ENGINEERING TRIPOS PART IA

2006

Paper 1 MECHANICAL ENGINEERING

SECTION A

1 (a) Bernoulli along surface streamline gives:

$$\cancel{p_0} + \tfrac{1}{2}\rho V_*^2 + \rho g(h-d) = \cancel{p_0} + \tfrac{1}{2}\rho V^2 + \rho g h$$

$$\Rightarrow \quad \tfrac{1}{2}\rho V_*^2 = \rho g d + \tfrac{1}{2}\rho V^2$$

$$\Rightarrow \quad \underline{\underline{V_*^2 = 2gd + V^2}}$$

(b) Continuity gives:

$$hV = (h - \delta - d)V_* = (h - 2d)V_*$$

$$\Rightarrow \quad V_* = \frac{V}{1 - 2d/h}$$

$$\Rightarrow \quad \frac{V^2}{(1 - 2d/h)^2} = V^2 + 2gd$$

$$\Rightarrow \quad \underline{\underline{\frac{V^2}{2gd} = \left(1 + \frac{V^2}{2gd}\right)\left(1 - \frac{2d}{h}\right)^2}}$$

2 (a) Applying Bernoulli from nozzle exit to fluid leaving the bucket:

$$p + \tfrac{1}{2}\rho V^2 + \rho g y = \text{constant}$$

$$p = 0 \text{ and } y \approx \text{constant} \Rightarrow V = \text{constant}$$

(b) Applying the momentum equation to control volume shown below.

Force on fluid

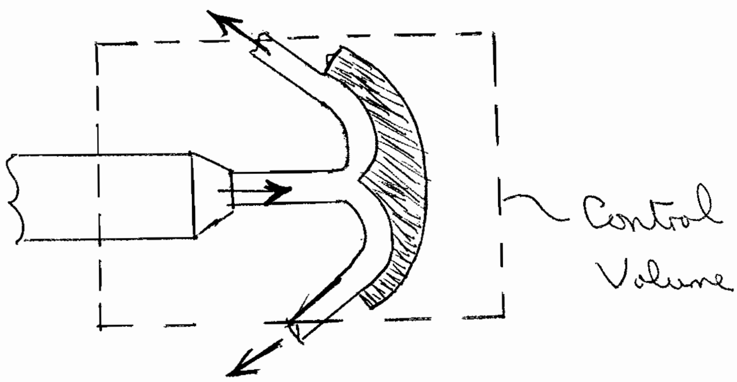
$$\sum_{OUT} \dot{m}\,\underset{\sim}{u} - \sum_{IN} \dot{m}\,\underset{\sim}{u} = \sum \underset{\sim}{F}$$

Apply in direction of flow leaving the Nozzle:

$$-\dot{m} V \cos\beta - \dot{m} V = -F$$

force on bucket

$$\underline{\underline{F = \dot{m} V (1 + \cos\beta)}}$$



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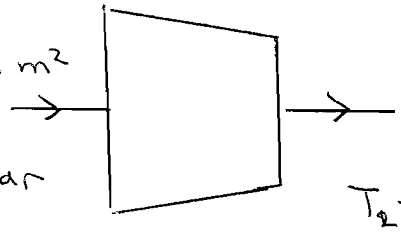
$A_1 = 0.1\ m^2$



$P_1 = 1$ Bar

$T_1 = 290K$

$c_1 = 18\ m/s$

$T_2 = 370K$

$c_2 = 6 m/s$

$$\begin{cases} c_p = 1.00\ kJ/kg/K \\ R = 0.287\ kJ/kg/K \end{cases}$$

(a) $$P_1 = \rho_1 R T_1 \Rightarrow \rho_1 = \frac{P_1}{RT} = \frac{10^5 N m^{-2}}{287\ J/kg/K \times 290K}$$

$$= \underline{\underline{1.201\ kg/m^3}}$$

$$\dot{m} = \rho_1 A_1 c_1 = \underline{\underline{2.163\ kg/s}}$$

(b) $$\dot{W} = -\dot{m}\left((h_2 - h_1) + \tfrac{1}{2}(c_2^2 - c_1^2)\right) \quad (\text{S.F.E.E.})$$

