Engineering Tripos Part IA 2006
Paper 2 Structures and Materials SOLUTIONS

1.

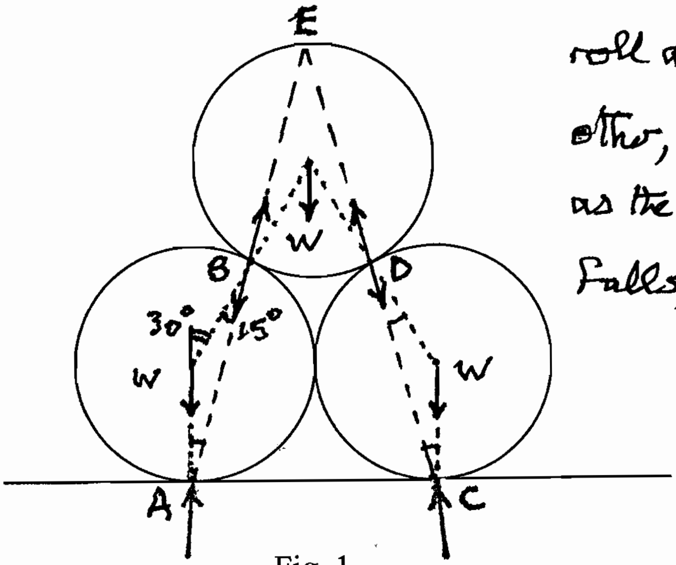


Fig. 1

In the limit, the two lower cylinders roll away from each other, losing contact, as the top cylinder falls, slipping at B & D.

If three forces act on a rigid body, their lines of action must pass through a point if the body is to be in moment equilibrium. Consider each of the lower cylinders. The line of action of the contact force at B must pass through A, and that of the contact force at D must pass through C.

The contact normals at B and D are at 60° to the horizontal (equilateral triangle), so 30° to the vertical. A trigonometric construction for the line of action AB shows it inclined at 15° to the vertical, and therefore 15° to the contact normal. We therefore require

$\underline{\mu \geq \tan 15^\circ \sim 0.268}$ for equilibrium

1. Performance average mark 4.0/10
34 perfect answers /1284

Alternative perfect solutions were obtained by putting $F = \mu N$ at contact point B, for example, and taking moments of force for equilibrium about point A so that only F & N come into the equation. The geometry gives $\mu = 1/(2+\sqrt{3}) = 2-\sqrt{3} = 0\cdot268$ as before. But F may accidentally be reversed in direction, causing 4 marks to be lost. Friction must always oppose relative motion, and by symmetry the top cylinder tends to fall vertically while the bottom cylinders move away from each other by rolling. The friction F must therefore oppose outward movement of the lower cylinders.

Unfortunately many candidates got stuck looking at the vertical equilibrium of the top cylinder which does not even need friction at B and D to support it, as many noticed. By missing the clue that the "lower cylinders may roll but not slip" on the rough floor, these candidates never studied moment equilibrium, and missed the point of the question.

2. a)

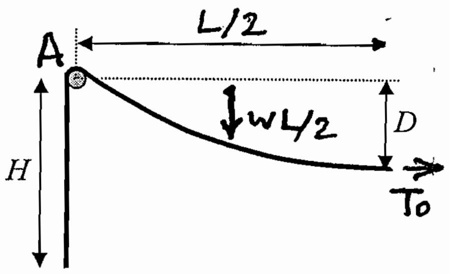


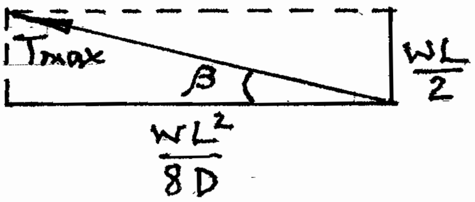
Fig. 2

Taking moments for the half-cable about peg A

$$T_o D = \frac{w(L/2)^2}{2}$$

(using $D << L$ to fix the centre of gravity at $L/4$ from the peg).

