Engineering Tripos Part IA FIRST YEAR

Paper 4: Mathematical Methods

Solutions to 2006 Tripos Paper

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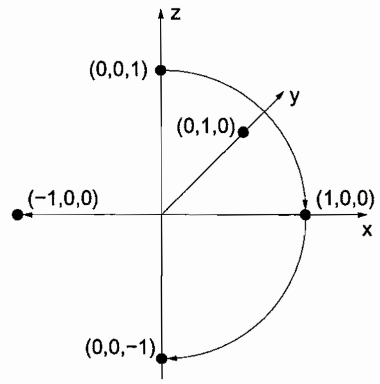
1. The transformation $\mathbf{R}$ is readily determined by considering its action on the principal unit vectors.



Thus

$$\mathbf{R}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix}\Leftrightarrow\mathbf{R}=\begin{bmatrix}0&0&-1\\0&1&0\\-1&0&0\end{bmatrix}$$

By inspection, the y axis is unchanged by this transformation. One eigenvector of $\mathbf{R}$ is therefore $[0\ \ 1\ \ 0]^T$ with a corresponding eigenvalue of 1. [10]

Examiner's remarks: This question asked the candidates to "find the 3×3 matrix $\mathbf{R}$ which represents a rotation of 90° about the y axis followed by a reflection in the $x = 0$ plane". There were many sign mistakes and very few candidates thought to find $\mathbf{R}$ in one step. Instead, they derived separate rotation and reflection matrices and then multiplied them together, sometimes in the wrong order. Despite the careful wording of the question, several candidates thought that the answer was a 2×2 matrix. On a brighter note, geometrical interpretation of the eigenvectors was good, with many candidates successfully writing down one eigenvector and eigenvalue of $\mathbf{R}$.

2. The unit eigenvectors are $(1/\sqrt{2})[1\ \ 1]^T$ and $(1/\sqrt{2})[1\ \ -1]^T$. Using the formula in the *Mathematics Data Book*, $\mathbf{A}$ can be constructed from its eigenvectors and eigenvalues as follows:

$$\mathbf{A}=\mathbf{U}\Lambda\mathbf{U}^T=\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}\begin{bmatrix}2&0\\0&3\end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix}1&1\\1&-1\end{bmatrix}=\frac{1}{2}\begin{bmatrix}5&-1\\-1&5\end{bmatrix}$$

$\mathbf{A}$ represents a stretch of 2 in the direction $[1 \;\; 1]^T$ and a stretch of 3 in the direction $[1 \;\; -1]^T$. [10]

Examiner's remarks: This question asked candidates to construct a 2×2 matrix from its (orthogonal) eigenvectors and eigenvalues, and to comment on the geometrical transformation it represents. Remarkably few candidates used the data book formula $\mathbf{A} = \mathbf{U}\Lambda\mathbf{U}^T$, the vast majority working from first principles but generally getting the right answer nevertheless. Not many candidates were able to interpret the transformation as a stretch along the eigenvectors.

3. The characteristic equation is

$$\lambda^2 = 1 - \lambda \Leftrightarrow \lambda^2 + \lambda - 1 = 0 \Leftrightarrow \lambda = \frac{-1 \pm \sqrt{5}}{2}$$

The general solution is therefore $x_n = A\left[\frac{1}{2}(\sqrt{5}-1)\right]^n + B\left[\frac{1}{2}(-\sqrt{5}-1)\right]^n$. By inspection, $A = 1$ and $B = 0$ satisfies the initial conditions. The solution is therefore $x_n = \left[\frac{1}{2}(\sqrt{5}-1)\right]^n$. [10]

Examiner's remarks: This question asked the candidates to solve an uncomplicated second order linear difference equation with friendly boundary values. Responses were very good, with most candidates scoring full marks.

