ENGINEERING TRIPOS PART IA

Tuesday 14 June 2005 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

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SECTION A

*Answer **all** questions in this section.*

1 **(short)** Express

$$\frac{\sin x}{\sqrt{1+2x^2}}$$

as a power series in x, up to and including the term in x^3. [10]

2 **(short)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + 10y = e^{-2x} \ .$$ [10]

3 **(short)** Find all the solutions of the equation $z^4 - z^2 + 1 = 0$. Show the location of each solution on an Argand diagram. [10]

4 (long) (a) Find the equation of the plane containing the line

$$\mathbf{r} = \begin{bmatrix} 2 \\ -3 \\ 1/2 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 2 \\ 3 \end{bmatrix}$$

and the point $(1, 0, 2)$. Express your answer in the form $ax + by + cz = d$. [10]

(b) Find the shortest distance between the plane

$$2x - y + z = 3$$

and the line

$$\frac{4x-8}{1} = \frac{y+3}{2} = \frac{2z-1}{3} \ .$$ [10]

(c) The point $\mathbf{y}$ is the reflection of the point $\mathbf{x}$ in the plane which passes through the origin and has unit normal $\mathbf{n}$. Show that

$$\mathbf{y} = \mathbf{x} - 2(\mathbf{x}.\mathbf{n})\mathbf{n} \ .$$

Hence, find a matrix $\mathbf{R}$ such that $\mathbf{y} = \mathbf{R}\mathbf{x}$. Calculate $\mathbf{R}$ for the particular plane $2x - y + z = 0$. What would you expect the determinant of $\mathbf{R}$ to be? Check that your answer has the expected determinant. [10]

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5 **(long)** (a) Solve the difference equation

$$x_{n+2} - 8x_{n+1} + 15x_n = 0 \, ,$$

with initial values $x_0 = 0$ and $x_1 = 1$. [7]

(b) Consider the simultaneous difference equations

$$\begin{aligned} x_{n+1} &= 0.6x_n \, , \\ y_{n+1} &= 0.1x_n + 0.9y_n + 0.2z_n \, , \\ z_{n+1} &= 0.3x_n + 0.1y_n + 0.8z_n \, , \end{aligned}$$

with initial values $x_0 = N$ and $y_0 = z_0 = 0$.

(i) Show that the equations can be written in the form $\mathbf{p}_{n+1} = \mathbf{A}\mathbf{p}_n$, where $\mathbf{p}_n = [x_n \; y_n \; z_n]^T$. Find the coefficients of the matrix $\mathbf{A}$. [3]

(ii) Find the eigenvalues and eigenvectors of $\mathbf{A}$. Explain why the eigenvectors are not orthogonal. [12]

(iii) Express $\mathbf{p}_0$ in terms of the eigenvectors of $\mathbf{A}$. Hence find the limiting value of $\mathbf{p}_n$ as $n \to \infty$. [8]

SECTION B

*Answer **all** questions in this section.*

6 **(short)** Solve using Laplace Transforms

$$\frac{dx}{dt} + x = e^{-2t} ,$$

where $x(0) = 1$. [10]

7 **(short)** The height h of a hill is given by

$$h = (x^2 + y)e^{-2x} .$$

Find the unit vector pointing in the direction of steepest descent at $x = y = 0$. How steep is the descent? [10]

8 **(short)** 30% of the population carry the MRSA bacterium. A diagnostic test detects MRSA in 95% of carriers and (incorrectly) 2% of non-carriers. I have just tested positive for MRSA: what is the probability that I am a carrier? [10]

(TURN OVER

9 **(long)** (a) Explain how the principle of superposition can be applied to linear systems. [4]

(b) A linear system has step response $1 - e^{-t}$. Find its response to the input

$$x(t) = \begin{cases} 0 & t < 0\,, \\ 1 & 0 \leq t \leq T\,, \\ -1 & t > T\,. \end{cases}$$

Check that your answer behaves as expected in the limit $T \to 0$. [8]

(c) Now use convolution to find the system's response to the input

$$x(t) = \begin{cases} 0 & t < 0\,, \\ 1 - e^{-t} & 0 \leq t \leq T\,, \\ 1 - e^{-T} & t > T\,. \end{cases}$$

Check that your answer is continuous at $t = T$. [18]

10 **(long)** (a) The following functions are all equal to $x - x^2$ in the range $0 \le x \le 0.5$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin 2n\pi x$$

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos 2n\pi x$$

$$h(x) = \sum_{n=1}^{\infty} c_n \sin(2n-1)\pi x$$

What are the periods of $f(x)$, $g(x)$ and $h(x)$? Sketch each function in the range $-1 \le x \le 1$, explaining clearly how you arrived at your sketch. Do not calculate any coefficients. [11]

(b) Which of $f(x)$, $g(x)$ and $h(x)$ would you expect to converge the slowest, and which the fastest? Justify your answers. [4]

(c) Find the coefficients a_0 and a_n ($n \ge 1$) for the function $g(x)$. [15]

SECTION C

*Answer **all** questions in this section.*

11 **(short)** Consider the following segment of C++ code.

```
int main() {
  int a, b = 5;
  if (factorial(b, a))
    cout << "The factorial of " << b << " is " << a << endl;
  return 0;
}
bool factorial (int x, int &y) {
  if (x < 0) return false;
  y = 1;
  while (x>1) { y = y*x; x = x-1; }
  return true;
}
```

When the program is executed, what is printed on the screen? What would be printed if the first parameter were passed to the `factorial` function by reference instead of by value? What is the algorithmic complexity of the `factorial` function? [10]

12 **(short)** The `float` variable `a` has the following IEEE standard single precision floating point representation.

1	10010111	10000000000000000000000

What is `a` in decimal? Assuming standard single precision arithmetic, what would be the result of adding `a` to the `float` variable `b = 0.5`? [10]

END OF PAPER