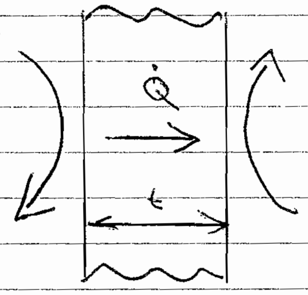
$$= -\dot{m}\left[c_p(T_2 - T_1) + \tfrac{1}{2}(c_2^2 - c_1^2)\right]$$

$$= -2.163\left[10^3 \times 80 - \tfrac{1}{2}(18^2 - 6^2)\right]$$

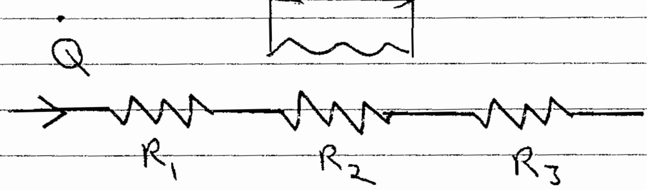
$$= -2163\left[80 - 0.144\right]$$

$$= \underline{\underline{-172.7 \times 10^3 \text{ Watts}}}$$

4 (a) Calculate thermal resistance



$$\begin{cases} \dot{Q} = hA\,\Delta T \\ \dot{Q} = -\lambda A \dfrac{\partial T}{\partial x} \end{cases}$$



$$\Rightarrow \begin{cases} R_1 = \dfrac{1}{hA} \quad R_2 = \dfrac{t}{\lambda A} \\ R_3 = \dfrac{1}{hA} \end{cases}$$

Total resistance : $\sum R = \frac{1}{A}\left(\frac{2}{h} + \frac{t}{\lambda}\right)$

$$\Rightarrow \quad \dot{Q} = \frac{T_f - T_a}{\sum R} = \frac{T_f - T_a}{\frac{1}{A}\left(\frac{2}{h} + \frac{t}{\lambda}\right)}$$

$$\Rightarrow \quad \dot{q} = \frac{\dot{Q}}{A} = \frac{T_f - T_a}{\frac{2}{h} + \frac{t}{\lambda}}$$

5 (a)

$$W = \int F\,dx = \int PA\,dx = \int P\,dV$$

$$PV = mRT \Rightarrow \quad W = \int_{V_a}^{V_b} \frac{mRT}{V}\,dV = mRT \ln\frac{V_b}{V_a}$$

$$= P_a V_a \ln\frac{V_b}{V_a}$$

Ideal gas $\Rightarrow$ $U = U(T)$

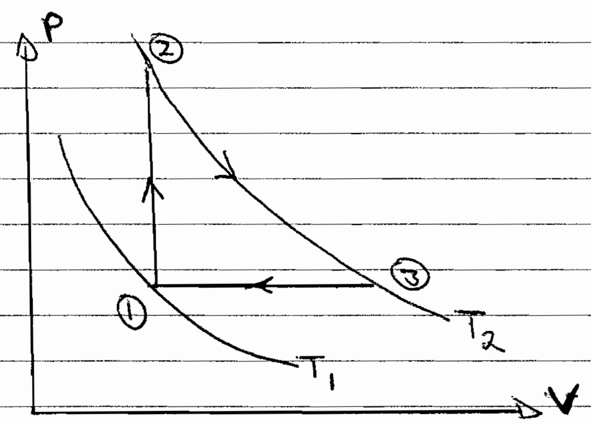
T constant $\Rightarrow$ U = constant

1st Law: $Q - W = \Delta U \Rightarrow Q = W = P_a V_a \ln\frac{V_b}{V_a}$

(Heat into gas)

(b)

(i)



$1 \to 2$: $PV = mRT$, V = const $\Rightarrow$ $\frac{T_2}{T_1} = \frac{P_2}{P_1} = r$

$3 \to 1$: $PV = mRT$, P = const $\Rightarrow$ $\frac{V_3}{V_1} = \frac{T_2}{T_1} = r$