4. (a) The auxiliary equation is

$$\lambda^2 + (3+a)\lambda + 3a = 0 \Leftrightarrow (\lambda + 3)(\lambda + a) = 0$$

So, assuming $a \neq 3$ (repeated root), the complementary function is

$$y = Ae^{-3x} + Be^{-ax}$$

Assuming $a \neq 2$ (complementary function the same as the right hand side), the particular integral is $y = Ce^{-2x}$. Substitute into the differential equation to find C:

$$\begin{aligned} 4Ce^{-2x} + -2(3+a)Ce^{-2x} + 3aCe^{-2x} &= e^{-2x} \Leftrightarrow -2C + +aC = 1 \\ \Leftrightarrow C &= \frac{1}{a-2} \end{aligned}$$

The general solution is therefore

$$y = Ae^{-3x} + Be^{-ax} + \frac{e^{-2x}}{a-2}$$

The boundary conditions tell us that

$$\begin{aligned} 0 &= A + B + \frac{1}{a-2} \\ 0 &= -3A - aB - \frac{2}{a-2} \end{aligned}$$

Multiplying the top equation by 3 and adding the bottom one:

$$0 = (3-a)B + \frac{1}{a-2} \Leftrightarrow B = \frac{1}{(a-2)(a-3)}$$

Substituting into the top equation:

$$A = -\frac{1}{(a-2)(a-3)} - \frac{1}{a-2} = \frac{-1-a+3}{(a-2)(a-3)} = \frac{-(a-2)}{(a-2)(a-3)} = \frac{1}{3-a}$$

The solution satisfying the boundary conditions is therefore

$$y = \frac{e^{-3x}}{3-a} + \frac{e^{-ax}}{(a-2)(a-3)} + \frac{e^{-2x}}{a-2}$$ [12]

(b) When $a = 2$, the first term in the above expression is simply e^{-3x}. For the other terms, we substitute $\epsilon = a - 2$ to get

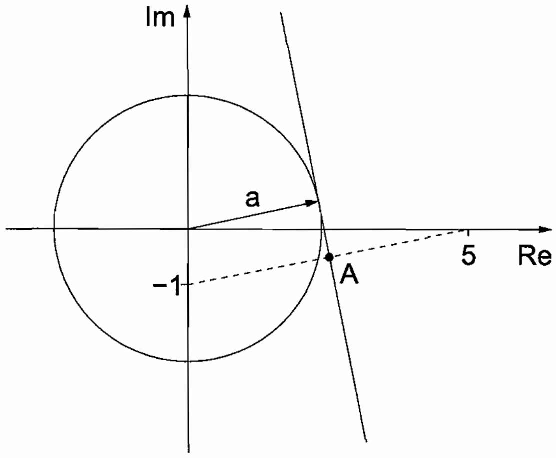
$$\begin{aligned}
y &= e^{-3x} + \frac{e^{-(\epsilon+2)x}}{\epsilon(\epsilon-1)} + \frac{e^{-2x}}{\epsilon} = e^{-3x} - \frac{e^{-(\epsilon+2)x}(1-\epsilon)^{-1}}{\epsilon} + \frac{e^{-2x}}{\epsilon} \\
&= e^{-3x} - \frac{e^{-2x}e^{-\epsilon x}(1+\epsilon+O(\epsilon^2))}{\epsilon} + \frac{e^{-2x}}{\epsilon} = e^{-3x} - \frac{e^{-2x}}{\epsilon}\left[e^{-\epsilon x}(1+\epsilon+O(\epsilon^2)) - 1\right] \\
&= e^{-3x} - \frac{e^{-2x}}{\epsilon}\left[(1-\epsilon x+O(\epsilon^2))(1+\epsilon+O(\epsilon^2)) - 1\right] \\
&= e^{-3x} - \frac{e^{-2x}}{\epsilon}\left[(1-\epsilon x+O(\epsilon^2)+\epsilon) - 1\right] = e^{-3x} - e^{-2x}(1-x+O(\epsilon)) \\
&= e^{-3x} + (x-1)e^{-2x} \text{ in the limit as } \epsilon \to 0
\end{aligned}$$ [12]

(c) Instead of taking the limit, we could have returned to the particular integral in (a) and tried a function of the form Cxe^{-2x}. For the case $a = 3$, we could take a limit as in (b), or return to (a) and change the complementary function to

$$y = (A + Bx)e^{-3x}$$ [6]

