ENGINEERING TRIPOS PART IA

Tuesday 13 June 2006 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

*Answers to questions in each section should be tied together and handed in separately.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

SECTION A

1 **(short)** Find the $3 \times 3$ matrix $\mathbf{R}$ which represents a rotation of $90°$ about the $y$ axis followed by a reflection in the $x = 0$ plane. The sense of the rotation is clockwise when viewed outwards along the positive $y$ axis. Write down one eigenvector of $\mathbf{R}$ and its corresponding eigenvalue. [10]

2 **(short)** Find the $2 \times 2$ matrix $\mathbf{A}$ which has eigenvalues $\lambda_1 = 2$, $\lambda_2 = 3$, and corresponding eigenvectors $\mathbf{u}_1 = [1 \;\; 1]^T$, $\mathbf{u}_2 = [1 \;\; -1]^T$. What geometrical transformation does $\mathbf{A}$ represent? [10]

3 **(short)** Solve the difference equation

$$x_{n+1} = x_{n-1} - x_n$$

with initial values $x_0 = 1$ and $x_1 = \frac{1}{2}(\sqrt{5} - 1)$. [10]

4 **(long)** (a) Consider the differential equation

$$\frac{d^2y}{dx^2} + (3+a)\frac{dy}{dx} + 3ay = e^{-2x}$$

with boundary conditions

$$\frac{dy}{dx} = y = 0 \text{ at } x = 0$$

Assuming the constant $a$ does not equal 2 or 3, derive the solution

$$y = \frac{e^{-3x}}{3-a} + \frac{e^{-ax}}{(a-2)(a-3)} + \frac{e^{-2x}}{a-2}$$

You must derive the solution: **do not** simply check that the given solution satisfies the differential equation and the boundary conditions. [12]

(b) The solution is indeterminate when $a = 2$ or $a = 3$. By writing $\epsilon = a - 2$ and taking the limit as $\epsilon \to 0$, find the solution when $a = 2$. [12]

(c) What alternative technique could you use to find the solution when $a = 2$? How would you go about finding the solution when $a = 3$? Do not actually determine any further solutions, just describe the appropriate methods. [6]

5 **(long)** (a) Consider the following system of simultaneous equations:

$$|z-5| = |z+i|$$
$$|z| = a \text{ , where } a \text{ is a positive real number.}$$

(i) Each equation defines a curve in the Argand plane. Sketch the two curves on the same set of axes. [5]

(ii) For what value of $a$ will the system have precisely one solution? [10]

(iii) Solve the equations for the value of $a$ found in (ii). [5]

(b) Determine all values of $z$ for which $\sinh z = 2$. [10]

SECTION B

6 **(short)** A linear system has step response $1 - e^{-2t}$ for $t \geq 0$. Find its response to the input

$$x(t) = \begin{cases} e^{-3t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

[10]

7 **(short)** Locate the stationary points of the function $z = y(1+x)(2-y)$. Do not classify the stationary points, just find their locations. [10]

8 **(short)** Using the information on page 24 of the *Electrical and Information Data Book*, find a Fourier series for the function

$$x(t) = \begin{cases} -t & -2 \leq t < 0 \\ t & 0 \leq t < 2 \end{cases}$$

where $x(t)$ is periodic with period 4. [10]

9 **(long)** A second order linear system is governed by the differential equation

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 3y = x(t)$$

where $y(t)$ is the system's output and $x(t)$ is its input.

(a) Using Laplace transforms, **and not otherwise**, show that the system's impulse response $g(t)$ is given by

$$g(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}(e^{-t} - e^{-3t}) & t \geq 0 \end{cases} \qquad [10]$$

(b) Now consider the system input

$$x(t) = \begin{cases} 0 & t < 0 \\ \sin t & t \geq 0 \end{cases}$$

Show that the Laplace transform of the output is given by

$$Y(s) = \frac{1}{2(s^2+1)}\left(\frac{1}{s+1} - \frac{1}{s+3}\right)$$

and hence determine $y(t)$. [15]

(c) Check that your answer to (b) has the expected boundary values of $y(t)$ and $\dot{y}(t)$ at $t = 0$. [5]

10 **(long)** (a) Eight players enter a tennis tournament. The players are paired off for the four first round matches. How many possible pairings are there? [6]

(b) Consider the following probabilistic analysis of a tennis rally. Players A and B take it in turn to hit the ball to each other, starting with A, who serves. Each time a player hits the ball, there is a fixed probability that they win the rally, either by hitting an unreturnable shot or by forcing their opponent into a foul return. This probability is $p$ for the first shot (the serve) and $q$ for all subsequent shots.

(i) Show that the probabilities that each player wins the rally on their $n^{\text{th}}$ shot are given by the following expressions:

$$\begin{aligned} \text{P(A wins rally on A's } n^{\text{th}} \text{ shot)} &= \begin{cases} p & n = 1 \\ (1-p)q(1-q)^{2n-3} & n > 1 \end{cases} \\ \text{P(B wins rally on B's } n^{\text{th}} \text{ shot)} &= (1-p)q(1-q)^{2n-2} \end{aligned}$$

[8]

(ii) Hence find the probability $a$ that A wins the rally, and the probability $b$ that B wins the rally. Check that your answers behave as expected for the case $p = 1$ and for the case $p = 0$, $q = 1$. [10]

(iii) Derive a condition on $p$ and $q$ for $a$ to equal $b$. Comment on your result. [6]

SECTION C

11 **(short)** The recursive C++ function below is designed to solve the difference equation $x_{n+2} - 8x_{n+1} + 15x_n = 0$ with initial values $x_0 = 0$ and $x_1 = 1$.

```
int term (int n)
// Returns the nth term in the series generated by
// the difference equation
{
  if (n == 0) return 0;
  else if (n == 1) return 1;
  else return 8*term(n-1) - 15*term(n-2);
}
```

The function is called with the parameter 4. Draw a tree diagram showing the resulting pattern of function calls. What is the algorithmic complexity of this function? [10]

12 **(short)** Consider the following C++ data definition:

```
struct point {
  float x;
  float y;
};
struct line {
  point points[100];
};
struct graph {
  line lines[5];
};
graph g;
```

How many bytes of memory does g occupy? How would you access the $x$ coordinate of the eighth point on the third line of the graph g? [10]

**END OF PAPER**