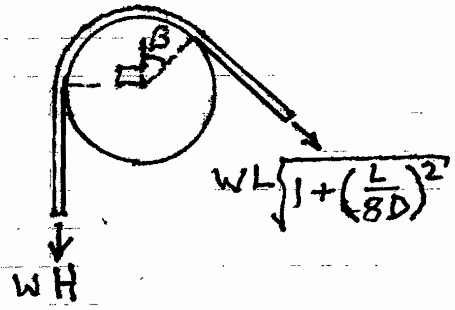
At the peg, the cable has to carry a horizontal component To and a vertical component $wL/2$



$$\therefore T_{max} = \sqrt{\left(\frac{wL}{2}\right)^2 + \left(\frac{wL^2}{8D}\right)^2}$$

$$\text{so } \underline{T_{max} = \frac{wL}{2}\sqrt{1+\left(\frac{L}{4D}\right)^2}}$$

b)



The cable wraps $\left(\frac{\pi}{2}+\beta\right)$ around the peg, where

$$\tan\beta = \frac{wL/2}{(wL^2/8D)} = \frac{4D}{L}$$

For slippage of a cable around a peg, we know

$$\frac{T_{big}}{T_{small}} = \exp(\mu\theta) \qquad \text{where } \theta = \frac{\pi}{2}+\beta \text{ in this case.}$$

Considering the "drop" pulling down ($T_{big} = wH$) and being pulled up ($T_{small} = wH$) and cancelling w we get, for equilibrium

$$\frac{L}{2}\sqrt{1+\left(\frac{L}{4D}\right)^2}\exp\left[-\mu\left(\frac{\pi}{2}+\tan^{-1}\frac{4D}{L}\right)\right] < H$$

$$H < \frac{L}{2}\sqrt{1+\left(\frac{L}{4D}\right)^2}\exp\left[+\mu\left(\frac{\pi}{2}+\tan^{-1}\frac{4D}{L}\right)\right]$$

2. Performance average mark 5·6/10
6 perfect solutions / 284

3 marks were awarded for part (b) and no actual calculations were required. But for full marks, candidates should have mentioned that there is an exponential variation of tension in a cable slipping around a rough peg and that there were two extremes — H too large causing outward slip and the cable to pull straight, and H too small causing inward slip and the cable to sag and fall between the pegs.

2 marks were lost in part (a) if candidates only calculated the horizontal component of tension and mistook this for the maximum tension.

Unfortunately, some candidates first found the total vertical reaction on a peg and then used it in a moment equilibrium equation for a segment of cable in the central part. This violates vertical equilibrium for the central part. It is best to imagine cutting the cable just inside the point of contact with pegs to define the equilibrium of the central part of width L. The outer lengths H can not enter into the equilibrium of the central zone unless the cable is slipping.

3. a)

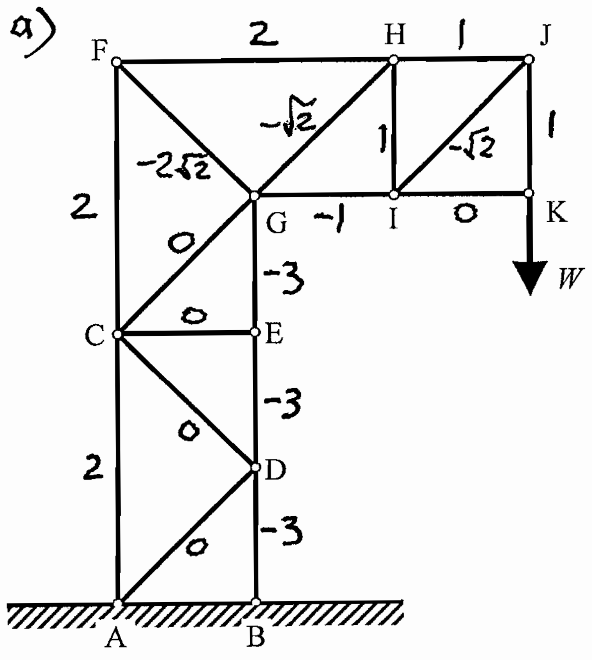


Fig. 3

A horizontal section anywhere through the tower ABGC shows zero force in the diagonals is required by horizontal equilibrium. Equilibrium at joints E and K requires $T_{CE} = T_{IK} = 0$. Considering whole structure, moments about A give $T_{BD} = -3W$, while vertical equilibrium gives $T_{AC} = 2W$.