Examiner's remarks: This question asked the candidates to solve the differential equation $\frac{d^2y}{dx^2} + (3+a)\frac{dy}{dx} + 3ay = e^{-2x}$. The question initially asked the candidates to assume $a \neq 2$ (so the complementary function has nothing in common with the right hand side) and $a \neq 3$ (so the auxiliary equation does not have repeated roots). Almost everyone derived the auxiliary equation $\lambda^2 + (3+a)\lambda + 3a = 0$ but then distressingly few spotted the obvious factorisation, launching instead into the standard formula for solving a quadratic equation. Nevertheless, since the candidates knew where they were going in part (a), which was a "show that" question, algebraic slips were corrected and almost everyone scored full marks. In part (b), candidates were asked to find the solution when $a = 2$ by writing $\epsilon = a - 2$ and taking the limit as $\epsilon \to 0$. Responses were very poor, with only a handful of students finding the right answer using power series. Most instead tried to use L'Hôpital's rule, which can produce the right answer (a few succeeded) but of course you first have to rewrite the indeterminate part of the expression as a single numerator over a single denominator, with both numerator and denominator tending to zero in the limit: most candidates seemed unaware of this prerequisite. In part (c), candidates were asked how else they might find the solution when $a = 2$ or $a = 3$. Only a few suggested trying a different particular integral or complementary function, with most instead suggesting power series in place of L'Hôpital's rule, or vice versa.

5. (a) (i) The first equation defines the locus of points z in the Argand plane equidistant from 5 and $-i$. It is therefore the solid straight line in the diagram below. The second equation defines the locus of points z in the Argand plane which lie a distance a from the origin. It is therefore a circle of radius a, centred at the origin. One such circle is shown in the diagram below.



[5]

(ii) The solutions to the simultaneous equations are at the points where the line and circle intersect. For the specific value of a shown above, there is just one solution, at the point where the line passes closest to the origin. For smaller a there will be no solutions, for larger a there will be two solutions.

So the value of a which gives precisely one solution is the shortest distance from the line to the origin. To find this distance, we need to find one point $\mathbf{p}$ on the line and the unit normal $\hat{\mathbf{n}}$ to the line. The shortest distance is then $\mathbf{p}.\hat{\mathbf{n}}$.

The point A, midway between 5 and $-i$, is one point on the line. Its coordinates are $0.5(5 - i)$. By inspection of the diagram, the normal to the line is in the direction $5 + i$. The shortest distance from the line to the origin is therefore

$$\frac{1}{\sqrt{26}}\begin{bmatrix} 5 \\ 1 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 5 \\ -1 \end{bmatrix} = \frac{12}{\sqrt{26}}$$

So the system has precisely one solution when $a = 12/\sqrt{26}$. [10]

(iii) For this value of a, the solution is at the point where the line passes closest to the origin, which is when

$$z = \frac{12}{\sqrt{26}} \times \frac{1}{\sqrt{26}}(5+i) = \frac{6}{13}(5+i)$$

Note that an algebraic approach to (ii) and (iii) is feasible, but the geometric approach is far quicker. [5]

(b) Let $z = x + iy$.

$$\begin{aligned} \sinh(x+iy) &= \sinh x \cos y + i \cosh x \sin y = 2 \\ \Leftrightarrow \sinh x \cos y &= 2 \;\text{ and }\; \cosh x \sin y = 0 \end{aligned}$$

The second equation is the more informative, since $\cosh x$ is never zero, so $\sin y = 0$, which implies $y = n\pi$ and $\cos y = (-1)^n$. The first equation then tells us that $x = (-1)^n \sinh^{-1} 2$. We conclude that if $\sinh z = 2$, $z = (-1)^n \sinh^{-1} 2 + n\pi i$. [10]

Examiner's remarks: Part (a) asked the candidates to sketch $|z-5| = |z+i|$ and $|z| = a$ in the Argand plane, determine the value of a for which the equations have precisely one solution (ie. when the line is tangent to the circle) and then solve the equations. The question was structured to suggest a geometrical approach, and it was pleasing to see many candidates taking the hint and succeeding. Others took the long, algebraic route, with a reduced success rate caused by algebraic slips. There were some poor solutions, with quite a few candidates thinking that $|z-5| = |z+i|$ is an ellipse and others thinking that $|z| = a$ is a circle of radius $\sqrt{a}$. In part (b), candidates were asked to solve $\sinh z = 2$. Those who approached the problem by trigonometric expansion generally did well, though there was endemic carelessness in the details of the answer: $z = (-1)^n \sinh^{-1} 2 + n\pi i$ is not the same as $z = \pm \sinh^{-1} 2 + n\pi i$. Those who expressed $\sinh z$ in terms of exponentials fared less well. They generally got as far as $z = \ln(2 \pm \sqrt{5})$ and then either left this as the answer, or stated that you can't have a log of a negative number, or (rarely) expressed $2 \pm \sqrt{5}$ in complex polar form and proceeded to the correct answer.

