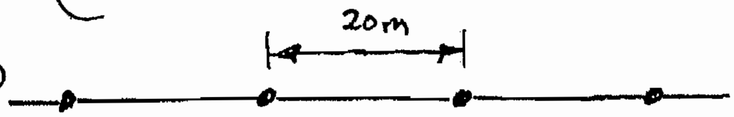
Engineering Tripos Part IA 2007

Paper 2, Structures — Solutions

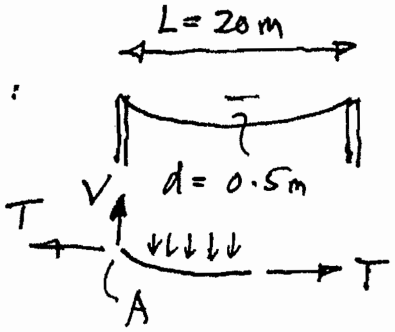
Q1 (short) [marks allocated]

(a)



plan

typical span in elevation :



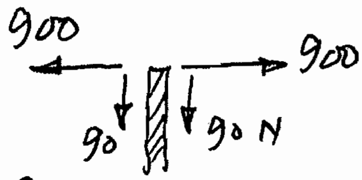
$w = 9\ kN/m$

(A) $T.d = w.\frac{L}{2}.\frac{L}{4}$

$\therefore\ T = \frac{9 \times 10 \times 5}{0.5} = \underline{900\ N}$ [3]

Vertical force V is $9 \times 10 = 90$ N.

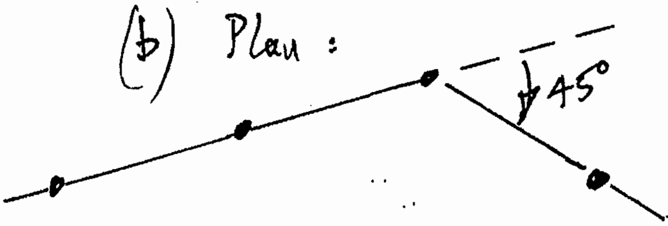
A typical pole has cable both sides :



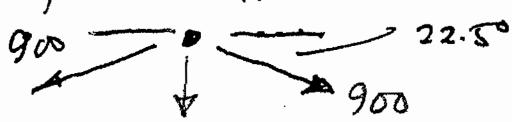
— so it sustains an axial downward force of $\underline{180N}$ (causing compression), but no other stress-resultant. [1]

(though there could be local forces due to the fastening).

(b) Plan :



Corner pole: applied horizontal forces



Resultant applied horizontal force is $2 \times 900 \sin(22.5°) = 689$ N

So stress resultants in pole 7 m down are:

axial compression 180 N

shear force 689 N

moment $689 \times 7 = 4.82$ kN m [5]

For deflection, use cantilever formula $WL^3/3EI$ where L is height of pole (? but rotation of foundation block). [1]

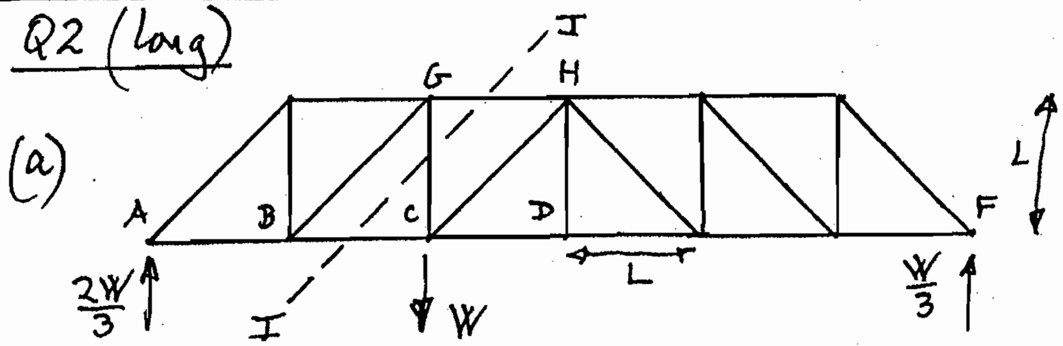
Q1 (cont'd)

Examiner's comment. Most candidates could do the very standard part on cable tension, but lots did not realise that a typical pole had cables on both sides, so that the **net** horizontal force **on the pole** was zero. Several omitted the axial compression of 180 N in the pole, and some contrived to have an upwards (tensile) force on the typical pole.

For the corner pole, it was surprising how few students, despite having sketched a plan view, realised that the resultant horizontal force on the corner pole bisected the angle between the two lines of poles. Many invented an x-y co-ordinate system with x parallel to one line of cables, and went on to find component shear forces and moments in this system – very laborious ! Mean mark was about 60%.

