ENGINEERING TRIPOS PART IA

PAPER 1: MECHANICAL ENGINEERING

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SECTION A

1 (a) Bernoulli 1 → 2, working with gauge pressure

$$P_1 + \tfrac{1}{2}\rho V_1^2 + \rho g y_1 = P_2 + \tfrac{1}{2}\rho V_2^2 + \rho g y_2$$

$y_1 = y_2$

$$P_1 - \underbrace{P_2}_{0} = \tfrac{1}{2}\rho V_2^2 - \tfrac{1}{2}\rho V_1^2 = -\tfrac{1}{2}\rho V_2^2\left[\frac{V_1^2}{V_2^2} - 1\right]$$

Continuity, $A_1 V_1 = A_2 V_2 \Rightarrow V_1/V_2 = A_2/A_1$

$$P_1 = -\tfrac{1}{2}\rho V_2^2\left[(A_2/A_1)^2 - 1\right]$$

(b) $P_1 = -\rho g h = -\tfrac{1}{2}\rho V_2^2\left[(A_2/A_1)^2 - 1\right]$

(hydrostatic balance)

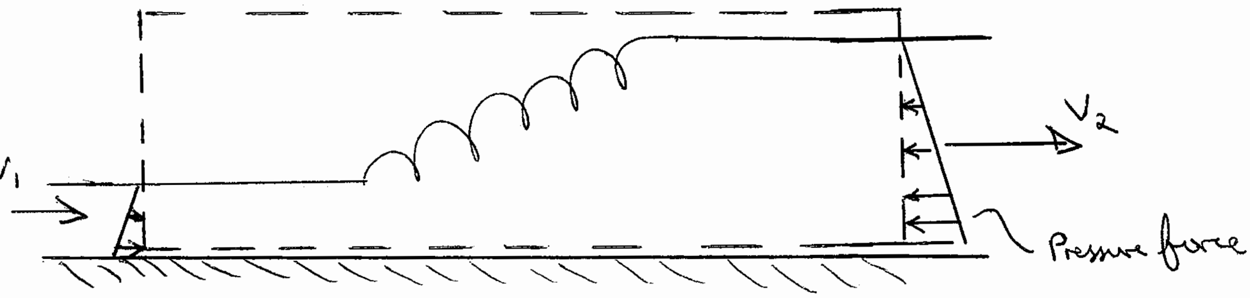
$$\Rightarrow \quad gh = \tfrac{1}{2}V_2^2\left[(A_2/A_1)^2 - 1\right] = \frac{15}{2}V_2^2$$

$$V_2 = \sqrt{\tfrac{2}{15}\, 9.81 \times 0.1} = 0.362 \text{ m/s}$$

2 (a) The flow at 1 and 2 is straight and parallel, so there is no vertical acceleration. The vertical force balance is between gravity and pressure, which leads to a hydrostatic pressure distribution.

(b) There is a conversion of mechanical energy into internal energy (heat) in the turbulence, and in such cases Bernoulli does not apply.

(c)



Apply momentum equation in the streamwise direction.