All other bar tensions found by sections or resolution of forces at joints. See Table.

b) If the tension in a member is T, the strain will be $(\alpha\sigma_y/E) \times$ sign of T, so the extension $e = (L\alpha\sigma_y/E) \times \frac{T}{|T|}$ and values of $eE/L\alpha\sigma_y$ are listed in the Table.

c) If the vertical displacement of joint K is δ_v, the Virtual Work Principle gives:

$$\underbrace{\overbrace{W\,\delta_v = \sum T}^{\text{equilibrium}}\,e}_{\text{compatible kinematics}}$$

In this case, both components are "real".

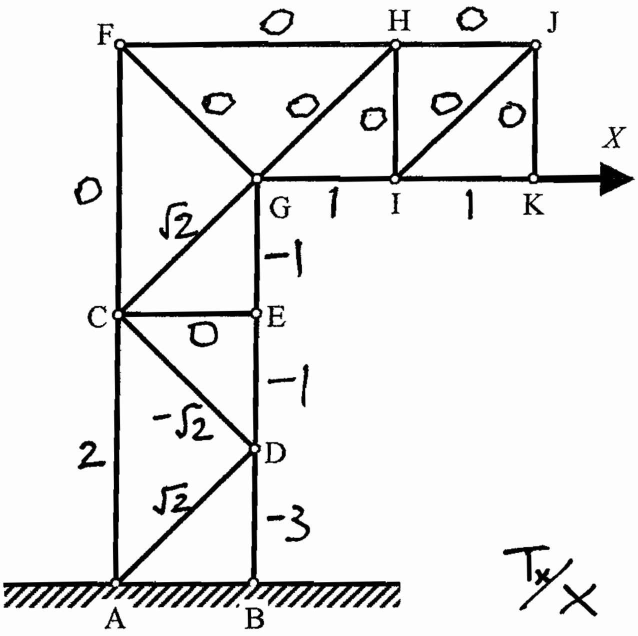
Using the Table: $\underline{\delta_v = 33\,\frac{L\alpha\sigma_y}{E}}$

We want the horizontal displacement δ_h of joint K, so apply force X horizontally at K.

Table for Question 3

bar	length	$\frac{T}{W}$	$\frac{eE}{L\alpha\sigma_y}$	$\frac{T}{W}\cdot\frac{eE}{L\alpha\sigma_y}$	$\frac{T_x}{X}$	$\frac{T_x}{X}\cdot\frac{eE}{L\alpha\sigma_y}$
AC	$2L$	2	2	4	2	4
AD	$\sqrt{2}L$	0	0	0	$\sqrt{2}$	0
BD	L	-3	-1	3	-3	3
CD	$\sqrt{2}L$	0	0	0	$-\sqrt{2}$	0
CE	L	0	0	0	0	0
CF	$2L$	2	2	4	0	0
CG	$\sqrt{2}L$	0	0	0	$\sqrt{2}$	0
DE	L	-3	-1	3	-1	1
EG	L	-3	-1	3	-1	1
FG	$\sqrt{2}L$	$-2\sqrt{2}$	$-\sqrt{2}$	4	0	0
FH	$2L$	2	2	4	0	0
GH	$\sqrt{2}L$	$-\sqrt{2}$	$-\sqrt{2}$	2	0	0
GI	L	-1	-1	1	1	-1
HI	L	1	1	1	0	0
HJ	L	1	1	1	0	0
IJ	$\sqrt{2}L$	$-\sqrt{2}$	$-\sqrt{2}$	2	0	0
IK	L	0	0	0	1	0
JK	L	1	1	1	0	0
				33		8

3. c) cont.



The force X is transmitted horizontally to the tower. A vertical section through the tower reveals the tensions T_x required for horizontal equilibri

Moments about A give $T_x(BD) = -3X$.

All other bar tensions are found by resolution of forces at the joints; they are listed as T_x/X in the Table.

Virtual Work gives:

$$\overbrace{X\,\underbrace{\delta_h \;=\; \Sigma\, T_x\, e}_{\text{real kinematics}}}^{\text{virtual forces}}$$

Using the Table : $\underline{\delta_h = 8\,\frac{L\alpha\sigma_y}{E}}$

Point of interest. The structure is 3L wide and 5L high, and all bars are allowed to strain by $(\alpha\sigma_y/E)$. Blindly applying the strain to the structure as a whole, one would have got movements an order of magnitude smaller than they actually are. This is due to "bending" of the structure.

3. Performance average mark 17.9/30 ≏ 60%
11 perfect solutions / 284

In part (a), many candidates were inefficient in their calculation of bar forces. The method of sections, plus the free body diagram, was more effective than 9 consecutive considerations of joint equilibrium. Most understood the value of the bracing members.