Q2 (long)

(a)



Take moments about A or F for whole structure ⇒ reactions [2]

Consider section II, equilibrium of part on left:

– vertical forces: $T_{CG} = +\,2W/3$ (+ve sign ≡ tension)

– moments about G: $\frac{2W}{3}\cdot 2L = T_{BC}\cdot L \Rightarrow T_{BC} = +\frac{4W}{3}$

– resolve horizontally: $T_{GH} = -\,4W/3$

Consider cut vertically through BG: $T_{BG} = -\frac{2\sqrt{2}}{3}W$ [8]

(b) (i) Virtual work, considering unit load at C (solved above)

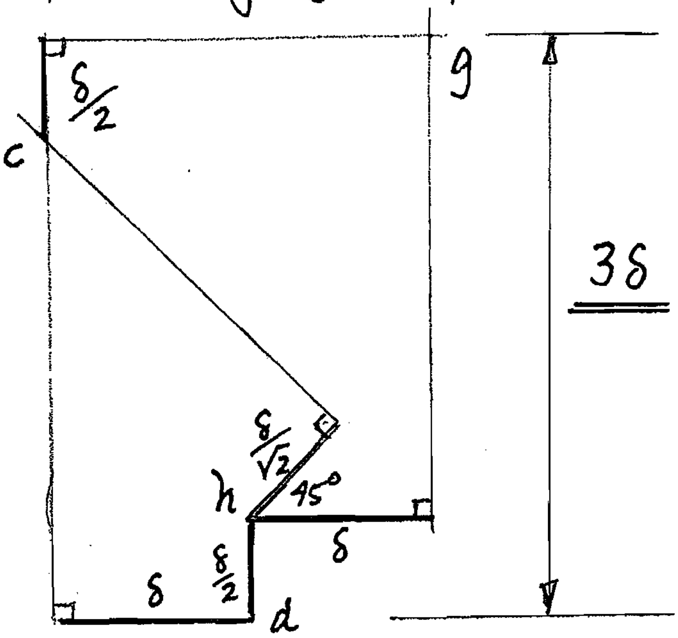
	e	P	product
BC	$+\Delta$	$+4/3$	$+4\Delta/3$
BG	$-\Delta/\sqrt{2}$	$-2\sqrt{2}/3$	$+2\Delta/3$
GH	-2Δ	$-4/3$	$+8\Delta/3$
CG	$+\Delta/2$	$+2/3$	$+\Delta/3$
		total	5Δ = displacement of C

[8]

(ii) All members in CHFD have unchanged lengths – so it is a rigid body. So the downward displacement of D is $3L/4L$ of that of C $= \underline{15\Delta/4}$ [4]

Q2 (cont'd)

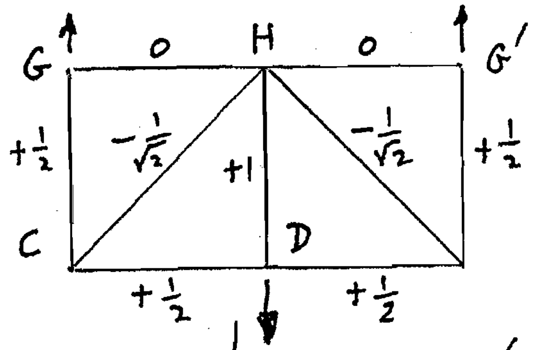
(c) Method 1 : displacement diagram Better to start at H or D, since by symmetry HD remains vertical.



No need to draw to scale – with 45° and 90° angles, calculation by trigonometry is easy

[8]

Method 2 : virtual work



Local equilibrium system, unit load at D, reactions at G and G′ — using symmetry, this will give the required relative displacement δ_{DG}.

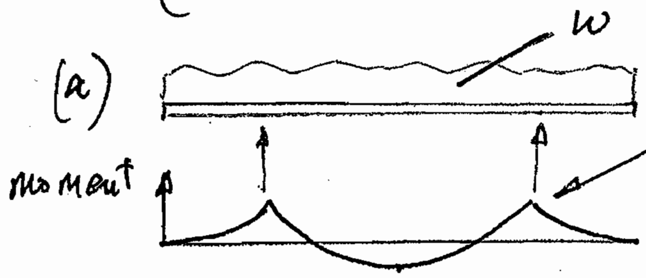
$$\delta_{DG} = 2\cdot\frac{1}{2}\cdot\frac{\delta}{2} + 2\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{\delta}{\sqrt{2}}\right) + 2\cdot\frac{1}{2}\cdot\delta + 1\cdot 1\cdot\frac{\delta}{2}$$