$$\sum F = \sum \dot{m} V_{OUT} - \sum \dot{m} V_{IN}$$

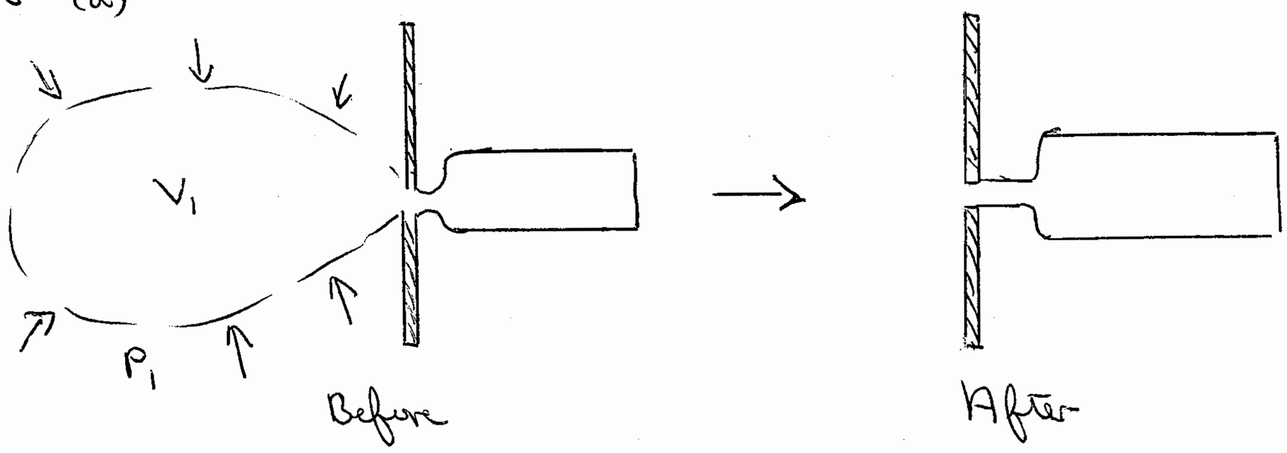
pressure forces

unit depth into page: $\left(\frac{1}{2}\rho g h_1\right) h_1 - \left(\frac{1}{2}\rho g h_2\right) h_2 = \dot{m}(V_2 - V_1)$

mean pressure at 1   mean pressure at 2 $\rho h_1 V_1 = \rho h_2 V_2$

$$\underline{\underline{\frac{1}{2}\rho g h_1^2 - \frac{1}{2}\rho g h_2^2 = \rho h_1 V_1 (V_2 - V_1)}}$$

3 (a)



Work by system $= \int P\,dV = -P_1 V_1 = -P_1 m v_1$ (negative)

1st law : $-W + \cancel{Q} = \Delta U = m c_v (T_2 - T_1)$

$$\Rightarrow \quad m P_1 v_1 = m c_v (T_2 - T_1)$$

(b) Ideal gas law,

$$P_1 v_1 = R T_1$$

$$\Rightarrow \quad R T_1 = c_v (T_2 - T_1)$$

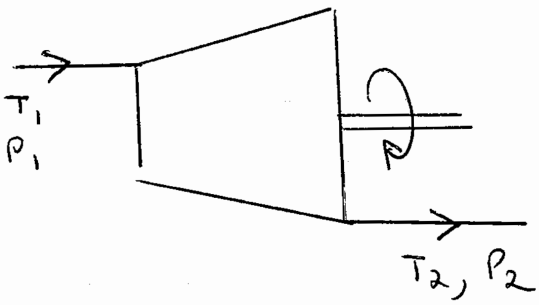
$$\Rightarrow \quad c_v T_2 = (R + c_v) T_1$$

$$\Rightarrow \quad T_2 = \frac{R + c_v}{c_v} T_1$$

$$= \frac{0.52 + 1.71}{1.71} \, 300\,K$$

$$= 391\,K$$

4 (a)



Ideal gas law, $Pv = RT \Rightarrow P = \rho R T$

$$\rho_1 = \frac{P_1}{R T_1} = \frac{10^6}{287 \times 1400} = \underline{\underline{2.49 \text{ kg/m}^3}}$$

$$\dot{m} = \rho A u = 2.49 \times 0.6 \times 4 = \underline{\underline{5.97 \text{ kg/s}}}$$

(b) Steady flow energy equation,

$$\dot{Q} - \dot{W} = \dot{m}(h_2 - h_1) = \dot{m} c_p (T_2 - T_1)$$

$\dot{Q} \to 0$

$$\dot{W} = \dot{m} c_p (T_1 - T_2) = 5.97 \times 10^3 \times (1400 - 800)$$

$$= \underline{\underline{3.58 \text{ M Watts}}}$$

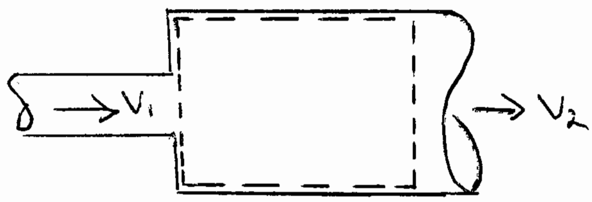
Perfect gas and adiabatic

$$\frac{P_2}{P_1} = \left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma - 1}}, \quad \gamma = \frac{c_p}{c_v} = 1.40$$

$$P_2 = 10^6 \left(\frac{800}{1400}\right)^{\frac{1.40}{0.4}} = \underline{\underline{1.41 \times 10^5 \text{ N/m}^2}}$$

5 (a) Apply momentum equation in streamwise direction.

