Paper 4: Mathematical Methods

Solutions to 2007 Tripos Paper

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1. Two vectors in the plane are $[0 \quad -1 \quad 3]^T - [1 \quad 2 \quad 1]^T = [-1 \quad -3 \quad 2]^T$ and $[2 \;\; 1 \;\; 0]^T - [1 \;\; 2 \;\; 1]^T = [1 \quad -1 \quad -1]^T$. The plane's normal is therefore

$$\begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \times \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

The equation of the line is therefore

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \lambda \begin{bmatrix} 5 \\ 1 \\ 4 \end{bmatrix}$$

or, equating the three expressions for λ,

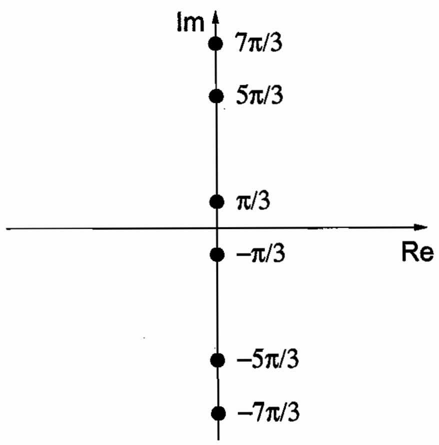
$$\frac{x-1}{5} = \frac{y-1}{1} = \frac{z-1}{4}$$ [10]

Examiner's remarks: This straightforward question tested the candidates' understanding of the geometry of lines and planes. It was very well answered, with most candidates arriving at the correct answer. Those who did not either made careless slips or struggled to convert the equation of a line from the form $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b}$ to the form $\frac{x-a}{p} = \frac{y-b}{q} = \frac{z-c}{r}$. Only a handful of candidates did not know where to start.

2. This is clearly a quadratic equation in e^z. Solving in the usual way, we get

$$e^z = \frac{1 \pm \sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i = e^{i(\pm\pi/3 + 2n\pi)}$$

Taking logs, we obtain the solutions $z = i(\pm\pi/3 + 2n\pi)$.



[10]

Examiner's remarks: This question asked candidates to solve $e^{2z} - e^z + 1 = 0$. Most spotted the quadratic, but far too many laboured over converting the solution to exponential form (this was contrived to be straightforward — the modulus was 1 and the argument was $\pi/3$). A disturbing number of candidates did not appear to know how to take the logarithm of a complex number in exponential form. Those who did then struggled to plot the solutions on an Argand diagram. For some reason, a large number of candidates thought that numbers of the form ki were somehow distributed around the unit circle.

3. The auxiliary equation is

$$\lambda^2 + 2\lambda + 5 = 0 \Leftrightarrow \lambda = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i$$

The particular integral is, by inspection, $y = 2$, so the general solution is

$$y = e^{-t}(A\cos 2t + B\sin 2t) + 2$$

Turning now to the boundary conditions, $y(0) = A + 2 = 2$, so $A = 0$ leaving $y = Be^{-t}\sin 2t + 2$. Then $\dot{y} = -Be^{-t}\sin 2t + 2Be^{-t}\cos 2t$ and $\dot{y}(0) = 2B = 2$, so $B = 1$. The solution is therefore $y = e^{-t}\sin 2t + 2$. [10]

Examiner's remarks: This question asked candidates to solve an under-damped linear second order differential equation without using Laplace transforms (one candidate found the right answer using Laplace transforms, but received zero marks for his or her efforts). Most candidates were evidently comfortable with this material, arriving at the correct solution and scoring full marks. Where candidates lost marks, it was generally for algebraic slips, particularly when substituting the initial conditions to find the constants in the complementary function. More worrying was a small number of candidates who did not know what to do when the auxiliary equation had complex roots.