$$= 3\delta$$

Examiner's comment. Mean mark just below 60%. Some had difficulty finding reactions (!) and bar forces in part (a). Part (b) (i) on virtual work was usually done well – designed to be straightforward – but part (b) (ii) was disappointing. Doing part (c) by a displacement diagram, many candidates made an injudicious choice of starting point, using G rather than H or D (better since bar HD does not rotate, by symmetry) – so there were few fully correct solutions. Some tried a local equilibrium system and virtual work, and one gave exactly the right system but went astray in the calculation. There was one heroic, and completely correct, solution, applying unit loads downwards at D and upwards at G and solving for forces throughout the structure! A distressing number of candidates thought that they needed to work out the bar extensions due to the forces calculated in part (a) – though no section or material properties had been given.

Q3 (short)

(a)



max hogging (tension top):

$w(0{\cdot}2L)(0{\cdot}1L) = \underline{\underline{0{\cdot}02\,wL}}$ [4]

max. sagging

$\frac{wL}{2}(0{\cdot}3L) - \frac{wL}{2}(0{\cdot}25L) = \underline{\underline{0{\cdot}025\,wL}}$

(Note that $\frac{d^2M}{dz^2} = w$ means that for constant w the curvature of the B.M. diagram has to be the same everywhere, and same sign — except at reaction points.)

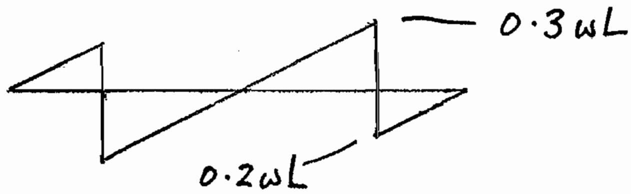
(b) For general h, max. hogging moment is $w\left(\frac{L-h}{2}\right)\left(\frac{L-h}{4}\right)$

and max. sagging is $w\frac{L}{2}\left(\frac{h}{2} - \frac{L}{4}\right)$ (free-body diagrams).

So for equality (of magnitude, opposite sign) $\frac{1}{8}(L^2 - 2hL + h^2) = \frac{1}{8}(2hL - L^2)$

$\Rightarrow \underline{\underline{h^2 - 4Lh + 2L^2 = 0}}$ [2]

(c)



[2]

(Note constant slope from $\frac{dS}{dx} = w$, except at reaction points.)

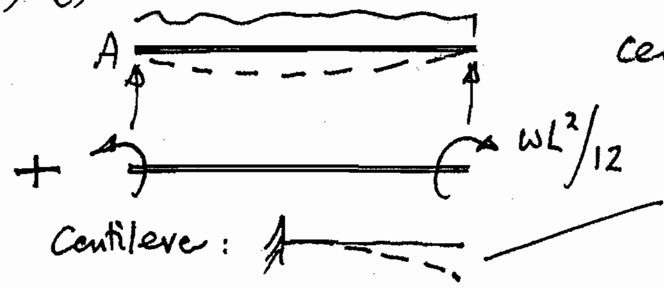
(d) (i) Not affected — stress resultants follow by equilibrium from applied loads only irrespective of cross-section or its uniformity.

(ii) Stress-resultants due to this load w are not affected since the beam is statically determinate — but might need to be added to moments due to the self-weight. [2]

Exaniner's comment. Simple and fundamental – but badly done, mean mark about 50%. Some sketches and diagrams were terrible, suggesting the candidate had never seen such a thing before – not symmetric, having substantial moment at the tips of the cantilevers, not curving in the right sense, being made to have zero bending moment over the supports, the moment diagram not having zero slope at the cantilever tip, etc etc Many wanted to determine the function M(z) instead of just working out the moment at a couple of salient points using Free Body Diagrams – a few did not even get the reactions right ! Given the problems over part (a), it was not surprising that very few made an effective attack on part (b) about the 'optimum' h/L – but one candidate went right through and solved the resulting quadratic equation (though not asked to).

Q4 (long)

(a) (i) Use Data Book coefficients : superpose

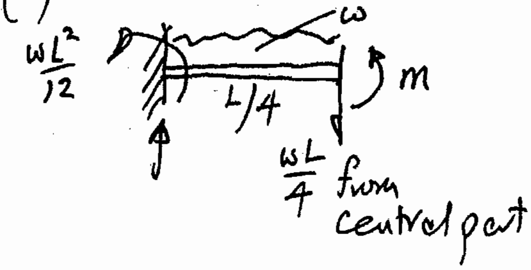


central deflection $\dfrac{5wL^4}{384EI}$

$$\frac{wL^2}{12}\cdot\frac{(L/2)^2}{2EI} = \frac{wL^4}{96EI}$$

subtract to get required central deflection $\underline{\underline{\dfrac{wL^4}{384EI}}}$ [8]