$$\underbrace{\Sigma F}_{\text{pressure forces}} = \dot{m} V_{OUT} - \dot{m} V_{IN}$$



$$P_1 A_2 - P_2 A_2 = \dot{m}(V_2 - V_1) = \rho V_2 A_2 (V_2 - V_1)$$

$$\Rightarrow \quad \underline{\underline{P_2 - P_1 = \rho V_2 (V_1 - V_2)}}$$

(b) Let H be Bernoulli's constant, $H = \cancel{\rho g y} + \frac{1}{2}\rho V^2 + P$

$$H_1 - H_2 = \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho V_2^2 + P_1 - P_2$$

$$= \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho V_2^2 - \rho V_2 (V_1 - V_2)$$

$$= \tfrac{1}{2}\rho V_1^2 - \tfrac{1}{2}\rho V_2^2 - \rho V_1 V_2 + \rho V_2^2$$

$$= \tfrac{1}{2}\rho \left[V_1^2 + V_2^2 - 2 V_1 V_2 \right]$$

$$= \tfrac{1}{2}\rho \left[V_1 - V_2 \right]^2$$

(c) Continuity: $V_1 = 10$ m/s, $A_2/A_1 = 2 \Rightarrow V_2 = V_1 \frac{A_1}{A_2} = 5$ m/s

$$H_1 - H_2 = \tfrac{1}{2} \times 10^3 \times [10 - 5]^2 = \underline{12{,}500 \ N/m^2}$$

Note there is a conversion of mechanical energy into 'internal energy' (heat) and Bernoulli does not apply in such cases, i.e. $H_1 \neq H_2$

(d) $\text{Power} = Q \Delta H = \frac{\dot{m}}{\rho}(H_1 - H_2) = \frac{10 \text{ kg s}^{-1}}{10^3 \text{ kg m}^{-3}} \times 12{,}500 \frac{N}{m^2}$

$$= 125 \frac{Nm}{s} = \underline{\underline{125 \ W.}}$$

(e) p has the same dimensions as ρV^2, c.f. Bernoulli eqn.

Thus $\dfrac{\Delta p}{\rho V^2}$ is dimensionless

One possibility is

$$c_p = \frac{\Delta p}{\rho V_2^2} = \frac{\cancel{\rho} V_2(V_1 - V_2)}{\cancel{\rho} V_2^2} = \frac{V_1}{V_2} - 1$$

Continuity $V_1/V_2 = A_2/A_1$

$$\underline{\underline{c_p = \frac{\Delta p}{\rho V_2^2} = \frac{A_2}{A_1} - 1}}$$

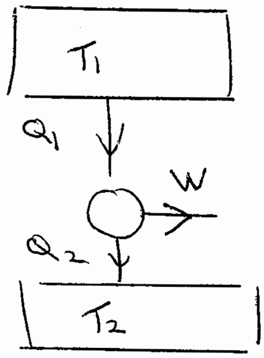
If we work with c_p then we have elliminated V_1 and V_2 from the problem, i.e. c_p is only a function of the geometry.

6 (a) (i) True, as it is independent of the path taken

(ii) False, as the first law tells us the heat transfer depends on the work done by the system.

(iii) False, it can always be returned by a suitable heat transfer.

(b)



Clausius $\oint \frac{dQ}{T} \leq 0$

$$\Rightarrow \frac{Q_1}{T_1} - \frac{Q_2}{T_2} \leq 0$$

1^{st} Law $\quad W = Q_1 - Q_2$

$$\Rightarrow \frac{Q_1}{T_1} - \frac{Q_1 - W}{T_2} \leq 0$$

$$\Rightarrow Q_1\left(\frac{1}{T_1} - \frac{1}{T_2}\right) + \frac{W}{T_2} \leq 0$$

$$\Rightarrow \frac{W}{T_2} \leq Q_1\left(\frac{1}{T_2} - \frac{1}{T_1}\right)$$

$$\Rightarrow \underline{\underline{\frac{W}{Q_1} \leq 1 - \frac{T_2}{T_1}}}$$

η (efficiency)

(c) 1^{st} law : $Q_3 = W + Q_2^*$

But $W \leq \left(1 - \frac{T_2}{T_1}\right) Q_1$ from above

and $\frac{Q_2^*}{T_2} - \frac{Q_3}{T_3} \leq 0$ from Clausius $\Rightarrow Q_2^* \leq \frac{T_2}{T_3} Q_3$

$$\Rightarrow \underline{\underline{Q_3 \leq \left(1 - \frac{T_2}{T_1}\right) Q_1 + \frac{T_2}{T_3} Q_3}}$$

(d)

$$Q_3\left(1-\frac{T_2}{T_3}\right)=\left(1-\frac{T_2}{T_1}\right)Q_1$$

$$\Rightarrow \quad \frac{Q_3}{Q_1}=\frac{1-T_2/T_1}{1-T_2/T_3}$$

$$Q_3 > Q_1 \quad \Rightarrow \quad \frac{T_2}{T_1} < \frac{T_2}{T_3} \quad \Rightarrow \quad T_1 > T_3$$