4. (a) To avoid messy fractions, let's work out the eigenvalues of $4\mathbf{A}$ first. The characteristic equation is

$$\begin{aligned} \begin{vmatrix} 3-\lambda & -1 & 0 \\ -1 & 3-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} &= 0 \Leftrightarrow (1-\lambda)\left[(3-\lambda)^2 - 1\right] = 0 \\ \Leftrightarrow (1-\lambda)(\lambda^2 - 6\lambda + 8) &= 0 \Leftrightarrow (1-\lambda)(\lambda-4)(\lambda-2) = 0 \end{aligned}$$

The eigenvalues of $4\mathbf{A}$ are therefore 4, 2 and 1, and those of $\mathbf{A}$ are 1, 1/2 and 1/4. Turning now to the eigenvectors, for $\lambda = 1$ we have

$$\frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{array}{l} \Leftrightarrow \; 3x - y = 4x\,, \;\; -x + 3y = 4y\,, \;\; z = 4z \\ \Leftrightarrow \; -y = x\,, \;\; z = 0 \Leftrightarrow \mathbf{u}_1 = \frac{1}{\sqrt{2}}[1 \;\; -1 \;\; 0]^T \end{array}$$

For $\lambda = 1/2$ we have

$$\frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{array}{l} \Leftrightarrow\ 3x - y = 2x\ ,\ \ -x + 3y = 2y\ ,\ \ z = 2z \\ \Leftrightarrow\ x = y\ ,\ \ z = 0 \Leftrightarrow \mathbf{u}_2 = \frac{1}{\sqrt{2}}[1\ \ 1\ \ 0]^T \end{array}$$

For $\lambda = 1/4$ we have

$$\frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \begin{array}{l} \Leftrightarrow\ 3x - y = x\ ,\ \ -x + 3y = y\ ,\ \ z = z \\ \Leftrightarrow\ 2x = y\ ,\ \ x = 2y \Leftrightarrow x = y = 0 \Leftrightarrow \mathbf{u}_3 = [0\ \ 0\ \ 1]^T \end{array}$$

$\mathbf{A}$ therefore describes a stretch by a factor of $1/4$ in the direction $[0\ \ 0\ \ 1]^T$ and a stretch by a factor of $1/2$ in the direction $[1\ \ 1\ \ 0]^T$. Vectors in the direction $[1\ \ -1\ \ 0]^T$ are unchanged. [12]

(b) After much repeated application of $\mathbf{A}$, the components of $\mathbf{x}$ along $\mathbf{u}_2$ and $\mathbf{u}_3$ will vanish (since the corresponding eigenvalues have magnitudes less than one) while the component of $\mathbf{x}$ along $\mathbf{u}_1$ will remain unchanged. We therefore just need to work out the component of $\mathbf{x}$ along $\mathbf{u}_1$.

$$\mathbf{x}.\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -\frac{1}{\sqrt{2}} \Rightarrow \mathbf{A}^{20}\mathbf{x} \approx -\frac{1}{\sqrt{2}}\mathbf{u}_1 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$ [6]

(c) (i) Like $\mathbf{A}$, $\mathbf{B}$ must be symmetric and we can therefore use the standard diagonal decomposition from the *Mathematics Data Book.*

$$\begin{aligned} \mathbf{AB} &= \mathbf{U}\Lambda_A\mathbf{U}^T\mathbf{U}\Lambda_B\mathbf{U}^T = \mathbf{U}\Lambda_A\Lambda_B\mathbf{U}^T \quad \text{(since } \mathbf{U} \text{ is orthogonal)} \\ &= \mathbf{U}\Lambda_B\Lambda_A\mathbf{U}^T \quad \text{(since diagonal matrices commute)} \qquad (1) \\ &= \mathbf{U}\Lambda_B\mathbf{U}^T\mathbf{U}\Lambda_A\mathbf{U}^T = \mathbf{BA} \end{aligned}$$ [6]

(ii) If we consider applying first $\mathbf{B}$ then $\mathbf{A}$ to one of $\mathbf{A}$'s eigenvectors, it's clear that it does not change direction but is simply scaled by first λ_B then λ_A . It follows that this is an eigenvector of $\mathbf{BA}$ with eigenvalue $\lambda_A\lambda_B$. More formally,

$$\mathbf{ABu}_i = \mathbf{A}\lambda_{Bi}\mathbf{u}_i = \lambda_{Ai}\lambda_{Bi}\mathbf{u}_i$$