Part (b) was misunderstood by most candidates. It simply intended that all extensions and compressions be proportional to the length of the member, except that bracing members should have no change in length because they have zero axial force with some finite sectional area. For the loaded bars, it remains important to distinguish extension (+ve) from compression (−ve). Most candidates reverted to the "usual" assumption that sectional areas were constant A. If they then quoted their deduced displacement units as WL/EA, e.g. $\delta_v = (55 + 12\sqrt{2})\,WL/EA$, they were not penalised in part (c). Harsher penalties were applied to candidates with dimensionally incorrect displacement units, of which $E/(W^2 L \alpha \sigma_y)$ was the worst example.

4. Moments about A give:

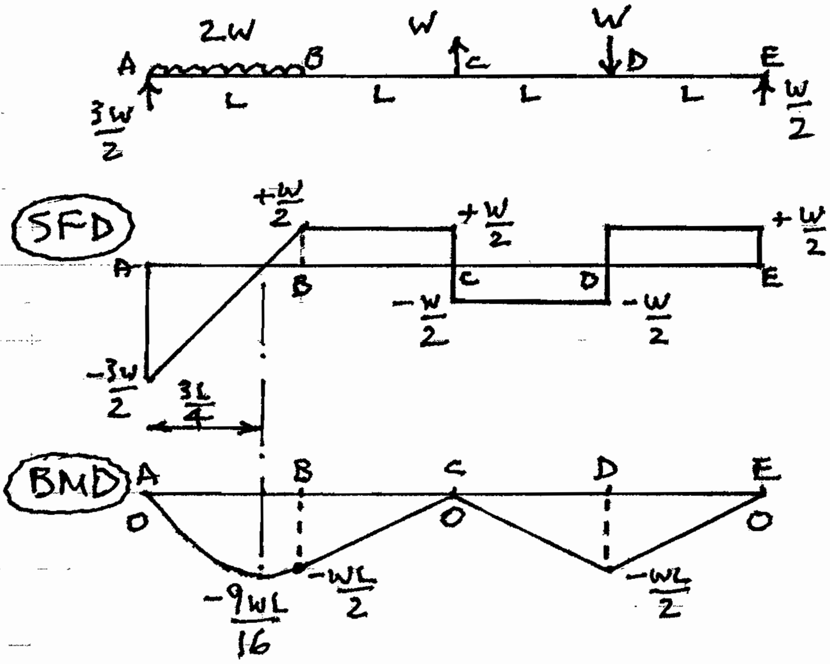
$$2W\,{}^{L}\!/_{2} \;-\; W.2L \;+\; W.3L \;-\; R_E.4L = 0$$

$$\therefore \quad \underline{R_E = W/2}$$

Resolve forces vertically to get:

$$2W - W + W = R_E + R_A$$

$$\therefore \quad \underline{R_A = 3W/2}$$



$$S = \frac{dM}{dx}$$

$$M = \int S\,dx$$

Between A and B the bending moment has a maximum when the shear force is zero, i.e at $\underline{3L/4 \text{ from } A}$.

$$M_{max} = \int_A^B S\,dx = -\frac{3W}{2}\cdot\frac{3L}{4}\cdot\frac{1}{2} = \underline{-\frac{9}{16}WL}$$

By inspection of the areas on the SFD, or by taking moments directly,

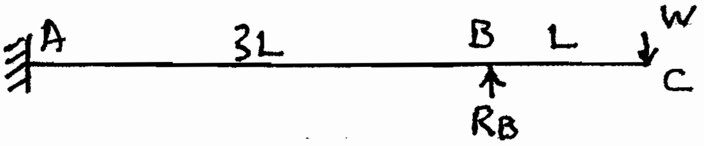
$$\underline{M_D = -\frac{WL}{2} \;;\quad M_C = 0 \;;\quad M_B = -\frac{WL}{2}}$$

4. <u>Performance</u> average mark 7.6/10
76 perfect solutions/284

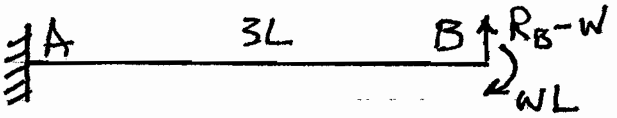
Where marks were lost, it was generally because the candidate did not hold firm to a consistent sign convention. Time was lost by candidates who decided to write down general expressions for bending moment at every point, rather than following the suggested sequence of SF & BM diagrams.