Engineering Tripos Part IA 2008

Paper 1 Section B

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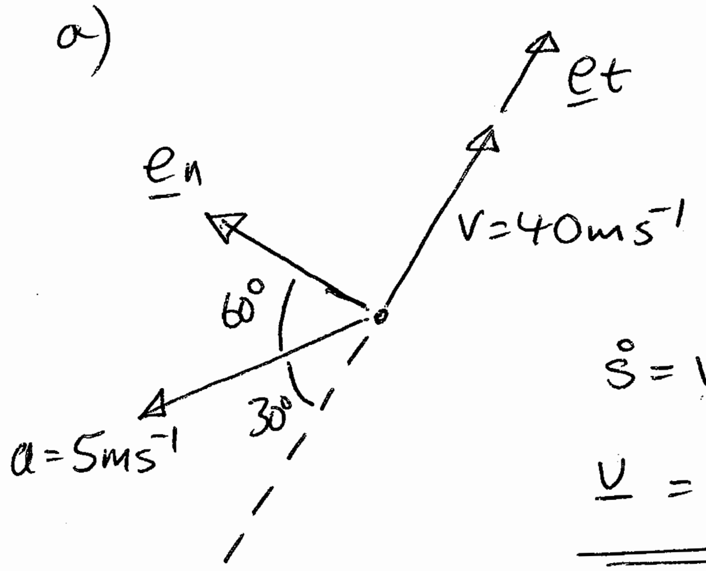
⑦ a)



$$\dot{s} = V = 40ms^{-1}$$

$$\underline{\underline{\underline{v} = \dot{s}\underline{e}t = 40\underline{e}t \ ms^{-1}}}$$

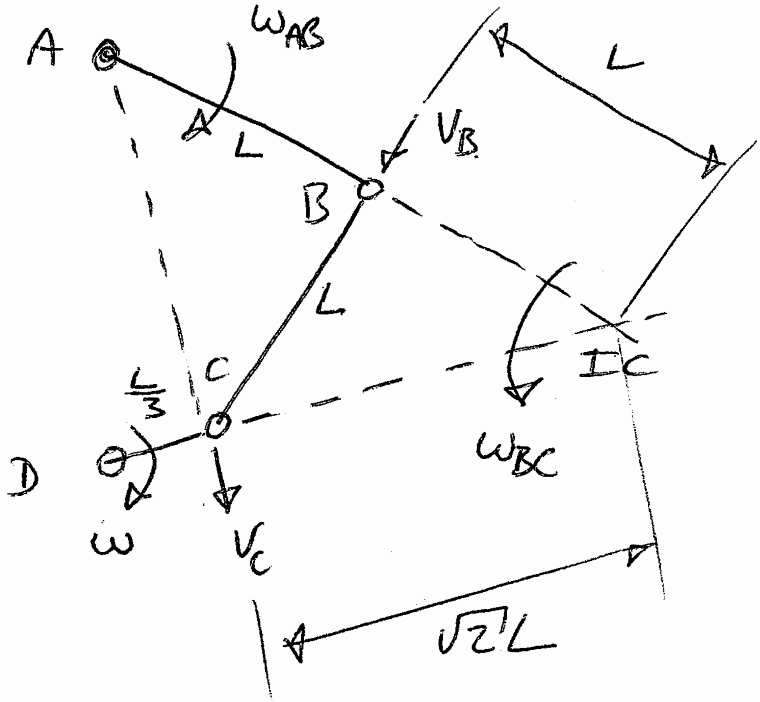
$$\underline{a} = \ddot{s}\,\underline{e}t + \frac{\dot{s}^2}{\rho}\underline{e}n$$

$$= -5\cos 30°\,\underline{e}t + 5\sin 30°\,\underline{e}n$$

$$\underline{\underline{\underline{a} = -\frac{5\sqrt{3}}{2}\underline{e}t + \frac{5}{2}\underline{e}n \quad ms^{-2}}}$$

b) $$\frac{\dot{s}^2}{\rho} = \frac{5}{2} \quad \therefore \rho = \frac{2\dot{s}^2}{5} = \frac{2.40^2}{5} = \underline{\underline{640m}}$$

8 a)



Locate IC as shown.

$$V_C = \frac{\omega L}{3}$$

$$\therefore \quad \underline{\underline{\omega_{BC}}} = \frac{V_C}{\sqrt{2}L} = \frac{\omega L}{3} \cdot \frac{1}{\sqrt{2}L} = \underline{\underline{\frac{\omega}{3\sqrt{2}}}}$$

$$V_B = \omega_{BC} \cdot L = \frac{\omega}{3\sqrt{2}}$$

$$\underline{\underline{\omega_{AB}}} = \frac{V_B}{L} = \underline{\underline{\frac{\omega}{3\sqrt{2}}}}$$