The eigenvalues of $\mathbf{AB}$ are therefore $\lambda_{Ai}\lambda_{Bi}$, where we found λ_{Ai} in part (b), and the eigenvectors are $\frac{1}{\sqrt{2}}[1\ \ -1\ \ 0]^T$, $\frac{1}{\sqrt{2}}[1\ \ 1\ \ 0]^T$ and $[0\ \ 0\ \ 1]^T$. This result is also evident by inspection of $\mathbf{AB}$'s decomposition in Equation (1). [6]

Examiner's remarks: In part (a), candidates were asked to find the eigenvalues and eigenvectors of

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Many candidates ignored the $\frac{1}{4}$ to simplify the algebra, but then forgot to put it back in later. Hardly anyone evaluated the determinant in the characteristic equation starting from the bottom row, which would have produced a ready-factorised cubic equation. Instead, most candidates laboured through the cubic equation the long way and many made careless mistakes (even though the standard issue calculators solve cubics at the touch of a button). When it came to the eigenvectors, $\mathbf{u}_3 = [0 \;\; 0 \;\; 1]^T$ proved particularly troublesome, since many candidates started by arbitrarily setting the x-component to 1. Equally distressing was the number of candidates who gave one of the eigenvectors as $[0 \;\; 0 \;\; 0]^T$, or who settled for a clearly non-orthogonal eigenvector set. Part (b) asked candidates to estimate $\mathbf{A}^{20}[1 \;\; 2 \;\; 1]^T$ and was also poorly answered. Most choose the long-winded method of evaluating $\mathbf{A}^{20} = \mathbf{U}\Lambda^{20}\mathbf{U}^T$ and generally forgot to normalize the eigenvectors in $\mathbf{U}$ and/or made a slip in the matrix multiplications. Only a handful decided to express $[1 \;\; 2 \;\; 1]^T$ in the eigenvector basis. In contrast, part (c), which explored properties of matrices sharing the same set of eigenvectors, was well answered. Most candidates demonstrated a sound understanding of what eigenvectors are and how they work, and were consequently able to deduce the required properties by algebraic manipulation or geometric argument.

5. (a) The characteristic equation is

$$\lambda^2 - (2+a)\lambda + 2a = 0 \Leftrightarrow (\lambda - 2)(\lambda - a) = 0$$

Assuming $a \neq 2$, the general solution is therefore $x_n = A2^n + Ba^n$. The initial values tell us that

$$\begin{aligned} 1 &= A + B \Leftrightarrow B = 1 - A \\ \text{and } b &= 2A + aB = 2A + a(1-A) = A(2-a) + a \\ \Leftrightarrow A &= \frac{b-a}{2-a}, \quad B = 1 - \frac{b-a}{2-a} = \frac{2-b}{2-a} \end{aligned}$$

The specific solution is therefore

$$x_n = \left(\frac{b-a}{2-a}\right) 2^n + \left(\frac{2-b}{2-a}\right) a^n$$

[10]

(b) Substituting $\epsilon = 2 - a$, we get

$$x_n = \left(\frac{b-2+\epsilon}{\epsilon}\right) 2^n + \left(\frac{2-b}{\epsilon}\right) (2-\epsilon)^n$$

We may now proceed using either power series expansions or l'Hôpital's rule. For completeness, both approaches are described here.

Power series expansions

$$\begin{aligned} x_n &= \left(\frac{b-2+\epsilon}{\epsilon}\right)2^n + \left(\frac{2-b}{\epsilon}\right)(2-\epsilon)^n \\ &= \left(\frac{b-2+\epsilon}{\epsilon}\right)2^n + \left(\frac{2-b}{\epsilon}\right)2^n\left(1-\frac{\epsilon}{2}\right)^n \\ &= \left(\frac{b-2+\epsilon}{\epsilon}\right)2^n + \left(\frac{2-b}{\epsilon}\right)2^n\left(1-\frac{n\epsilon}{2}+O(\epsilon^2)\right) \quad \text{(binomial expansion)} \\ &= \frac{2^n}{\epsilon}\left(b-2+\epsilon+2-b-(2-b)\frac{n\epsilon}{2}+O(\epsilon^2)\right) \\ &= \frac{2^n}{\epsilon}\left(\epsilon+(b-2)\frac{n\epsilon}{2}+O(\epsilon^2)\right) = 2^n\left(1+\tfrac{1}{2}(b-2)n+O(\epsilon)\right) \\ &= 2^{n-1}(2+(b-2)n) \text{ in the limit as } \epsilon \to 0 \end{aligned}$$