Those candidates who failed to calculate the location and magnitude of the maximum bending moment, but who sketched the correct shape of BM diagram, were only penalised 2 marks.

5. Let the reaction at B be R_B upwards.



Consider the bending of portion AB, transferring force W from C to B and compensating with a clockwise couple WL.



Solve for upward force $(R_B - W)$ and couple WL such that the deflection of B relative to A is zero. Using the Structures Data Book p. 6, we get:

$$\frac{(R_B - W)(3L)^3}{3EI} = \frac{WL(3L)^2}{2EI}$$

$$\therefore \quad R_B - W = \frac{W}{2}$$

$$\therefore R_B = \underline{\frac{3W}{2}}$$

5. Performance average mark 4.3/10
22 perfect answers / 284

This is a redundant structure which can only be solved by including equations of compatibility. Those many candidates who failed to realise this, and who resolved forces and took moments about various places until they had generated an error that permitted statics to "solve" the problem, had missed the point of the question and could only be given 1 or 2 charity marks. The majority did aim for elastic compatibility, but most of these made an important mistake. They replaced section BC by a moment WL at B, but forgot the extra shear-force W down at B, so they got $R_B = W/2$ instead of $3W/2$. A few realised that the last standard case listed on P6 of the Structures Data Book gave a two line solution. Since $M_B = WL$ and $M_A = \frac{1}{2} M_B$, $M_A = 1/2\ WL$ so moments about A give $R_B = 3W/2$.

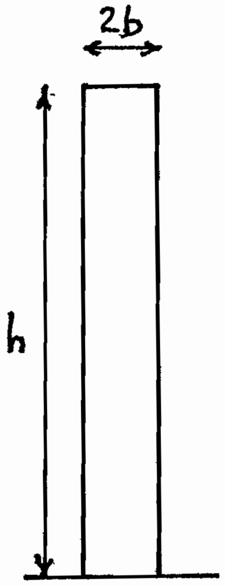
Those who understood elastic compatibility but who chose to answer the question as though W were applied and R_B adjusted to prevent tip C from moving, were only penalised 3 marks if they got their "own question" right.

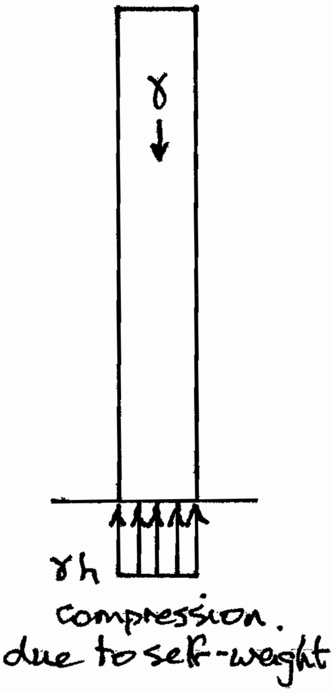
6. a) The axial compression in the chimney arises from the self-weight of material lying above the section of interest. So the maximum axial compression occurs at the base of the chimney, having increased linearly from the top.

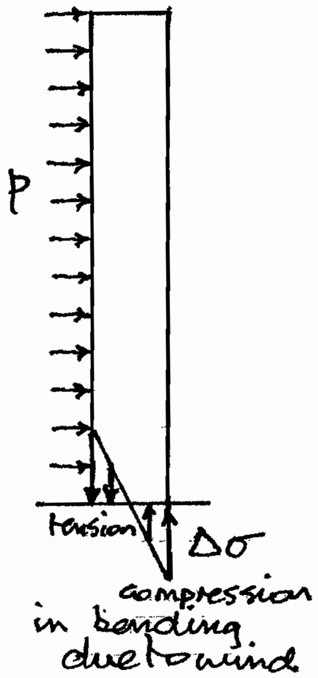
The bending moment in the chimney arises from the moment of wind forces applied above the section of interest. So the maximum bending moment also occurs at the base of the chimney. Bending moment increases by the square of the distance from the top.

Bending stresses are tensile (towards the wind-blown face) and compressive (on the opposing face). So greatest compressive stress (threatening crushing) occurs at the base and opposite the wind, where axial load and bending moment effects combine. And smallest compressive stress (threatening tension) also occurs first at the base, where the quadratic tension overcomes the linearly increasing compression.