(ii) $1 \to 2$: V = constant $\Rightarrow W = 0 \Rightarrow Q = \Delta U$.

$$\Rightarrow Q = mc_v(T_2 - T_1)$$

$$\Rightarrow Q = mc_v T_1 (r-1)$$

$2 \to 3$: T = constant $\Rightarrow$ U = constant $\Rightarrow$ $Q = W$

$$\Rightarrow Q = mRT_2 \ln\frac{V_3}{V_2}$$

(from (a))

$$\Rightarrow Q = mRT_1 r \ln r$$

$3 \to 1$: $Q = W + \Delta U = -P_1(V_3 - V_1) - mc_v(T_2 - T_1)$

$$= -P_1V_1(r-1) - mc_vT_1(r-1)$$

$$\Rightarrow \quad Q_{31} = -mRT_1(r-1) - mc_vT_1(r-1)$$

$$= -m(R+c_v)T_1(r-1)$$

(iii) $$\eta = \frac{\Sigma W}{Q_{IN}} = \frac{\Sigma Q}{Q_{IN}}$$

$$\Sigma Q = mT_1\left[c_v(r-1) + R\,r\ln r - (R+c_v)(r-1)\right]$$

$$= mRT_1\left[r\ln r - (r-1)\right]$$

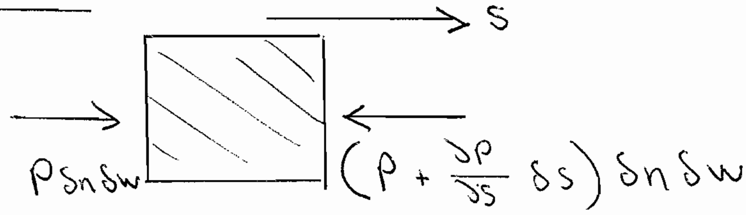
$$Q_{IN} = mT_1\left[c_v(r-1) + R\,r\ln r\right]$$

$$\Rightarrow \eta = \frac{R(r\ln r - (r-1))}{c_v(r-1) + R\,r\ln r}$$

$$\Rightarrow \eta = \frac{r\ln r - (r-1)}{r\ln r + (r-1)c_v/R}$$

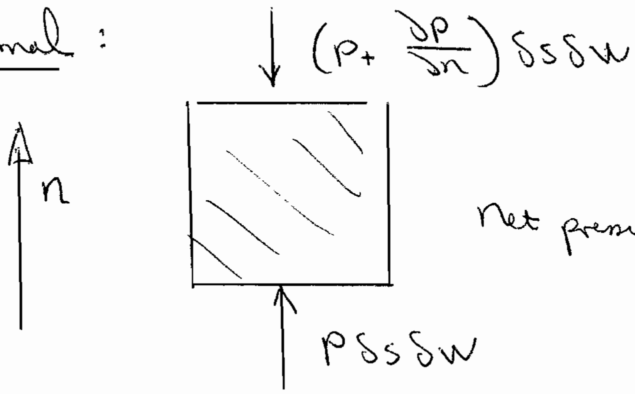
6 (a) (i)

Streamwise:



Net pressure force is $\underline{\underline{-\frac{\partial p}{\partial s}\,\delta s\,\delta n\,\delta w}}$

Normal:



Net pressure force is $\underline{\underline{-\frac{\partial p}{\partial n}\,\delta s\,\delta n\,\delta w}}$

(a) (ii) Newton II:

// direction: $\rho(\delta n\,\delta s\,\delta w)\,V\frac{\partial V}{\partial s} = F_{\parallel} = -\frac{\partial p}{\partial s}\,\delta s\,\delta n\,\delta w$

$$\Rightarrow \quad \underline{\underline{\rho V\frac{\partial V}{\partial s} = -\frac{\partial p}{\partial s}}}$$

$\perp$'ar direction: $\rho(\delta n\,\delta s\,\delta w)\left(-\frac{V^2}{R}\right) = -\frac{\partial p}{\partial n}\,\delta s\,\delta n\,\delta w$

$$\Rightarrow \quad \underline{\underline{\rho\frac{V^2}{R} = \frac{\partial p}{\partial n}}}$$