l'Hôpital's rule

$$\begin{aligned} x_n &= \left(\frac{b-2+\epsilon}{\epsilon}\right)2^n + \left(\frac{2-b}{\epsilon}\right)(2-\epsilon)^n \\ &= \frac{(b-2+\epsilon)2^n+(2-b)(2-\epsilon)^n}{\epsilon} = \frac{f(\epsilon)}{g(\epsilon)} \end{aligned}$$

It is clear that $f(\epsilon) = g(\epsilon) = 0$ when $\epsilon = 0$, so l'Hôpital's rule is applicable.

$$\begin{aligned} f'(\epsilon) &= 2^n - (2-b)n(2-\epsilon)^{n-1} \\ g'(\epsilon) &= 1 \\ \lim_{\epsilon\to 0} x_n &= \frac{f'(0)}{g'(0)} = 2^n - (2-b)n2^{n-1} = 2^{n-1}(2+(b-2)n) \end{aligned}$$ [10]

(c) For the case $a = b = 0.9$, the specific solution in (a) becomes

$$x_n = 0 \times 2^n + 1 \times 0.9^n$$

and we would therefore expect the computer to generate a sequence of increasing powers of 0.9, eventually displaying $x_{99}/x_{98} = 0.9$. However, things will in fact pan out rather differently because the algorithm used by the computer to enumerate the x_n values is unstable. When we initialise `x[1] = 0.9`, the computer's internal representation of `x[1]` will be close to, but not exactly equal to, 0.9: this is because 0.9 cannot be represented precisely with a 23-bit binary mantissa. The computer will therefore be evaluating the sequence

$$x_n = \epsilon_1 \times 2^n + (1+\epsilon_2) \times 0.9^n$$

where ϵ_1 and ϵ_2 are small. For large n, the first term will swamp the second, and the ratio of successive terms will be 2, not the intended 0.9. The computer will therefore most likely display the number 2 on its console. [10]

Examiner's remarks: In part (a), candidates were asked to derive the solution to the linear difference equation $x_{n+2} = (2+a)x_{n+1} - 2ax_n$ with given initial values. Almost all candidates knew how to go about this and, since this was a "show that" question, most obtained full marks after correcting minor algebraic slips. It was, however, surprising how few candidates factorised the characteristic equation $\lambda^2 - (2+a)\lambda + 2a = 0$, the vast majority opting for the standard quadratic formula. In part (b), candidates were asked to derive the limiting form of the solution when $a = 2$. This could be done in a few lines of algebra using either a power series expansion or l'Hôpital's rule. Those candidates who knew how to take a limit did not find this part of the question difficult and scored full marks. Unfortunately, the vast majority were evidently incompetent in this area and went astray after making the suggested substitution $\epsilon = 2 - a$. A few candidates made botched attempts at the limit and then magically arrived at the correct answer (presumably by re-solving the difference equation from the beginning, for this particular value of a, on scrap paper). Such blatant dishonesty was not appreciated by the Examiner! In part (c), the candidates were presented with a C++ algorithm to enumerate the difference equation and asked to comment on its behaviour in the case when it should have produced a decaying geometric progression. Despite having seen very similar algorithms in lectures and the examples paper, very few recognised this as a classic unstable algorithm. It was pleasing, however, that a good proportion commented that the program was unlikely to produce the mathematically correct result, because of floating point error accumulation.

6. The given input is a superposition of a step, magnitude 2, at $t = T$ and an impulse, magnitude -3, at $t = 2T$. We are provided with the system's step response, which we can differentiate to find the impulse response $g(t) = e^{-t}$. The response to the input is therefore

$$y(t) = \begin{cases} 0 & t < T \\ 2(1 - e^{-(t-T)}) & T \leq t < 2T \\ 2(1 - e^{-(t-T)}) - 3e^{-(t-2T)} & t \geq 2T \end{cases}$$

The step response is continuous at $t = 0$, but the impulse response is not: this is what we'd expect from a first order system. Second order systems have impulse responses that are continuous at $t = 0$. [10]

Examiner's remarks: This question asked candidates to find the response of a system to an input consisting of a step and an impulse. Unfortunately, the vast majority of candidates failed to spot that the input had this simple form and attempted to find the solution by convolution (rather than a quick superposition) with varying degrees of success.