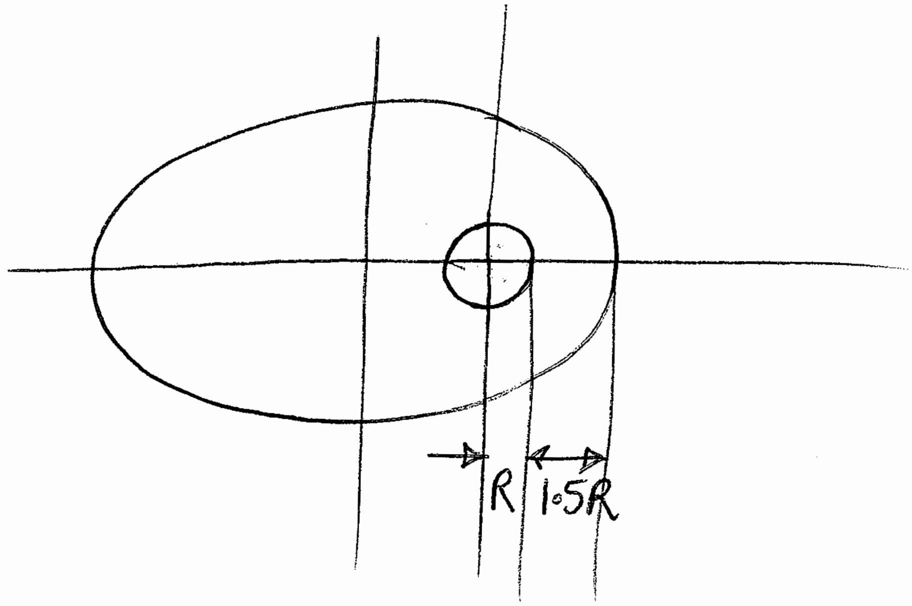
b) Power

$$T \cdot \omega = T_{AB} \cdot \omega_{AB}$$

$$T_{AB} = \frac{T\omega}{\omega_{AB}} = T\omega \cdot \frac{3\sqrt{2}}{\omega}$$

$$\underline{\underline{T_{AB} = 3\sqrt{2}\,T}}$$

(9)



a) $r_p = 2{\cdot}5R$

$$\frac{r_A}{r_p} = \frac{1+e}{1-e}$$

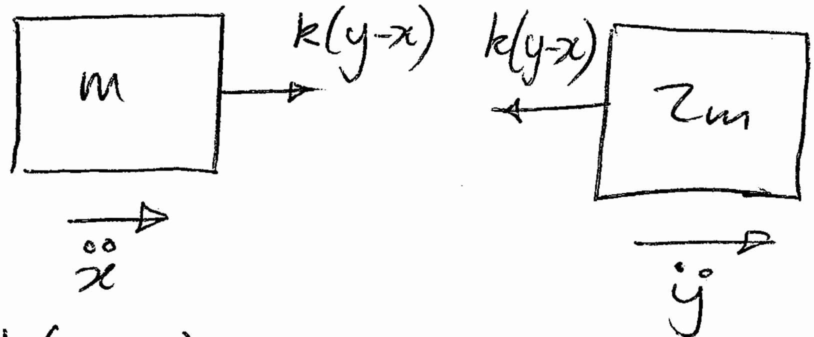
$$\therefore r_A = \frac{1{\cdot}6}{0{\cdot}4} r_p = 4 . 2{\cdot}5R = \underline{\underline{10R}}$$

$\therefore$ $\underline{\underline{\text{altitude at apogee} = 9R}}$

b) $r_p V_p = r_A V_A$

$$\therefore V_A = \frac{r_p}{r_A} . V_p = \frac{2{\cdot}5R}{10R} . V = \underline{\underline{\frac{V}{4}}}$$