(a) (iii) $$\frac{\partial p}{\partial s} + \rho V\frac{\partial V}{\partial s} = 0 \Rightarrow \frac{\partial}{\partial s}\left(p + \tfrac{1}{2}\rho V^2\right) = 0$$

Integrate: $\underline{\underline{p + \tfrac{1}{2}\rho V^2 = \text{const.}}}$ along s-line

(b) (i) $\begin{cases} 4 \text{ variables} & h-h_0,\ g,\ \omega,\ r \\ 2 \text{ dimensions}, & \text{length, time} \end{cases}$

Π theorem tells us 2 dimensionless groups.

By inspection they are, $h-h_0/r$ and $\omega^2 r/g$

Thus $$\frac{h-h_0}{r} = G\left(\frac{\omega^2 r}{g}\right)$$

(ii) $$\frac{\partial p}{\partial n} = \rho \frac{V^2}{R} \Rightarrow \underline{\underline{\frac{\partial p}{\partial r} = \rho \frac{V^2}{r} = \rho \omega^2 r}}$$

Integrate $$\underline{\underline{p = p_0 + \tfrac{1}{2}\rho \omega^2 r^2}}$$

(iii) No vertical acceleration $\Rightarrow$ $p(z)$ is hydrostatic.

Thus pressure on base is $p = \rho g h$.

$$\Rightarrow \quad \rho g h = \rho g h_0 + \tfrac{1}{2}\rho \omega^2 r^2$$

$$\Rightarrow \quad \underline{h - h_0 = \frac{\omega^2 r^2}{2g}}$$

ENGINEERING TRIPOS PART IA, 2006

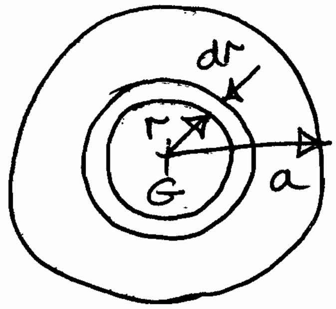
PAPER 1 MECHANICAL ENGINEERING

SECTION B MECHANICS

SOLUTIONS

PREPARED BY D J COLE 28-6-06

7 a)



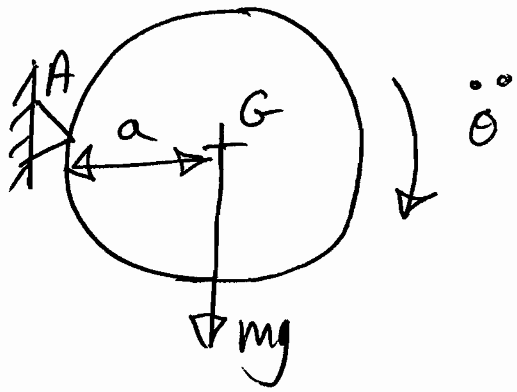
mass per unit area

$$I_G = \underbrace{\frac{m}{\pi a^2}}_{\text{mass per unit area}} \int_{r=0}^{r=a} 2\pi r \cdot r^2 \, dr$$

$$= \frac{2\pi m}{\pi a^2}\left[\frac{r^4}{4}\right]_0^a$$

$$I_G = \frac{ma^2}{2}$$

b)