6. b)





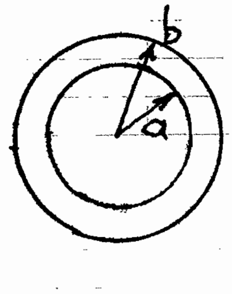


The bending moment is a maximum at the base.

The total lateral load $= p \cdot 2b \cdot h$

$$M = p \cdot 2b \cdot h \cdot \frac{h}{2} \quad = \underline{pbh^2.}$$

The cross section:



$$I = \frac{\pi b^4}{4} - \frac{\pi a^4}{4} = \frac{\pi}{4}(b^4 - a^4)$$

Let $\Delta\sigma$ be the maximum bending stress (+ve and -ve).

$$\frac{\Delta\sigma}{b} = \frac{M}{I} = \frac{pbh^2}{\frac{\pi}{4}(b^4-a^4)} = \frac{4}{\pi}\frac{p.bh^2}{(b^4-a^4)}$$

$$\text{so} \quad \underline{\Delta\sigma = \frac{4}{\pi} p.h^2.\frac{b^2}{(b^4-a^4)}}$$

6. cont.

Consider Limit ①. Self-weight compression plus bending compression must be less than σ_c.

$$\gamma h + \frac{4}{\pi} p h^2 \frac{b^2}{(b^4 - a^4)} < \sigma_c$$

$$\therefore \quad p < \frac{\pi}{4} \frac{(b^4 - a^4)}{b^2 h^2} (\sigma_c - \gamma h) \qquad (1)$$

Consider Limit ②. Self-weight compression must not be entirely eliminated by tension due to bending.

$$\frac{4}{\pi} p \frac{h^2 b^2}{(b^4 - a^4)} < \gamma h$$

$$\therefore \quad p < \frac{\pi}{4} \frac{(b^4 - a^4)}{b^2 h^2} \cdot \gamma h \qquad (2)$$

IF $\gamma g h > \frac{\sigma_c}{2}$ (1) will be critical, else (2).

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6. Performance average mark 16.7/30 ≃ 56%
14 perfect answers

In part (a), some candidates restated the conclusion instead of explaining the reason.

In part (b), almost all candidates got the theory for first cracking correct. Many then ignored first crushing, or incorrectly ignored compression due to bending.

Less than half the candidates correctly used the Mechanics Data Book P.18 case 5.4.3 to deduce $I_{xx} = \pi/4(b^4-a^4)$, or could use integration or the perpendicular axis theorem to derive it. Most confused I_{xx} and I_{TT} (or I_{ZZ}), and a few lapsed into J and even got the mass of the chimney involved in moment of inertia.

Carelessness with radii and diameters was endemic. And many candidates interpreted "area of elevation" as something other than 2bh, wrongly taking πbh or $2\pi bh$. Only 1 or 2 marks were deducted.

Candidates who wrongly thought the chimney could topple, or who introduced bending deflection or buckling, lost most of the available marks.

PART 1A ENGINEERING TRIPOS 2006
PAPER 2 STRUCTURES AND MATERIALS

Q7)
(a) The tensile fracture of brittle materials such as ceramics occurs by fast fracture from small (usually intrinsic) defects. In compression, the defects close-up, and fast fracture cannot occur. Instead, crack faces shear relative to one another, generating wing cracks at the crack ends. The wing cracks tend to be aligned parallel to the loading direction. Failure occurs by the progressive spreading and interlinking of cracks, leading to a general crushing failure at a much higher stress.

5

(b) The beam is stronger in bending for the following reasons.
(i) Only half the beam is in tension, so the volume which is subjected to tensile stress is only half that in the lengthwise loaded beam.
(ii) The tensile stress varies from 0 at the neutral axis to 500 MPa at the upper face, meaning that the average effective tensile stress is much less than 500 MPa.

5

Q8)
(a) The stainless steel is protected by a compact, strongly adherent film of chromium oxide, which is an excellent barrier to the diffusion of oxygen ions from the air environment to the steel proper. Since the oxidation rate is proportional to the rate of oxygen diffusion through the oxide film, in the case of stainless steel the oxidation rate is extremely small. The oxide formed on the surface of un-alloyed steel is not compact, strongly adherent or a good barrier to oxygen diffusion, so the oxidation rate is large.