10 (a)



$$m\ddot{x} = k(y-x) \qquad 2m\ddot{y} = -k(y-x)$$

$$\Rightarrow \begin{bmatrix} m & 0 \\ 0 & 2m \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{y} \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

eigenvalues

$$\left| -\omega^2 M + K \right| = 0$$

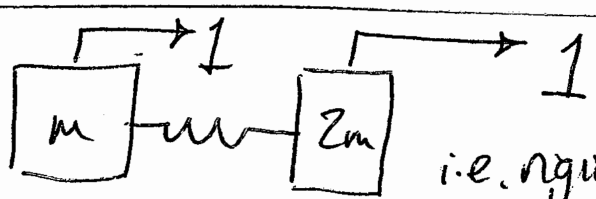
$$\begin{vmatrix} -\omega^2 m + k & -k \\ -k & -\omega^2 2m + k \end{vmatrix} = 0$$

$$(-\omega^2 m + k)(-2\omega^2 m + k) - k^2 = 0$$

$$2\omega^4 m^2 - 3\omega^2 mk + k^2 - k^2 = 0$$

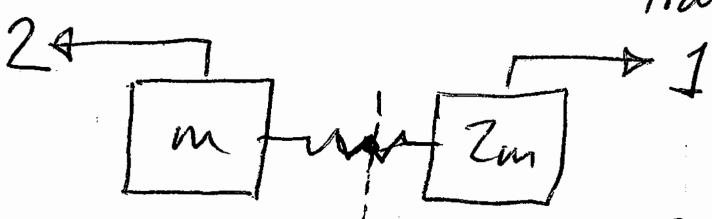
$$\therefore \quad \omega^2 = 0 \quad \text{or} \quad \omega^2 = \frac{3k}{2m}$$

(b) $\omega = 0$



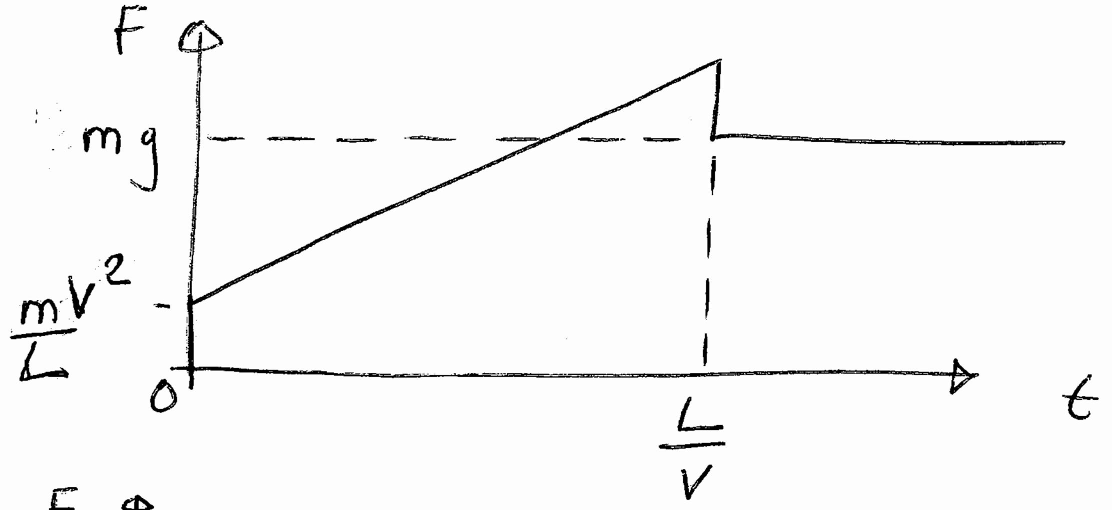
i.e. rigid body translation.

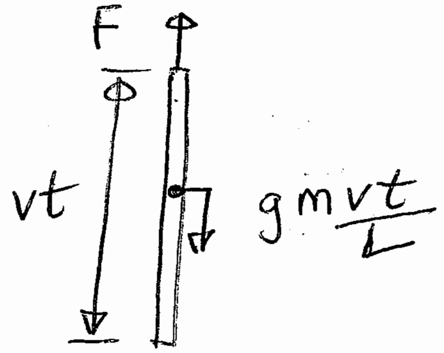
$\omega^2 = \frac{3k}{2m}$



nodal point $\frac{2}{3}$ way along spring.

(11) (a) (i)





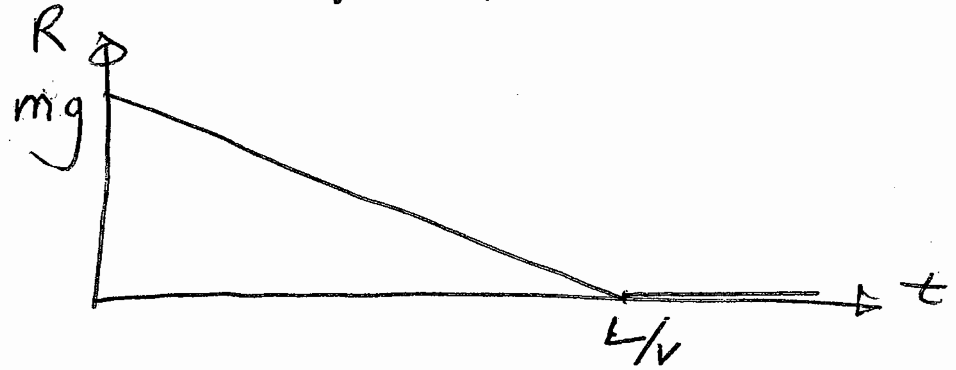
$\dot{m}v$ where $\dot{m} = \frac{m}{L}v$