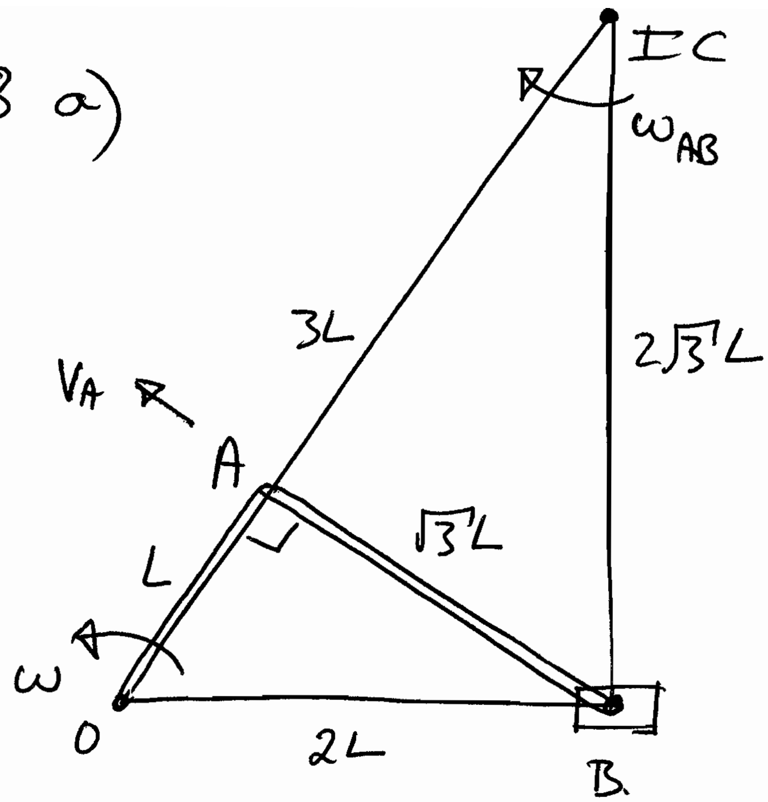
parallel axes $\quad I_A = I_G + ma^2 = \frac{3}{2}ma^2$

moments about A $\quad T = I_A \ddot{\theta}$

$$mga = \frac{3}{2}ma^2\ddot{\theta}$$

$$\ddot{\theta} = \frac{2}{3}\frac{g}{a}$$

8 a)



By similar triangles $IC.A = 3L$

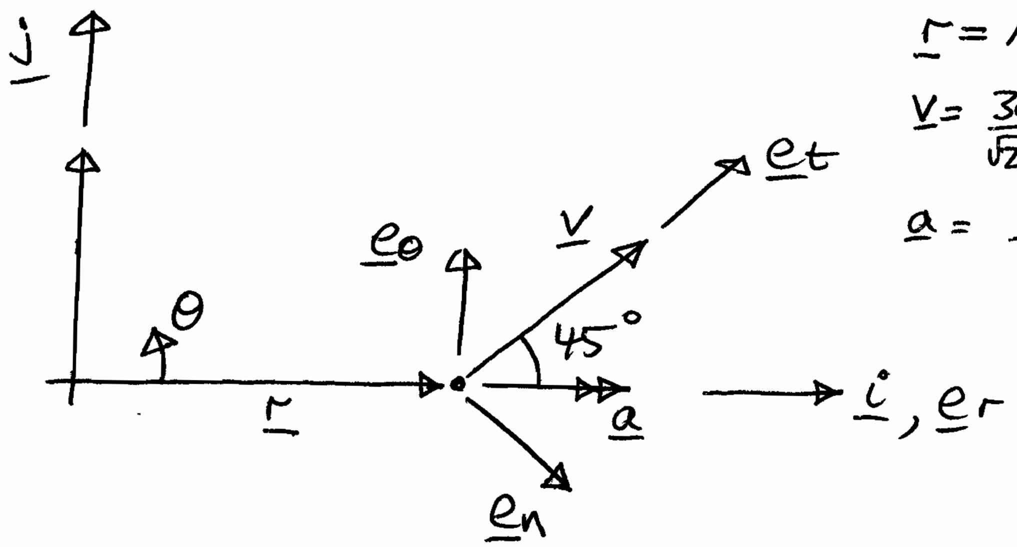
hence $\omega_{AB} = \dfrac{V_A}{IC.A} = \dfrac{\omega L}{3L} = \underline{\underline{\dfrac{\omega}{3}}}$

b) $T.\omega = \left(\omega + \dfrac{\omega}{3}\right) Q$

$\underline{\underline{T = \dfrac{4}{3} Q}}$

angular velocity between OA and AB

9



$\underline{r} = 1000\,\underline{i}$

$\underline{v} = \frac{300}{\sqrt{2}}\underline{i} + \frac{300}{\sqrt{2}}\underline{j}$