3

(b) The table of standard electrode potentials given in the Materials Data Book (p37) shows that iron is more reactive (lower potential) than copper. This means that in a galvanic couple between iron and copper, the iron will become the anode (and will corrode) whereas the copper will become the cathode (and will be protected from corrosion). In addition, because the surface area of the cathode (the roofing sheet) is much more than that of the anode (the steel nails), the rate at which electrons are removed from the nails (and hence the rate at which they corrode) will be extremely large.

3

(c)

$$\frac{P}{a} \approx H \approx 3\sigma_y, P \approx 3a\sigma_y$$

$$\frac{F}{a} \approx k \approx \frac{\sigma_y}{2}, F \approx ak \approx \frac{a\sigma_y}{2}$$

$$F = \mu P \Rightarrow \mu = \frac{F}{P} \approx \frac{a\sigma_y}{2} . \frac{1}{3a\sigma_y} \approx \frac{1}{6}$$

4

Q9)
(a)
See lecture notes, or Ashby and Jones *Engineering Materials 1*, 3rd edition, Butterworth-Heinemann, 2005, Figs 9.3 and 9.4.

4

(b)
(i) When crystals yield, dislocations move through them on specific slip planes. Most crystals have several slip planes. Dislocations on these intersecting planes collide, and obstruct one another. This increases the shear stress needed to make the dislocations move, and hence increases the hardness.

2

(ii) The zinc atoms dissolve in the copper matrix to form a substitutional solid solution. The zinc atoms are bigger than the copper atoms, so a strain field is generated around each zinc atom. The strain fields interact with the dislocations, and obstruct their movement. Hence the alloy is harder than the pure metal.

2

(iii) The hard particles obstruct the movement of the dislocations. The dislocations cannot cut through the particles (because they are hard), and must bow between them instead. This increases the shear stress needed to make the dislocations move. Hence the precipitation hardened alloy is harder than the pure metal.

2

Q10)
(a)
(i) 510 MPa, 710 MPa, 30%.
(ii) n/a, 27 MPa, 6%.
(ii) 17 MPa, 23 MPa, 160%.

3

(b)
PMMA breaks by brittle fracture before it can yield. The graph shows linear elastic behaviour over the whole nominal stress range. Steel and Nylon 66 both show yielding, where the graph deviates from the straight (linear elastic) line. Both materials show strain hardening. However, after 15% strain, steel starts to develop an unstable neck (at the tensile strength). After a further 15% strain the cross section of the neck has decreased to the point at which final failure occurs. Nylon 66 shows stable necking. In the neck, the molecular chains become aligned in the loading direction. The neck grows into the neighbouring material, aligning the molecular chains as it spreads. After an extension of 160%, the whole specimen has necked, and failure then occurs by brittle tensile fracture of the molecular chains.

7

Q11) (a)
Since the cracks break the outer surface of the vessel, dye-penetrant inspection can be used. Since the vessel is made from steel, magnetic-particle testing can be used. Because the plane of the crack is perpendicular to the outer surface, an angled ultrasonic probe will detect the crack. X-ray detection will also work, provided an X-ray film can be placed inside the vessel under the welded seam (this depends on sufficient access to the interior of the vessel). Hydrostatic pressure testing and fracture mechanics calculations can also be used to establish the maximum size of crack.

8

(b) (i)

$$K_{\mathrm{IC}} = \sigma\sqrt{\pi a_{\mathrm{c}}}, a_{\mathrm{c}} = \frac{1}{\pi}\left(\frac{K_{\mathrm{IC}}}{\sigma}\right)^2$$

$$\sigma = \frac{pr}{t} = \frac{10\text{ MPa} \times 500\text{ mm}}{20\text{ mm}} = 250\text{ MPa}$$

$$a_{\mathrm{WP}} = \frac{1}{\pi}\left(\frac{3125\text{ MPa }\sqrt{\text{mm}}}{250\text{ MPa}}\right)^2 = 49.74\text{ mm}$$

3

(ii)

$$\sigma = \frac{pr}{t} = \frac{15\text{ MPa} \times 500\text{ mm}}{20\text{ mm}} = 375\text{ MPa}$$

$$a_{\mathrm{TP}} = \frac{1}{\pi}\left(\frac{3125\text{ MPa }\sqrt{\text{mm}}}{375\text{ MPa}}\right)^2 = 22.11\text{ mm}$$

