errors when adding a small number to a much larger one.

Andrew Gee
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