$\underline{a} = 1\underline{i}$

a) $\underline{v} = 300\,\underline{e_t}$ m/s where $\underline{e_t} = \frac{\underline{i}+\underline{j}}{\sqrt{2}}$

$\underline{a} = \frac{1}{\sqrt{2}}\underline{e_t} + \frac{1}{\sqrt{2}}\underline{e_n}$ m/s^2

b) $\underline{a} = \ddot{s}\,\underline{e_t} + \frac{\dot{s}^2}{\rho}\underline{e_n}$

$\therefore \quad \rho = \sqrt{2}\,\dot{s}^2 = \sqrt{2}\,300^2 = 90\sqrt{2}$ km

c) $\underline{v} = \dot{r}\,\underline{e_r} + r\dot{\theta}\,\underline{e_\theta}$

$\underline{v} = \frac{300}{\sqrt{2}}\underline{e_r} + \frac{300}{\sqrt{2}}\underline{e_\theta}$ m/s

$\underline{a} = 1.\,\underline{e_r} + 0.\,\underline{e_\theta}$ m/s^2

d) $\underline{a} = (\ddot{r} - r\dot{\theta}^2)\,\underline{e_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\underline{e_\theta}$

In the $\underline{e_r}$ direction

$$\ddot{r} - r\dot{\theta}^2 = 1$$

from (c) $\quad r\dot{\theta} = \dfrac{300}{\sqrt{2}}$

$$\therefore \quad \dot{\theta} = \frac{300}{\sqrt{2}\,1000} = \frac{0.3}{\sqrt{2}}$$

$$\therefore \quad \ddot{r} = 1000\left(\frac{0.3}{\sqrt{2}}\right)^2 + 1 = 1 + \frac{90}{2} = \underline{\underline{46\ m/s^2}}$$

In the $\underline{e_\theta}$ direction

$$2\dot{r}\dot{\theta} + r\ddot{\theta} = 0$$

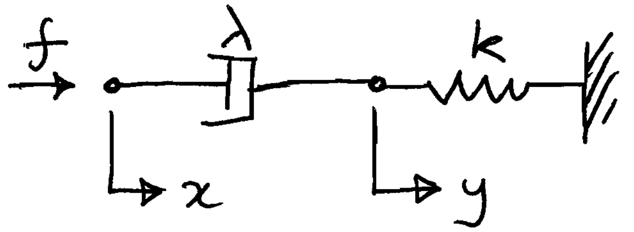
$$\ddot{\theta} = \frac{-2\dot{r}\dot{\theta}}{r}$$

$$= -2 \cdot \frac{300}{\sqrt{2}} \cdot \frac{0.3}{\sqrt{2}} \cdot \frac{1}{1000}$$

$$\underline{\underline{\ddot{\theta} = -0.09\ rad/s^2}}$$

i.e. 0.09 rad/s^2 in clockwise direction.

10 a)



force in spring = force in damper = f

spring $f = ky$, $\dot{f} = k\dot{y}$, $\dot{y} = \dfrac{\dot{f}}{k}$

damper $f = (\dot{x} - \dot{y})\lambda$, eliminate $\dot{y}$: $f = \dot{x}\lambda - \dot{f}\dfrac{\lambda}{k}$