6. The impulse response is

$$g(t) = \frac{dh(t)}{dt} = 2e^{-2t}$$

By convolution, the response to $x(t)$ is therefore

$$\begin{aligned} y(t) &= \int_0^t x(\tau) g(t-\tau) d\tau = 2\int_0^t e^{-3\tau} e^{-2(t-\tau)} d\tau \\ &= 2e^{-2t} \int_0^t e^{-\tau} d\tau = 2e^{-2t} \left[-e^{-\tau}\right]_0^t = 2e^{-2t}(1 - e^{-t}) \end{aligned}$$ [10]

Examiner's remarks: This question asked the candidates to compute a convolution for a system, given the system's step response. Almost all candidates were able to do this correctly, with a large number of perfect responses.

7. First take the partial derivatives with respect to x and y:

$$\begin{aligned}\frac{\partial z}{\partial x} &= y(2-y)\\ \frac{\partial z}{\partial y} &= (1+x)(-y+2-y) = 2(1+x)(1-y)\end{aligned}$$

For a stationary point, the x derivative tells us that $y = 0$ or $y = 2$. For $y = 0$, the y derivative tells is that $x = -1$. For $y = 2$, the y derivative again tells is that $x = -1$. The stationary points are therefore $(-1, 0)$ and $(-1, 2)$. Note that the stationary points are located at the intersections of the $z = 0$ contours. While this is a sufficient condition for a stationary point, it is not a necessary condition. It is therefore important to evaluate the partial derivatives and check that there are no further stationary points. [10]

Examiner's remarks: This question asked the candidates to compute the stationary points of a function of two variables. Most candidates did this correctly, although several found the stationary points by considering the intersections of the $z = 0$ contours without checking for further stationary points by computing the partial derivatives. Other candidates failed to realise that both partial derivatives had to be zero simultaneously.

8. Differentiating $x(t)$, we obtain

$$x'(t) = \begin{cases} -1 & -2 < t < 0 \\ 1 & 0 < t < 2 \end{cases}$$

This is the square wave on page 24 of the *Electrical and Information Data Book*, with period $T = 4$. Hence

$$x'(t) = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\frac{\pi}{2}t}{2n-1}$$

Integrating, we get

$$x(t) = 1 - \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos(2n-1)\frac{\pi}{2}t}{(2n-1)^2}$$

where the constant of integration (1) is the mean value of the triangular wave $x(t)$ given in the question. [10]

Examiner's remarks: This question asked candidates to compute the Fourier series of a triangular wave that differed in phase and mean value from the example in the data book. Large numbers of candidates had difficulty with this question. Errors included: thinking that the function given was in fact a square wave rather than a triangular wave; thinking that the function was the same as the example triangular wave given in the data book; and ignoring the data book and attempting to compute the series from first principles, with many opportunities for errors.

9. (a) To find the impulse response $g(t)$, we need to solve the differential equation

$$\frac{d^2g}{dt^2} + 4\frac{dg}{dt} + 3g = \delta(t)$$

with boundary values $g(0_-) = \dot{g}(0_-) = 0$. Taking the Laplace transform of both sides, we get

$$\begin{aligned} s^2G + 4sG + 3G &= 1 \Leftrightarrow G(s+3)(s+1) = 1 \\ \Leftrightarrow G &= \frac{1}{(s+3)(s+1)} = \frac{-1/2}{s+3} + \frac{1/2}{s+1} \end{aligned}$$

Looking up the inverse Laplace transforms in the *Mathematics Data Book*, we find

$$g(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}\left(e^{-t} - e^{-3t}\right) & t \geq 0 \end{cases}$$ [10]

(b) The output is given by the convolution integral

$$y(t) = \int_0^t x(\tau)g(t-\tau)\,d\tau$$

Taking Laplace transforms of both sides, and using the convolution property in the *Mathematics Data Book*, we get $Y(s) = X(s)G(s)$. We've already found $G(s)$ in (a), and for $x(t) = \sin t$ we can look up $X(s) = 1/(s^2+1)$. So

$$Y(s) = X(s)G(s) = \frac{1}{s^2+1}\left(\frac{-1/2}{s+3} + \frac{1/2}{s+1}\right) = \frac{1}{2(s^2+1)}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)$$