$$\underline{F = gm\frac{vt}{L} + \dot{m}v} \qquad 0 < t < \frac{L}{v}$$

max F when $t = \frac{L}{v} \Rightarrow \underline{F = mg + \left(\frac{m}{L}\right)v^2}$

$\underline{F = mg}$ when $t > \frac{L}{v}$ $(\dot{m} = 0)$

ii) Ground reaction force is just the weight of the chain not yet lifted.



(b) (i) chain drops freely $\therefore$ speed $= gt$

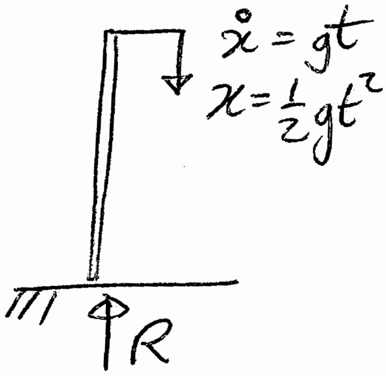
drop $x = \frac{1}{2}gt^2$

max speed when $\frac{1}{2}gt^2 = L$

$$t = \sqrt{\frac{2L}{g}}$$

hence max speed $= g\sqrt{\frac{2L}{g}} = \underline{\underline{\sqrt{2gL}}}$

(ii)



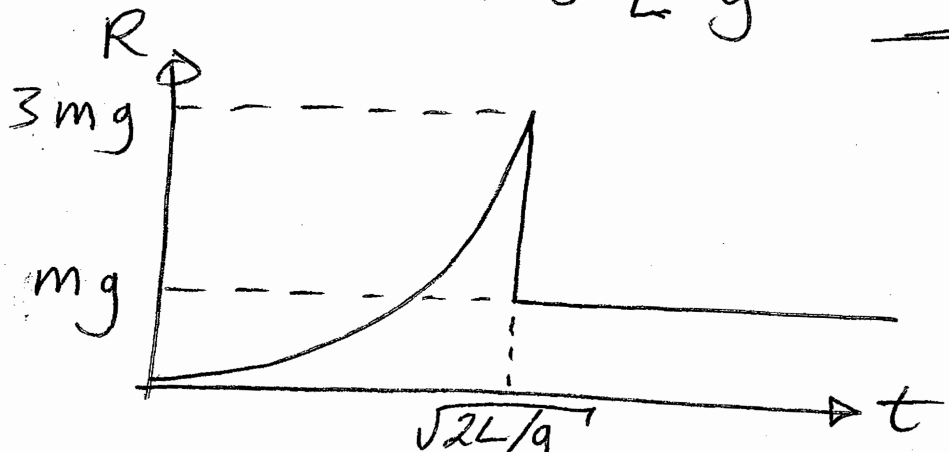
now $\dot{m} = \frac{m}{L}\dot{x}$

$$\begin{aligned} R &= \text{weight of chain on ground} + \dot{m}\dot{x} \\ &= \frac{x}{L}mg + \left(\frac{m}{L}\dot{x}\right)\dot{x} \\ &= \frac{1}{2}g^2\frac{m}{L}t^2 + \frac{m}{L}g^2t^2 \end{aligned}$$

$$\underline{\underline{R = \frac{3}{2}g^2\frac{m}{L}t^2}}$$

max R when $t = \sqrt{\frac{2L}{g}}$

$\therefore R = \frac{3}{2}g^2\frac{m}{L}\frac{2L}{g} = \underline{\underline{3mg}}$



12 (a) Rigid : $|f| = \omega^2 b = 1000^2 \cdot 10^{-3}$

$= 1000 \text{ N}$

$\lambda = k = 0$ $\therefore$ $|y| = \dfrac{|f|}{m\omega^2} = \dfrac{b}{m} = \dfrac{10^{-3}}{100} = 0.01 \text{mm}$