(ii) Consider cantilever of length $L/4$: first need end moment m



Moments about root (left end)

$$m + \frac{wL^2}{12} = \frac{wL}{4}\cdot\frac{L}{4} + \frac{wL}{4}\cdot\frac{L}{8}$$

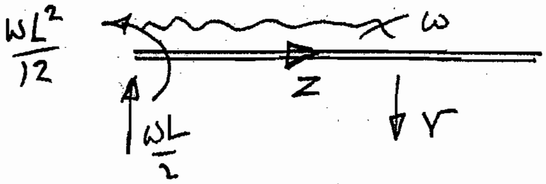
$$\therefore \quad m = wL^2/96$$

Now use coefficients : $EI.\Delta = \dfrac{wL}{4}\cdot\dfrac{(L/4)^3}{3} + \dfrac{w}{8}\cdot\left(\dfrac{L}{4}\right)^4 - \dfrac{wL^2}{96}\cdot\dfrac{(L/4)^2}{2}$

$$\Rightarrow \quad \Delta = \underline{\underline{3wL^4/2048EI}}. \qquad [6]$$

Note : This question can be done in various other ways e.g. by considering a cantilever of length $L/2$ fixed at the left end (A above) ; or a cantilever of length $L/2$ fixed against rotation at midspan. In any case it is <u>essential</u> to get the loading (bending moments etc.) on the cantilever correct.

Alternative route : solve the differential equation for deflection:

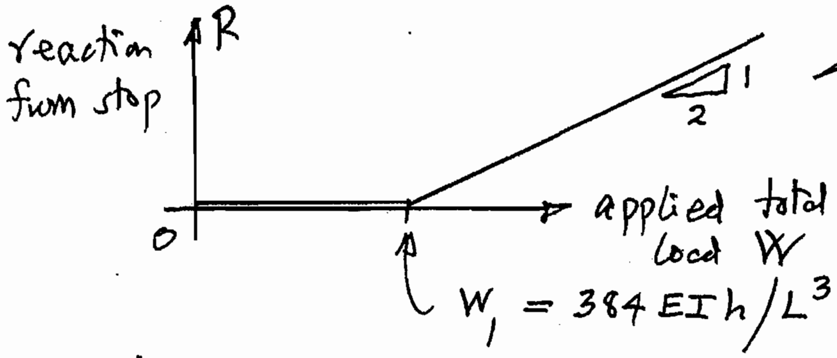


$$M(z) = \frac{wL^2}{12} - \frac{wL}{2}z + \frac{wz^2}{2} = EI\frac{d^2v}{dz^2}$$

Integrating twice and putting in the boundary condition (zero slope and deflection at $z=0$), neatly gives $v(z)$ from which the deflections at $\frac{L}{2}$ and $\frac{L}{4}$ follow.

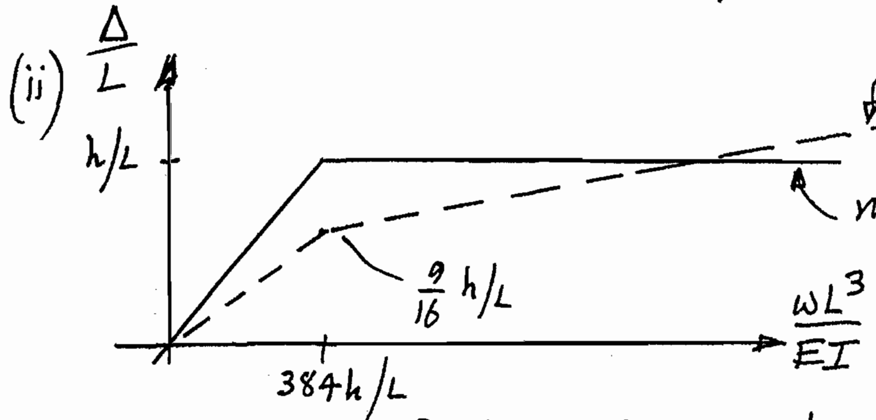
Q4 (cont'd)

(b) (i) Note that after the beam hits the stop at midspan it behaves for further loading like two fixed-ended beams (by symmetry) of span $L/2$ — so we already have formulae for the deflection. [3]



Reaction from stop is half the load added after beam hits stop. [4]

(ii)



[7]

(obvious dimensionless measure of w)

All graphs made of straight lines — linear-elastic, small deflection

For further load, $\Delta\left(\frac{L}{4}\right) = \frac{1}{384}\cdot\frac{w_e (L/2)^4}{EI}$, so when $\Delta_e = \frac{7}{16}h$ [2]