To find $y(t)$, we first need to decompose $Y(s)$ into simple terms using partial fractions. Starting with the first term in $Y(s)$,

$$\frac{1/2}{(s^2+1)(s+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} = \frac{(A+B)s^2 + (B+C)s + A + C}{(s^2+1)(s+1)}$$

Comparing the numerators, we get the following three equations

$$A + B = 0\,,\ B + C = 0\,,\ A + C = 1/2$$

which are readily solved to yield $A = 1/4$, $B = -1/4$ and $C = 1/4$. Now for the second term in $Y(s)$,

$$\frac{-1/2}{(s^2+1)(s+3)} = \frac{A}{s+3} + \frac{Bs+C}{s^2+1} = \frac{(A+B)s^2 + (3B+C)s + A + 3C}{(s^2+1)(s+3)}$$

Comparing the numerators, we get the following three equations

$$A + B = 0\,,\ 3B + C = 0\,,\ A + 3C = -1/2$$

which are readily solved to yield $A = -1/20$, $B = 1/20$ and $C = -3/20$. Bringing the two terms together, we get

$$\begin{aligned} Y(s) &= \frac{\frac{1}{4}}{s+1} + \frac{-\frac{1}{4}s}{s^2+1} + \frac{\frac{1}{4}}{s^2+1} + \frac{-\frac{1}{20}}{s+3} + \frac{\frac{1}{20}s}{s^2+1} + \frac{-\frac{3}{20}}{s^2+1} \\ &= \frac{\frac{1}{4}}{s+1} - \frac{\frac{1}{20}}{s+3} + \frac{\frac{1}{10}}{s^2+1} - \frac{\frac{1}{5}s}{s^2+1} \end{aligned}$$

Looking up the inverse Laplace transforms, we get

$$y(t) = \frac{1}{4}e^{-t} - \frac{1}{20}e^{-3t} + \frac{1}{10}\sin t - \frac{1}{5}\cos t \qquad (t \geq 0)$$ [15]

(c) For an initially quiescent second order system driven by a finite input, we should find $y(0) = \dot{y}(0) = 0$. From our answer in (b), we get

$$y(0) = \frac{1}{4} - \frac{1}{20} - \frac{1}{5} = 0 \ , \ \dot{y}(0) = -\frac{1}{4} + \frac{3}{20} + \frac{1}{10} = 0$$

So the answer in (b) does indeed have the expected boundary values at $t = 0$. [5]

Examiner's remarks: This question asked candidates to compute the impulse response of a system defined in terms of a second order linear differential equation, and to then compute the response to a sine wave gated at $t = 0$. Almost all candidates were able to correctly determine the impulse response in part (a), although many took a circuitous route via the step response. A very large number of candidates were able to work their way through the somewhat tricky set of partial fractions in part (b) and went on to correctly provide the four components of the answer. Candidates were also able to compute the value and derivative in part (c) and indicate that these should be zero, with several indicating surprise that they had successfully handled the algebraic complexity in part (b)!

10. (a) The first player out of the hat has a choice of 7 opponents. The first remaining player has a choice of 5 opponents, the next has a choice of 3, and this leaves one pair with no choice. The number of possible pairings is therefore $7 \times 5 \times 3 = 105$. [6]

(b) (i) The probability that A wins the rally on A's first shot (the serve) is p. To win on A's second shot, A must not win on the serve and B must not win on the return. The probability is therefore $(1-p)(1-q)q$. Similarly, the probability that A wins on A's third shot is $(1-p)(1-q)(1-q)(1-q)q$. For $n > 1$, it is clear that the expressions have a common term $(1-p)q$ and are expanding by a multiple of $(1-q)^2$ each time. Therefore,

$$\text{P(A wins rally on A's } n^{\text{th}} \text{ shot)} = \begin{cases} p & n = 1 \\ (1-p)q(1-q)^{2n-3} & n > 1 \end{cases}$$