(b) $m\ddot{y} = -ky - \lambda\dot{y} + \omega^2 b e^{i\omega t}$

$m\ddot{y} + \lambda\dot{y} + ky = \omega^2 b e^{i\omega t}$

$\dfrac{m}{k}\ddot{y} + \dfrac{\lambda}{k}\dot{y} + y = \dfrac{\omega^2 b}{k} e^{i\omega t}$

then let $\omega_n = \sqrt{\dfrac{k}{m}}$, $\dfrac{2\zeta}{\omega_n} = \dfrac{\lambda}{k}$

$\therefore \zeta = \dfrac{\lambda}{2\sqrt{km}}$

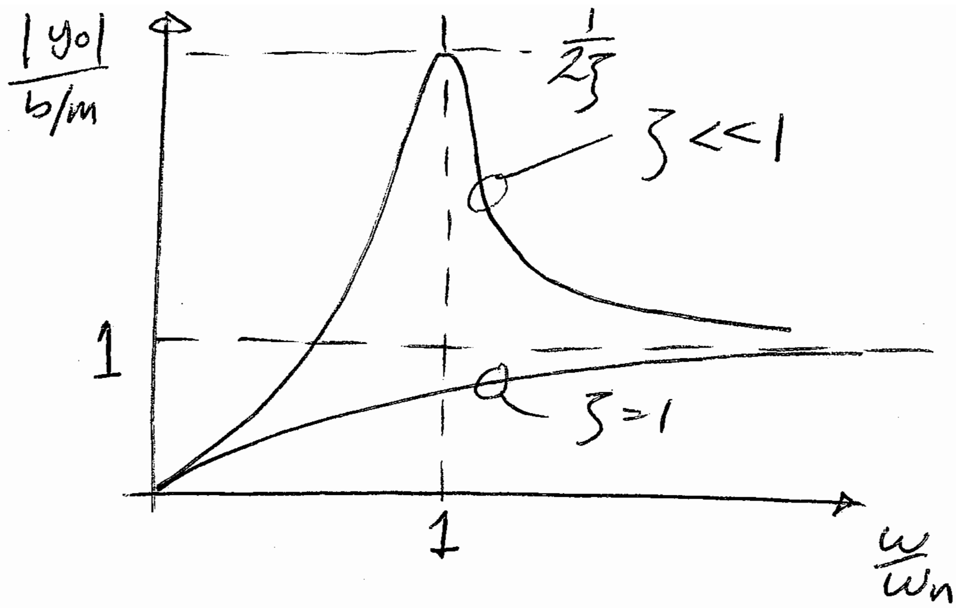
and let $\dfrac{-\ddot{x}}{\omega_n^2} = \dfrac{\omega^2 b}{k} e^{i\omega t}$

integrating twice $x = \omega_n^2 \dfrac{b}{k} e^{i\omega t} = \dfrac{b}{m} e^{i\omega t}$

(c) data book case (b)

let $y = y_0 e^{i\omega t}$ $\Rightarrow$ $y_0\left[-\left(\dfrac{\omega}{\omega_n}\right)^2 + 2i\zeta\dfrac{\omega}{\omega_n} + 1\right] = \dfrac{b}{m}\left(\dfrac{\omega}{\omega_n}\right)^2$

$\therefore |y_0| = \dfrac{\dfrac{b}{m}\left(\dfrac{\omega}{\omega_n}\right)^2}{\sqrt{\left(1-\left(\dfrac{\omega}{\omega_n}\right)^2\right)^2 + \left(2\zeta\dfrac{\omega}{\omega_n}\right)^2}}$



(d) $\omega_n = \sqrt{\frac{250 \cdot 10^3}{100}} = 50 \text{ rad/s}$

$\zeta = \frac{1}{2} \frac{\lambda}{\sqrt{mk}} = \frac{1}{2} \frac{500}{\sqrt{100 \cdot 250 \cdot 10^3}} = 0.05$

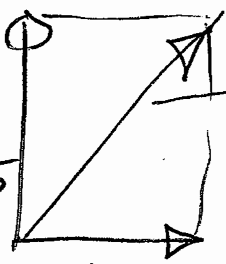
$\frac{b}{m} = \frac{0.001}{100} = 10^{-5} \text{ m}$

$\frac{\omega}{\omega_n} = \frac{1000}{50} = 20 \gg 1$

hence assume $|y_0| = \frac{b}{m} = 10^{-5} \text{ m}$

use phasor diagram to calculate force:

$|y| i\omega\lambda$
$= 10^{-5} \cdot 1000 \cdot 0.05$
$= 5N$



resultant $= \sqrt{5^2 + 2.5^2} = \underline{5.59 N}$

$|y| k = 10^{-5} \cdot 250 \cdot 10^3 = 2.5N$