hence $$\underline{\underline{\dot{x}k = \dot{f} + f\frac{k}{\lambda}}}$$

b) unit step in $\dot{x}$ $\rightarrow$ $\dot{x} = 1$ $t > 0$

$$\dot{f} + f\frac{k}{\lambda} = k$$

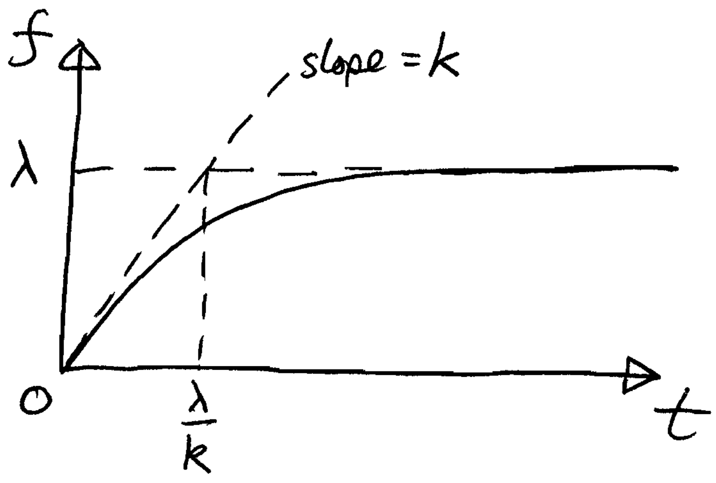
C.F. $f = Ae^{-\frac{k}{\lambda}t}$

P.I. $f = \lambda$

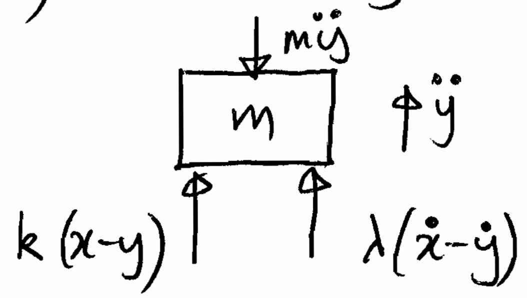
Gen. soln. $f = Ae^{-\frac{k}{\lambda}t} + \lambda$

but $f = 0$ at $t = 0$ $\therefore$ $A = -\lambda$.

so $$\underline{\underline{f = \lambda\left(1 - e^{-\frac{k}{\lambda}t}\right)}}$$



11 a) free body diagram

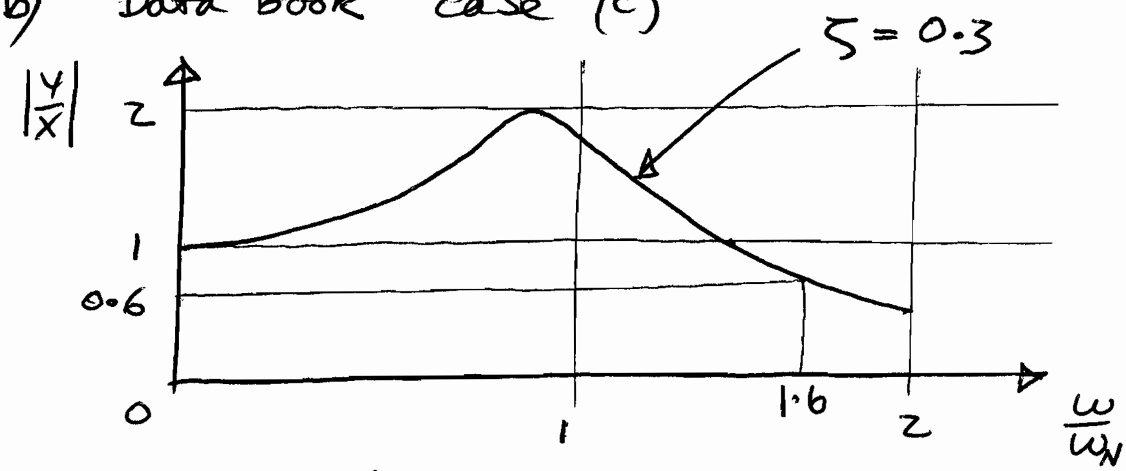


Newton's 2nd Law :

$$m\ddot{y} = k(x-y) + (\dot{x}-\dot{y})$$

$$\underline{\underline{m\ddot{y} + \lambda\dot{y} + ky = \lambda\dot{x} + kx}}$$

b) Data book case (c)