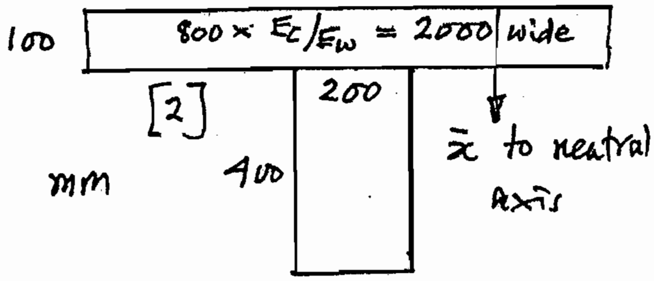
$$\frac{w_e L^3}{EI} = 384 \times 7\,\frac{h}{L} \quad : \quad \text{total } \frac{wL^3}{EI} = 384 \times 8\,\frac{h}{L}$$

Examiner's comment A disaster – mean mark about 25 % ! Many students made an injudicious choice of Data Book cases – e.g. trying to use (twice) the formulae (giving rotation not deflection) for a couple at one end of a simply-supported beam, instead of considering a couple on the end of a cantilever of length L/2. Those considering a cantilever, rooted either at midspan or one support, often got some part of the loading wrong – omitting a tip shear force, or not getting the beam midspan bending moment correct. Both parts of (a) can be done neatly, together, by solving the differential equation for deflection as suggested – but those on this route often omitted the end reaction wL/2 on the beam. The main problems were again in *equilibrium* – getting the force and moment system correct.

In part (b) one surprise, in a linear-elastic small-deflection problem, was how many sketched graphs were curved – instead of just consisting of two straight lines. Few realised, despite the hint on boundary conditions for further loading, that the two halves of the beam are now fixed-ended by symmetry, so the quoted formulae can be applied direct – and the reaction on the central stop goes up at half the rate of the total load on the beam. The requested 'suitable dimensionless measure of the applied load' caused some students to reach for Buckingham's theorem – when it seems obvious, given the request to plot δ/L and available formulae for deflection δ, that wL^3/EI should be used.

Q5 (short)

(a) Transform to timber (since stress in timber is asked for)



$A = 280{,}000 \ mm^2$

$280{,}000 \, \bar{x} = 100 \times 2000 \times 50 + 200 \times 400 \times 300$

$\Rightarrow \ \bar{x} = 121{\cdot}4 \ mm$ [2]

(so concrete is entirely in compression, no worries on cracking)

Second moment of area $I = \dfrac{2000 \times 100^3}{12} + 2000 \times 100 \times (71{\cdot}4)^2$

$+ \ 200 \times \dfrac{400^3}{12} + 200 \times 400 \times (178{\cdot}6)^2$

$= 4{\cdot}804 \times 10^9 \ mm^4$ [2]

Bottom fibre stress $\sigma = \dfrac{My}{I} = \dfrac{200 \times 10^6 \times 378{\cdot}6}{4{\cdot}804 \times 10^9} = \underline{\underline{15{\cdot}8 \ MPa}}$ [2]

(b) $S = \dfrac{FA\bar{y}}{I} = \dfrac{70 \times 10^3 \times 2000 \times 100 \times 71{\cdot}4}{4{\cdot}804 \times 10^9} = \underline{\underline{208 \ kN/m}}$

[2]

Examiner's comment Best done question on the whole Section, mean about 67 %, with 31 full marks. In transforming using the ratio of E's, some candidates transformed the depth not the width ! - and since the concrete is stiffer than the timber ($E_c = 2.5 \ E_w$), transforming to timber should widen the concrete flange, not make it narrower (as many did). Some had evidently forgotten how to find the centroid of an area with only one axis of symmetry, adding a ghostly bottom flange to get round that problem – and there were frequent errors in applying the parallel axis theorem to find I (some squared the area not the distance, some did not square anything).

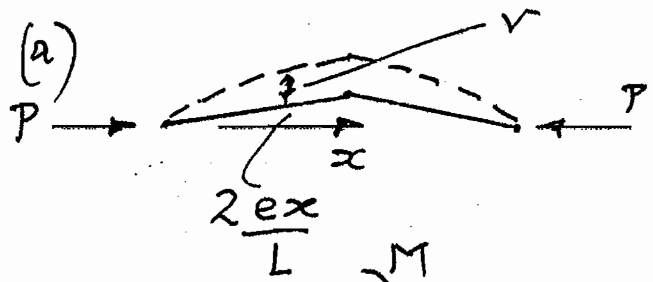
Q6 (short)

Apologies for the ambiguity in the displayed formula in part (b) – this troubled about a dozen candidates. It could have been queried in the 10-minute reading time. Would better have been set out:

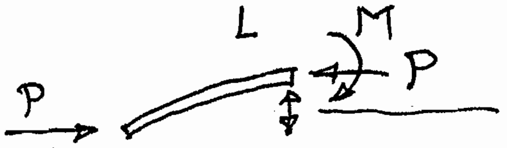
$$v = 2e \left\{ \frac{\sin \alpha x}{\alpha L \cos(\alpha L/2)} - \frac{x}{L} \right\}$$

Q6 (short) (cont'd)

(a)



central crookedness e
so for $x \leq L/2$
initial displacement (linear)
is $2ex/L$



$v + \frac{2ex}{L}$ displaced position

clearly $M = P\left(v + \frac{2ex}{L}\right)$ [2]

(b) $EI \frac{d^2v}{dx^2} = -M$ (check sign : vital)

$$\therefore \quad EI \frac{d^2v}{dx^2} + Pv = -P . \frac{2ex}{L}$$

Method 1 : back substitution Given $v = 2e\left\{\frac{\sin\alpha x}{\alpha L \cos\left(\frac{\alpha L}{2}\right)} - \frac{x}{L}\right\}$

$$\frac{d^2v}{dx^2} = -\frac{2e . P . \sin\alpha x}{EI . \alpha L \cos\left(\frac{\alpha L}{2}\right)} \qquad \alpha^2 = P/EI$$

so that substituting shows that the differential equation is satisfied.
However it is also necessary to check that the given solution satisfies the boundary conditions ($v = 0$ when $x = 0$, slope $dv/dx = 0$ when $x = L/2$) — as indeed it does.

Method 2 : standard solution of the differential equation

$$v = A \sin\alpha x + B\cos\alpha x - \frac{2ex}{L}, \quad \alpha^2 = \frac{P}{EI}$$

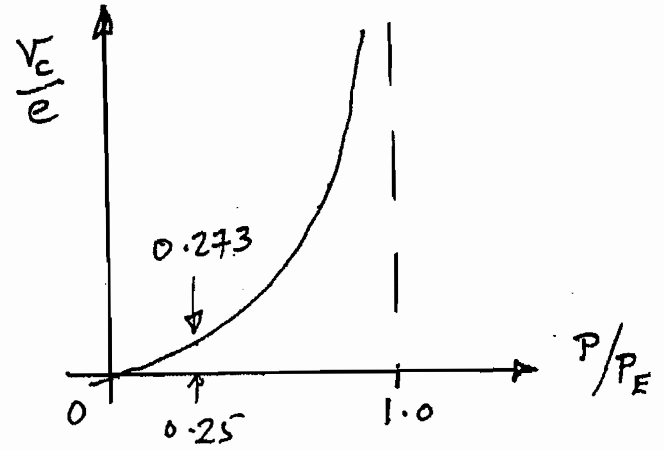
$x = 0$, $v = 0$ $\quad \therefore \quad B = 0$

$x = L/2$, $\frac{dv}{dx} = 0$ $\quad \therefore \quad 0 = A\alpha \cos\frac{\alpha L}{2} - \frac{2e}{L}$ [5]

whence $v = 2e\left\{\frac{\sin\alpha x}{\alpha L \cos\left(\frac{\alpha L}{2}\right)} - \frac{e}{L}\right\}$ as required.

Q6 (short)

(c)



$$P_E = \frac{\pi^2 EI}{L^2}$$

$$\alpha^2 = P/EI.$$

when $P = P_E/4$ $\qquad \alpha^2 L^2 = \frac{1}{4}\cdot\frac{\pi^2 EI}{L^2}\cdot\frac{L^2}{EI} = \frac{\pi^2}{4}$

$$\therefore \quad \alpha L = \pi/2$$

$$\therefore \quad v_c = 2e\left\{\frac{\sin(\pi/4)}{\frac{\pi}{2}\cos(\pi/4)} - \frac{1}{2}\right\} = e\left(\frac{4}{\pi} - 1\right)$$

$$\therefore \quad \frac{v_c}{e} = \underline{0.273} \qquad [3]$$

Examiner's comment Reasonably well done – mean about 50% but only one full mark. This was because nearly all those who correctly back-substituted, to show that the given v(x) satisfied the differential equation, neglected to show that it also satisfied the boundary conditions on displacement.