7. Taking Laplace transforms of both sides, we get

$$s^2X + 2sX + X = (s+1)^2X = \frac{1}{s}$$

$$\Leftrightarrow X = \frac{1}{s(s+1)^2} = \frac{1}{s} + \frac{As+B}{(s+1)^2} = \frac{s^2+2s+1+As^2+Bs}{s(s+1)^2}$$

Hence $A = -1$ and $B = -2$, giving

$$X = \frac{1}{s} - \frac{(s+2)}{(s+1)^2} = \frac{1}{s} - \frac{(s+1)}{(s+1)^2} - \frac{1}{(s+1)^2} = \frac{1}{s} - \frac{1}{(s+1)} - \frac{1}{(s+1)^2}$$

Inverting the Laplace transform using the *Mathematics Data Book*, we arrive at the solution $x = 1 - e^{-t} - te^{-t}$. [10]

Examiner's remarks: This question asked candidates to solve a differential equation using Laplace transforms. The question was well answered by the majority of candidates, though a few had difficulty with the partial fractions.

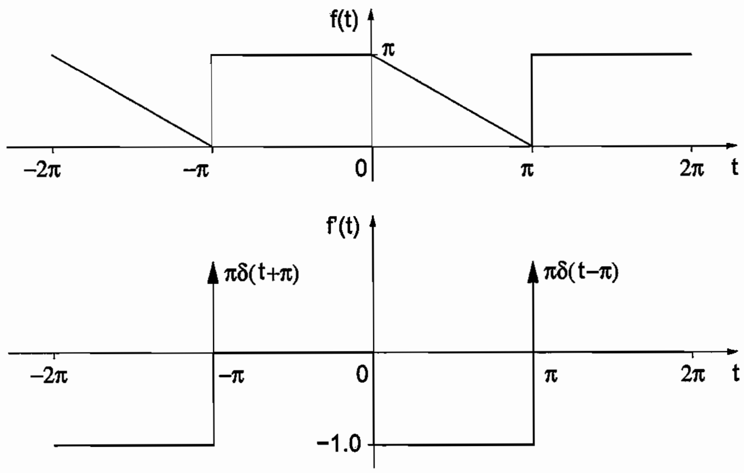
8. With no ties, the number of rank orders is simply $4 \times 3 \times 2 \times 1 = 24$. With arbitrary ties, the general problem is much harder but, fortunately, with just four students we can enumerate the various patterns. In the table below, the top three rows show the possible tie patterns and the bottom row shows the numbers of ways of allocating the students to the top three rows.

●●●●	●●●	●	●●	●●	●	●
	●	●●●	●●	●	●●	●
				●	●	●●
1	4	4	$^4C_2$	$^4C_2 \times 2$	$^4C_2 \times 2$	$^4C_2 \times 2$

Adding the numbers in the bottom row gives 51, plus the 24 tie-free rank orders making a total of 75. [10]

Examiner's remarks: This question asked candidates to count the number of ways that four students can be ranked, both with and without ties. Many made a reasonable start but very few were able to reach the correct answer in the case where ties were allowed. The most common failures were to assume that the answer was 4^4 or to miss the case where the four students break down into two pairs with equal scores.

