ENGINEERING TRIPOS PART IA

Tuesday 12 June 2007 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (short) The plane P passes through the points $(1,2,1)$, $(0,-1,3)$ and $(2,1,0)$. Find the equation of the line that is perpendicular to P and passes through the point $(1,1,1)$. Express your answer in the form

$$\frac{x-a}{p}=\frac{y-b}{q}=\frac{z-c}{r}$$ [10]

2 (short) Find all values of z that satisfy the equation

$$e^{2z}-e^{z}+1=0$$

On an Argand diagram, mark the locations of the six solutions that lie closest to the origin. [10]

3 (short) Without using Laplace transforms, find the solution of the differential equation

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+5y=10$$

with initial conditions $y(0)=\dot{y}(0)=2$. [10]

4 (long)

(a) Find the eigenvalues and eigenvectors of the matrix $\mathbf{A}$, where

$$\mathbf{A} = \frac{1}{4}\begin{bmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence describe the geometrical transformation represented by $\mathbf{A}$. [12]

(b) If $\mathbf{x} = [1 \;\; 2 \;\; 1]^T$, what, approximately, is $\mathbf{A}^{20}\mathbf{x}$? [6]

(c) The matrix $\mathbf{B}$ has the same eigenvectors as $\mathbf{A}$ but distinct eigenvalues λ_{B1}, λ_{B2} and λ_{B3}, which are each different from those of $\mathbf{A}$.

(i) Prove that $\mathbf{AB} = \mathbf{BA}$. [6]

(ii) What are the eigenvalues and eigenvectors of $\mathbf{AB}$? [6]

(TURN OVER

5 (long)

(a) Consider the linear difference equation

$$x_{n+2} = (2+a)x_{n+1} - 2ax_n$$

with initial values $x_0 = 1$ and $x_1 = b$, where a and b are constants. Assuming $a \neq 2$, show that the solution is

$$x_n = \left(\frac{b-a}{2-a}\right)2^n + \left(\frac{2-b}{2-a}\right)a^n$$

You must derive the solution: **do not** simply check that the given solution satisfies the difference equation and the initial values. [10]

(b) The solution is indeterminate when $a = 2$. By writing $\epsilon = 2 - a$ and taking the limit as $\epsilon \to 0$, **and not otherwise**, find the solution when $a = 2$. [10]

(c) The following C++ code segment is used to enumerate the first 100 values of x_n for the case $a = b = 0.9$, and display the ratio x_{99}/x_{98}.

```
float x[100]; int n;
x[0] = 1.0; x[1] = 0.9;
for (n=0; n<98; n++) x[n+2] = 2.9*x[n+1] - 1.8*x[n];
cout << x[99]/x[98] << endl;
```

When the code is run, what will be displayed on the console? Justify your answer. [10]

SECTION B

6 **(short)** A linear system has step response $1 - e^{-t}$ for $t \geq 0$. Find its response to the input

$$x(t) = \begin{cases} 0 & t < T \\ 2 - 3\delta(t - 2T) & t \geq T \end{cases}$$

where $T > 0$. Is this a first or second order system? Justify your answer. [10]

7 **(short)** Solve using Laplace transforms

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + x = 1$$

with initial conditions $x(0) = \dot{x}(0) = 0$. [10]

8 **(short)** Four students are ranked according to their examination results. How many possible rank orders are there (i) assuming no two students receive the same marks and (ii) without this restriction? [10]

(TURN OVER

9 (long) Consider the function

$$f(t) = \begin{cases} \pi & -\pi \leq t < 0 \\ \pi - t & 0 \leq t < \pi \end{cases}$$

where $f(t)$ is periodic with period 2π.

(a) Sketch $f(t)$ and $f'(t)$ in the range $-2\pi \leq t \leq 2\pi$. Take care to mark the locations and magnitudes of any δ functions on the sketch of $f'(t)$. [6]

(b) What is the mean value of $f'(t)$? [4]

(c) Show that the Fourier series for $f'(t)$ is

$$f'(t) = \sum_{n=1}^{\infty} \left((-1)^n \cos nt + \left(\frac{(-1)^n - 1}{n\pi} \right) \sin nt \right)$$ [10]

(d) Find a Fourier series for $f(t)$. [6]

(e) Verify that the series for $f'(t)$ and $f(t)$ converge, or otherwise, at the expected rates. [4]

10 (long)

(a) Consider the gradient function

$$\nabla f = \left[\frac{1}{x^2-4} + e^y\right]\mathbf{i} + \left[xe^y + 5y^2 + 4\right]\mathbf{j}$$

(i) Find $f(x,y)$. [10]

(ii) Calculate $\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$. How do your findings relate to part (i)? [5]

(iii) In which direction does the contour of constant f pass through the origin? [5]

(b) A surface is defined by the equation

$$\mathbf{r} = u^2 v\,\mathbf{i} + v^2\,\mathbf{j} + u\,\mathbf{k}$$

Find the normal to the surface at the point $(1,1,1)$. [10]

(TURN OVER

SECTION C

11 **(short)** A C++ program contains the line `float a = 98304.0`. Express 98304.0 in the form $p \times 2^{q-127}$, where $1 \leq p < 2$ and q is an integer. Hence, write down the 32-bit binary IEEE standard floating point representation of `a`. A second variable `b` is initialised with the statement `float b = 98304.004`. Assuming standard single precision arithmetic, what would be the result of subtracting `a` from `b`? [10]

12 **(short)** Explain what is meant by *algorithmic complexity*. A computer sorts 10^5 items in one minute using QuickSort and three minutes using exchange sort. Estimate how long it would take to sort 10^6 items using each of the two algorithms. [10]

END OF PAPER