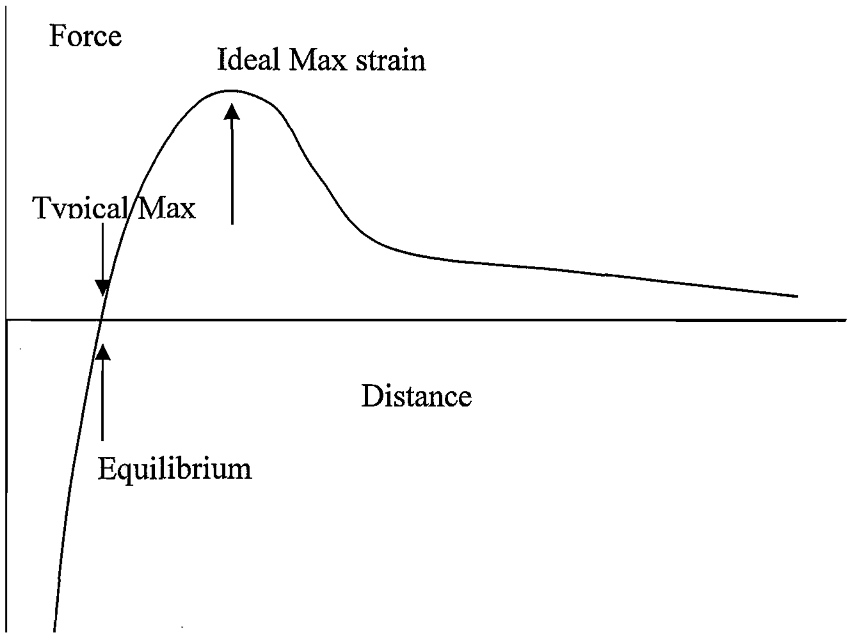
Most students could do part (a) on equilibrium correctly, though some contrived to have transverse end reactions despite the symmetry, or couples at the ends. The biggest problem in part (b) was not getting the correct sign in the equation relating moment to second derivative of deflection – which then led on to a differential equation with a minus sign, with solution involving exponential or hyperbolic rather than trigonometric functions, and so confusion. A surprise was the number of candidates who got hung up on initial curvature – despite the facts (i) that v(x) was defined to be measured from the initial crooked position and (ii) that the initial curvature was in any case zero.

The final graph was only moderately done – many did not show limitless deflection at the Euler load, and many began with $v_c = e$ (rather than zero) at zero load P, despite the definition of v in the question.

CT Morley July 2007

Crib Paper 2 Materials 1A 2007

7 a)



b) Let the radius of curvature on the neutral axis be R. The stress varies linearly with distance y from the neutral axis and reaches σ_y when y=0.5b. Hence $\sigma=\sigma_y y/0.5b$.
The elastic energy/vol at any point is $0.5\sigma^2/E=(0.5/E)(\sigma_y y/0.5b)^2$ so the total elastic energy/length is:Then the strain y from the neutral axis is $\varepsilon=y/R$ and if it just yields at the surface $Eb/2R=\sigma_y$.
The energy per unit volume a distance y from the axis is 0.5 so the total energy/length is $2a\int_0^{b/2}\frac{2y^2\sigma_y^2 dy}{b^2}=\frac{ab\sigma_y^2}{6E}$

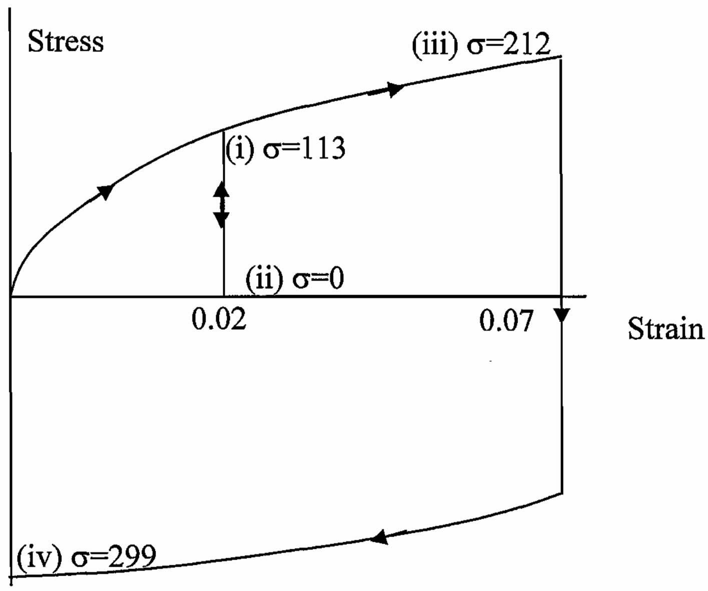
The total energy/vol is $\frac{\sigma_y^2}{6E}$.
In simple tension the stress can reach σ_y over the whole cross section so the stored energy/vol is $\frac{\sigma_y^2}{2E}$ which is three times greater.
However this can only be achieved by using very small displacements and large forces which is inconvenient for practical purposes.
Torsion also has a lower energy density as the strain is lower at the centre. However it is used in the form of coil springs which also can give large displacements.

c) Since a watch spring must be compact energy per unit volume is the first requirement. Energy per unit weight is less important. From fig 3.3 in the data book carbides and alumina are the best in this respect. However they are very brittle and difficult to form as a coil spring, we really need a metal. CFRP is a possibility but is likely to be more expensive and also difficult in small sizes. This leaves steels and titanium alloys as the best, with the steels being cheaper and easier to fabricate as they can be formed in a soft state and then hardened.
Other important properties are resistance to fracture, fatigue and creep.