9. (a)



[6]

(b) Looking at the sketch of $f'(t)$, the area under each δ function is π and the area under each rectangular block below the t-axis is $-\pi$. Hence, $\int_0^{2\pi} f'(t)\,dt = 0$. The mean value of $f'(t)$ is therefore zero. [4]

(c) Note that $f'(t)$ is neither even nor odd, so we'll need both sine and cosine terms. However, we have established that $f'(t)$ has zero mean so we will not need an a_0 term. When evaluating the Fourier coefficients, we should integrate from 0 to 2π and not from $-\pi$ to π, to avoid having the δ functions at the integration limits. From the *Mathematics Data Book*,

$$f'(t) = \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt)$$

where

$$\begin{aligned}
a_n &= \frac{1}{\pi}\int_0^{2\pi} f'(t) \cos nt\, dt \\
&= \frac{1}{\pi}\left(\int_0^{\pi} -\cos nt\, dt + \pi \cos n\pi\right) \quad \text{(sifting property of } \delta(t-\pi)) \\
&= \frac{1}{\pi}\left[-\frac{1}{n}\sin nt\right]_0^{\pi} + (-1)^n = (-1)^n
\end{aligned}$$

and

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_0^{2\pi} f'(t) \sin nt\, dt \\
&= \frac{1}{\pi}\left(\int_0^{\pi} -\sin nt\, dt + \pi \sin n\pi\right) \quad \text{(sifting property of } \delta(t-\pi)) \\
&= \frac{1}{\pi}\left[\frac{1}{n}\cos nt\right]_0^{\pi} + 0 = \frac{1}{n\pi}(\cos n\pi - 1) = \frac{1}{n\pi}((-1)^n - 1)
\end{aligned}$$

Hence

$$f'(t) = \sum_{n=1}^{\infty}\left((-1)^n \cos nt + \frac{((-1)^n - 1)}{n\pi}\sin nt\right)$$ [10]

(d) To go from $f'(t)$ to $f(t)$, we just need to integrate and work out the constant of integration, which will be the mean value of $f(t)$, this being $3\pi/4$ by inspection of the sketch. Hence

$$\begin{aligned}
f(t) &= \int \sum_{n=1}^{\infty}\left((-1)^n \cos nt + \frac{((-1)^n - 1)}{n\pi}\sin nt\right) dt \\
&= \frac{3\pi}{4} + \sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n}\sin nt - \frac{((-1)^n - 1)}{n^2\pi}\cos nt\right)
\end{aligned}$$ [6]

(e) $f(t)$ has discontinuities in value, and should therefore converge as $1/n$, as indeed it does. $f'(t)$ has δ functions, and therefore should not converge, as indeed is the case. [4]

Examiner's remarks: This question asked candidates to sketch a periodic function and its derivative, and hence derive the Fourier series for the derivative (given in the question) and then the function. Most candidates were able to do this well, but some were unable to handle the delta functions that appeared in the derivative correctly, leading to missing parts of the answer. Because the Fourier series of the derivative was given in the question, a large number of candidates attempted to "cheat" by just jumping to the correct answer part way through their derivation. In many cases, this was an obvious falsehood given earlier mistakes.

10. (a) (i) The gradient function tells us that

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{1}{x^2-4}+e^y\\ \frac{\partial f}{\partial y} &= xe^y+5y^2+4\end{aligned}$$

Integrating the first equation, we find

$$\begin{aligned}f &= \int \frac{1}{(x+2)(x-2)}\,dx + xe^y + g(y)\\ &= \int\left(\frac{-1/4}{(x+2)}+\frac{1/4}{(x-2)}\right)dx + xe^y + g(y)\\ &= \frac{1}{4}\ln\left(\frac{x-2}{x+2}\right) + xe^y + g(y)\end{aligned}$$

where $g(y)$ is an arbitrary function of y. Integrating the second equation, we find

$$f = xe^y + \frac{5}{3}y^3 + 4y + h(x)$$

where $h(x)$ is an arbitrary function of x. The two equations for f are reassuringly compatible. Hence

$$f = xe^y + \frac{5}{3}y^3 + 4y + \frac{1}{4}\ln\left(\frac{x-2}{x+2}\right) + C$$

where C is an arbitrary constant. [10]

(ii) The cross-derivatives are given by

$$\begin{aligned}\frac{\partial^2 f}{\partial y\partial x} &= \frac{\partial}{\partial y}\left(\frac{1}{x^2-4}+e^y\right) = e^y\\ \frac{\partial^2 f}{\partial x\partial y} &= \frac{\partial}{\partial x}\left(xe^y+5y^2+4\right) = e^y\end{aligned}$$

and are therefore equal. Equality of the cross-derivatives is a condition for f to be a well-behaved function[1]. We would not be able to find f unless the cross-derivatives were equal, since the expression given in the question would not in fact correspond to the gradient of a well-behaved function. [5]