3

(c) (i)

$$\frac{\mathrm{d}a}{\mathrm{d}N} = A(\Delta K)^3 = A(\Delta\sigma\sqrt{\pi a})^3 = A(\Delta\sigma)^3\pi^{3/2}a^{3/2}$$

$$\int_{a=22.11}^{a=49.74} \frac{\mathrm{d}a}{a^{3/2}} = A(\Delta\sigma)^3\pi^{3/2}\int_0^N \mathrm{d}N$$

$$\left[-\frac{2}{a^{1/2}}\right]_{a=22.11}^{a=49.74} = A(\Delta\sigma)^3\pi^{3/2}N$$

$$0.142 = 2\times10^{-13}(250)^3\pi^{3/2}N$$

$$N = 8149$$

10

(ii) Take the present case as an example. If the vessel is hydrostatically tested to 1.5 times working pressure and does not fail, then the largest crack is $a = 22.11$ mm (i.e. physical length of crack = 44.22 mm). If in subsequent service the vessel undergoes 8149 cycles of (0 to 10 to 0) MPa, the largest crack could be a = 49.74 mm (i.e. physical length = 99.48 mm). This would be on the point of causing fast fracture at the working pressure. To have a factor of safety, one should therefore subject the vessel to its next hydrostatic test after say 4000 cycles. If the vessel passes this test, then the largest crack is still $a = 22.11$ mm. This test/service cycle can be repeated until the vessel fails a hydrostatic test.

6

Q 12)

(a) (i)

$$p_{\mathrm{b}} = 0.3E\left(\frac{t}{r}\right)^2$$

$$m_{\mathrm{b}} = 4\pi r^2 t\rho$$

$$\left(\frac{t}{r}\right)^2 = \frac{p_{\mathrm{b}}}{0.3E}$$

$$\frac{t}{r} = \left(\frac{p_{\mathrm{b}}}{0.3E}\right)^{1/2}$$

$$t = \left(\frac{p_{\mathrm{b}}}{0.3E}\right)^{1/2} r$$

$$m_{\mathrm{b}} = 4\pi r^3 \rho\left(\frac{p_{\mathrm{b}}}{0.3E}\right)^{1/2} = 22.9 r^3 p_{\mathrm{b}}{}^{1/2}\left(\frac{\rho}{E^{1/2}}\right)$$

Materials performance index is $\left(\frac{E^{1/2}}{\rho}\right)$

8

(ii)

$$p_{\mathrm{f}} = 2\sigma_{\mathrm{f}}\left(\frac{t}{r}\right)$$

$$m_{\mathrm{f}} = 4\pi r^2 t\rho$$

$$\left(\frac{t}{r}\right) = \frac{p_{\mathrm{f}}}{2\sigma_{\mathrm{f}}}$$

$$t = \frac{p_{\mathrm{f}} r}{2\sigma_{\mathrm{f}}}$$

$$m_{\mathrm{f}} = \frac{4\pi r^3 p_{\mathrm{f}}\rho}{2\sigma_{\mathrm{f}}} = 2\pi r^3 p_{\mathrm{f}}\left(\frac{\rho}{\sigma_{\mathrm{f}}}\right)$$

Materials performance index is $\left(\frac{\sigma_{\mathrm{f}}}{\rho}\right)$

8

(b) (i)

Mass from result above.

$$t_{\mathrm{b}} = \frac{m_{\mathrm{b}}}{4\pi r^2 \rho}$$

(ii)

Mass from result above.

$$t_{\mathrm{f}} = \frac{m_{\mathrm{f}}}{4\pi r^2 \rho}$$

Set $r = 1$ m and $p_{\mathrm{b}} = p_{\mathrm{f}} = 200$ MPa. Values for ρ, E and σ_{f} are to be taken from the Table of data.

Material	m_b (tonne)	t_b (mm)	m_f (tonne)	t_f (mm)	Mechanism
Alumina	2.02	41	0.98	20	Buckling
Glass	3.18	97	1.63	50	Buckling
Steel	5.51	56	4.90	50	Buckling
Titanium	4.39	74	4.92	83	Compressive
Aluminium	3.30	97	6.79	200	Compressive

8

(c) Limiting failure mechanism is that which gives the larger of the two values of thickness for each material.

4

(d) Optimum material is alumina, giving the least mass of the five materials.

2

D R H Jones
Michaelmas Term, 2005