Comments
An almost universal error was to put the practical elastic limit where the graph became non-linear. The elastic limit is not determined by any of the parameters on this graph, but by dislocation pinning. Many candidates did not use the properties chart in the data book and of those who did, many moved the line in the wrong direction. For the last part many candidates just put down every parameter they could think of, relevant or not.

8) Plastic deformation occurs when the elastic limit of a material is exceeded and irreversible movement of atomic layers takes place. The effect is that atomic layers slide over each other in a shear deformation, so that the density remains constant. However for this to occur at the observed stresses it must be mediated by the movement of defects such as dislocations. This allows it to occur at a much lower stress than the ideal value at which complete planes would slide over each other. Dislocations are impeded by any other defects including other dislocations which form jogs where they cross. As the deformation increases the number of dislocations increases and it becomes more difficult for them to move. This causes 'work hardening'. Since it is caused by the shear of planes it makes no difference which direction they move so that tensile and compressive stresses have the same effect. Nominal stress strain curves differ because a tensile strain reduces the cross section so increasing the true stress for a given load while compression increases the cross section and reduces the true stress.

(b)



At (iv) the total plastic strain is 2x.07=0.114

(c) Whatever the initial tensile strain at any point, an equal compressive strain must be imposed to straighten the bar. The first stress will be $\sigma_0\sqrt{\varepsilon}$ so the final stress will be $\sigma_0\sqrt{2\varepsilon}$. Hence all stresses will be a factor $\sqrt{2}$ greater and this is the extra strength required i.e. about 40%. In addition it is difficult to straighten a bent bar as it tends to rotate in your hands. A smaller effect is tht the ends are closer together so the moment less

9) When two surfaces come in contact asperities deform plastically until the normal force is borne by the sum of the local stresses. The effective area is given by $A_{eff}=W/3\sigma_y$ where σ_y is the yield stress of the softer material. The frictional force comes from the shear stress of the oxide film τ_y. Hence the frictional force $F=\tau_y W/(3\sigma_y)$. If the mean pressure is $P=W/A_{nominal}$ where $A_{nominal}$ is the nominal area of contact $F= \tau_y P A_{nominl}/(3\sigma_y)$ so for a given pressure the force is proportional to the area. To minimise friction choose hard materials with a weak oxide layer, and materials which do not react with, or dissolve in, each other.

10) To avoid fast fracture for a given working stress the maximum crack size must be below a value determined by the fracture toughness. Since cracks grow with every cycle, it is necessary to work back from this value to the largest allowable initial crack size for a given number of cycles. A proof stress equal to the stress required to cause fast fracture if a crack of this size is present is applied and if the structure survives it is safe against fracture for the required number of cycles.
To avoid fatigue cracks sharp corners should be avoided. It is also desirable that the structure should yield, or a pressure vessel leak, before fast fracture which is much more dangerous. For high temperatures creep is the main problem, this is a function of time and stress, not the number of cycles.

11)
i) Stiffness is determined by the elastic modulus. Young's modulus is the most common one. It can be measured by bending a beam, in a tensile test with a strain gauge or from the velocity of sound. In metals it is not very temperature dependent.
ii) Hardness is determined by the yield stress. This is normally measured in a tensile test. A hardness test, such as the Vickers, is not at all accurate and mainly used for qualitative testing. In fcc metals the yield stress is not very temperature dependent but in other metals it increases significantly at low temperatures.
iii) Brittleness is determined by fracture toughness. This is measured in a Charpy impact test. Since a low yield stress leads to a large plastic zone and therefore a large fracture toughness, other things being equal, the fracture toughness goes in the opposite direction from the yield stress. Hard materials are usually brittle. Hence fcc metals do not become brittle at low temperatures, others do.

12) For one year and one sq m mass gain=$\sqrt{(2x10^{-8}x3600x24x365)}$=0.79 kg
Atomic wt of O is 16 and Fe 56 so 16 kg O added removes 56 kg Fe
0.79kg added removes 2.78 kg Fe
Density of Fe= 7870 kg m^{-3}
Thickness loss =0.35mm

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