For B to win on B's first shot, A must not win the serve. The probability is therefore $(1-p)q$. For B to win on B's second shot, A must not win on the serve, B must not win on the return and A must not win on the next shot. The probability is therefore $(1-p)(1-q)(1-q)q$. Again, it is clear that the expressions have a common term $(1-p)q$ and are expanding by a multiple of $(1-q)^2$ each time. Therefore,

$$\text{P(B wins rally on B's } n^{\text{th}} \text{ shot)} = (1-p)q(1-q)^{2n-2}$$ [8]

(ii) The B expression is the simplest to work with, since it doesn't have a special case for $n = 1$. The probability b that B wins the rally is given by

$$b = \sum_{n=1}^{\infty} \text{P(B wins rally on B's } n^{\text{th}} \text{ shot)}$$

This is the sum to infinity of a geometric series, with first term $(1-p)q$ and common ratio $(1-q)^2$. Therefore

$$b = \frac{(1-p)q}{1-(1-q)^2} = \frac{(1-p)q}{2q-q^2} = \frac{1-p}{2-q}$$

Given that a and b must sum to one, we have

$$a = 1 - \frac{1-p}{2-q} = \frac{2-q-1+p}{2-q} = \frac{1+p-q}{2-q}$$

When $p = 1$, A should always win the rally on the serve itself. The above expressions give $b = 0$ and $a = 1$, as expected. When $p = 0$ and $q = 1$, B should always win the rally on the return of serve. The above expressions give $b = 1$ and $a = 0$, as expected. [10]

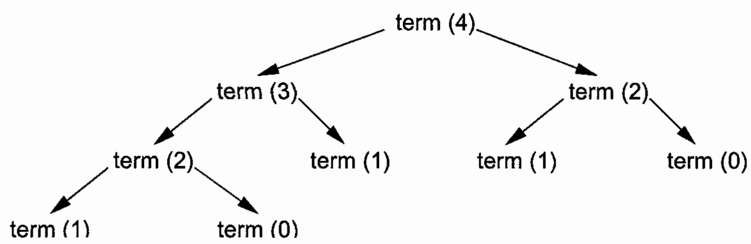
(iii) For a to equal b, we require

$$\frac{1-p}{2-q} = \frac{1+p-q}{2-q} \Leftrightarrow -p = p - q \Leftrightarrow q = 2p$$

So, for each player to have an equal chance of winning the rally, A must have such a weak serve that the chances of winning the rally on a return shot are twice those of winning the rally on the serve itself. This is because A has the advantage of going first: to make up for this advantage, A's first shot, the serve, must be weak. In fact, for most players with a decent serve $p > q$, demonstrating how strongly the odds are stacked in favour of the server. [6]

Examiner's remarks: This question asked candidates to solve a set of combinatorial and probability questions. Part (a) involved a combinatorial calculation which the vast majority of candidates were unable to compute correctly. Part (b)(i) asked candidates to derive given expressions for two probability series: this was well handled by the vast majority. Part (b)(ii) then asked candidates to sum these series. While many were able to do this correctly, a large number found this difficult with many unable even to spot that a sum over the series was required. Finally, in part (b)(iii) candidates were asked to perform a calculation on the expressions given by these sums. This was only really attempted in a plausible manner by those who had completed part (b)(ii), and then there were a significant number of algebraic errors.

11. The pattern of function calls is as follows.



There are $\mathcal{O}(n)$ levels of recursion, and the number of function calls doubles at each level. The complexity is therefore $\mathcal{O}(2^n)$. [10]

Examiner's remarks: This question tested understanding of recursive function calling and algorithmic complexity. Operational understanding of the code was reasonable even amongst those who didn't answer the specific question well. Few bookwork definitions of complexity were offered; there were many guessed, unjustified answers.

12. A `float` requires 32 bits, or 4 bytes, of storage. Each `point` is two `float`'s, or 8 bytes. Each `line` is 100 `point`'s, or 800 bytes. Each `graph` is 5 `line`'s, or 4000 bytes. The variable `g`, which is of type `graph`, therefore requires 4000 bytes of storage. The x coordinate of the eighth point on the third line of `g` would be accessed using the notation `g.lines[2].points[7].x` (remember array indices run from `0` to `n-1`, not `1` to `n`). [10]

Examiner's remarks: This question presented a nested data structure involving arrays, testing understanding of memory requirements and access methods. Conceptual difficulties with nested components were evident, as were revision deficiencies regarding bits, bytes, size of floats and zero-based indexing.

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