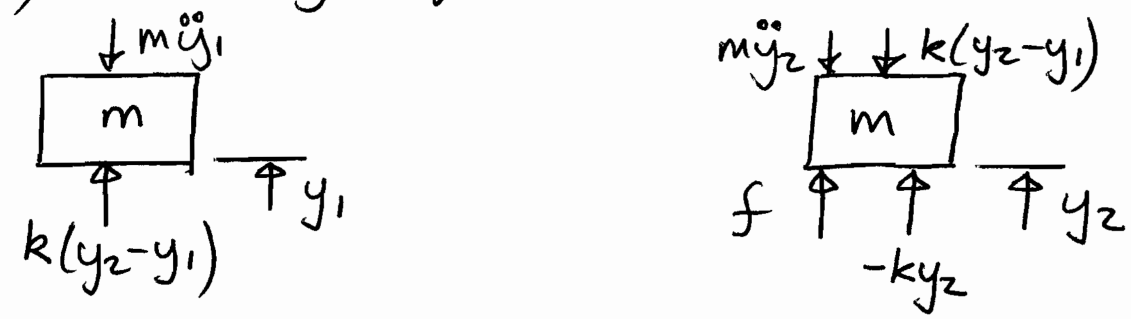


require $\left|\frac{Y}{X}\right|_{max} = 2$, so $\zeta = 0.3 = \dfrac{\lambda}{2\sqrt{km}}$

$$\therefore \lambda = \zeta\, 2\sqrt{km} = 0.3 \,.\, 2\sqrt{12.10^3 . 10}$$

$$\underline{\underline{\lambda = 207.9 \ \text{Ns/m}}}$$

12 a) Free body diagrams and Newton's 2nd Law:



$$m\ddot{y}_1 = k(y_2-y_1) \qquad m\ddot{y}_2 = f - ky_2 - k(y_2-y_1)$$

In matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ f \end{Bmatrix} \quad -1$$

b) $y_1 = Y_1 e^{i\omega t} \Rightarrow \ddot{y}_1 = -\omega^2 Y_1 e^{i\omega t}$

$y_2 = Y_2 e^{i\omega t} \Rightarrow \ddot{y}_2 = -\omega^2 Y_2 e^{i\omega t}$

hence

$$\left(-\omega^2\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}\right)\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

non-trivial solution when $\begin{vmatrix} k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{vmatrix} = 0$

$$(k-\omega^2 m)(2k-\omega^2 m) - k^2 = 0$$

the solutions of which are the natural frequencies squared ω_1^2 and ω_2^2.

$$\omega^4 m^2 - \omega^2 3km + k^2 = 0$$

$$\omega^2 = \frac{3km \pm \sqrt{9k^2m^2 - 4m^2k^2}}{2m^2}$$

$$\omega^2 = \frac{3 \pm \sqrt{5}}{2}\frac{k}{m}$$

$$\omega_1^2 = \frac{3-\sqrt{5}}{2}\frac{k}{m} \qquad \omega_2^2 = \frac{3+\sqrt{5}}{2}\frac{k}{m}$$

c) $y_1 = Y_1 e^{i\omega t} \Rightarrow \ddot{y}_1 = -\omega^2 Y_1 e^{i\omega t}$

$y_2 = Y_2 e^{i\omega t} \Rightarrow \ddot{y}_2 = -\omega^2 Y_2 e^{i\omega t}$

$f = F e^{i\omega t}$

putting these into ① and rearranging gives

$$\left(-\omega^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}\right) \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}$$

$$\begin{bmatrix} k-m\omega^2 & -k \\ -k & 2k-m\omega^2 \end{bmatrix} \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F \end{Bmatrix}.$$

premultiply both sides by inverse of matrix:

$$\begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \frac{1}{(k-m\omega^2)(2k-m\omega^2)-k^2} \begin{bmatrix} 2k-m\omega^2 & k \\ k & k-m\omega^2 \end{bmatrix} \begin{Bmatrix} 0 \\ F \end{Bmatrix}.$$

hence $$\frac{Y_2}{F} = \frac{k-m\omega^2}{(k-m\omega^2)(2k-m\omega^2)-k^2}$$

and from (b) $$\underline{\underline{\frac{Y_2}{F} = \frac{k-m\omega^2}{m^2(\omega^2-\omega_1^2)(\omega^2-\omega_2^2)}}}$$

d)