(iii) Contours of constant f are everywhere perpendicular to ∇f. At the origin, we have $\nabla f = \frac{3}{4}\mathbf{i} + 4\mathbf{j}$. The direction of the contour is therefore $4\mathbf{i} - \frac{3}{4}\mathbf{j}$. [5]

(b) By inspection, the surface passes through the point $(1,1,1)$ when $u = 1$ and $v = 1$. We can find the surface normal by computing the vector product of two tangents to the surface at this point. Clearly, two tangents are given by $\frac{\partial \mathbf{r}}{\partial u}$ and $\frac{\partial \mathbf{r}}{\partial v}$.

$$\begin{aligned} \frac{\partial \mathbf{r}}{\partial u} &= 2uv\,\mathbf{i} + \mathbf{k} = 2\mathbf{i} + \mathbf{k} \text{ at } (1,1,1) \\ \frac{\partial \mathbf{r}}{\partial v} &= u^2\,\mathbf{i} + 2v\,\mathbf{j} = \mathbf{i} + 2\mathbf{j} \text{ at } (1,1,1) \end{aligned}$$

The surface normal is therefore given by

$$\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$ [10]

Examiner's remarks: This question had two parts. The first asked candidates to integrate the two expressions for the gradient of a scalar field in order to recover the functional form of the field. Most candidates were able to make a good attempt at this but many had difficulties integrating $\frac{1}{x^2-4}$, failing to spot that this could be achieved by partial fractions. Some candidates performed the integration using a trigonometric substitution, which resulted in a convoluted form of the answer that many then failed to simplify. The second part of the question asked candidates to find the normal to a parametrically defined surface. Those that saw what to do were typically able to answer this question very quickly and generate the correct answer. However, a great many candidates were unable to recall (from an examples paper question) that the normal could be obtained from $\partial\mathbf{r}/\partial u \times \partial\mathbf{r}/\partial v$.

11. We can divide 98304.0 by two 16 times, leaving us with 1.5. So $98304.0 = 1.5 \times 2^{16} = 1.5 \times 2^{143-127}$. Also, $143 = 128 + 8 + 4 + 2 + 1$, so 143 in binary is 10001111. Remembering the implicit 1 in the mantissa, the floating point representation of `a` is

0	10001111	10000000000000000000000

The next floating point number above 98304.0 has a 1 in the final (23rd) bit of the mantissa, making it $98304.0 + 2^{-23} \times 2^{16} = 98304.0078125$. Since 98304.004 is closer to this number than 98304.0, this will be `b`'s internal representation. The subtraction result with therefore be 0.0078125. This is a large subtractive cancellation error. [10]

[1] First year engineers need not concern themselves with the precise mathematical definition of "well-behaved"!

Examiner's remarks: This question tested candidates' understanding of IEEE floating point arithmetic. Though few got full marks, over 75% of candidates produced the correct machine representation of a given decimal number. Most also appreciated the limitations of the representation, falling down only in the fine detail.

12. *Algorithmic complexity* describes how the number of operations scales with the size of the problem. For example, if an algorithm requires n^2 operations to solve a problem of size n, we would say that the algorithm has complexity $\mathcal{O}(n^2)$.

    The algorithmic complexity of QuickSort is $\mathcal{O}(n \log n)$ while that of exchange sort is $\mathcal{O}(n^2)$. For QuickSort, the time to sort 10^6 items is therefore approximately $\frac{10^6 \log 10^6}{10^5 \log 10^5} \times 1 = 12$ minutes. For exchange sort, the time is approximately $\frac{10^6 \times 10^6}{10^5 \times 10^5} \times 3 = 300$ minutes $= 5$ hours. [10]

    Examiner's remarks: This question was generally well answered. Several students tried to prove (rather than simply quote) the complexity of the exchange sort and QuickSort algorithms. The most common mistaken answers were $\mathcal{O}(n^{\frac{3}{2}})$ and $\mathcal{O}(n)$ respectively, though marks were dropped as much for sloppy arithmetic as for revision lapses.

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