A 51

YEAR	PART (of Tripos)	PAPER NUMBER & TITLE	NAME OF AUTHOR(S)
2008	IA	P4 – Mathematical Methods	Dr RS Cant

Final Crib

1)
$$\left.\begin{aligned} x+y+2z &= 3 \\ x+2y+z &= 2 \end{aligned}\right\} \qquad \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = \underline{i}(-3) - \underline{j}(-1) + \underline{k}.1$$

$$\text{put } x=0 : \quad \left.\begin{aligned} y+2z &= 3 \\ 2y+z &= 2 \end{aligned}\right] \quad y = 1/3, \ z = 4/3 \qquad \begin{pmatrix} 0 \\ 1/3 \\ 4/3 \end{pmatrix}$$

$$\text{put } y=0 : \quad \left.\begin{aligned} x+2z &= 3 \\ x+z &= 2 \end{aligned}\right] \quad x=1, \ z=1 \qquad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$$\text{put } z=0 : \quad \left.\begin{aligned} x+y &= 3 \\ x+2y &= 2 \end{aligned}\right] \quad x=4, \ y=-1 \qquad \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}$$

$$\therefore \quad \underline{l} = \underline{a} + \lambda \underline{b} = \begin{pmatrix} 0 \\ 1/3 \\ 4/3 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 1 \\ 1 \end{pmatrix}$$

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2) $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 5e^x$$

CF: $\lambda^2 + 3\lambda - 4 = 0$; $(\lambda+4)(\lambda-1) = 0$

$\lambda = -4$ or $\lambda = 1$

CF is $y = Ae^{-4x} + Be^x$

PI try $y = axe^x$

$$\frac{dy}{dx} = axe^x + ae^x$$

$$\frac{d^2y}{dx^2} = axe^x + 2ae^x$$

substitute: $\cancel{axe^x} + 2ae^x + \cancel{3axe^x} + 3ae^x - \cancel{4axe^x} = 5e^x$

$5ae^x = 5e^x \Rightarrow a = 1$

$\therefore$ general solution is $\underline{y = Ae^x + Be^{-4x} + xe^x}$

3) (a) $x_n = \dfrac{x_{n-1} + x_{n+1}}{2}$

Set $x_n = n$: $x_{n-1} = n-1$ and $x_{n+1} = n+1$

substitute : $\dfrac{n-1 + n+1}{2} = 2n/_2 = n$ QED

(b) Rearrange : $x_{n+1} = 2x_n - x_{n-1}$

If $x_0 = 0$ have $x_2 = 2x_1$

$x_3 = 2x_2 - x_1 = 3x_1$

$x_4 = 2x_3 - x_2 = 4x_1$

$\vdots$

$x_{10} = 10x_1$

but $x_{10} = 21$ $\therefore x_1 = 21/10$

$\therefore$ $x_n = n.21/10$ $\therefore$ $x_{10} = 21$

4) (a) $|z-5| = 6 \Rightarrow |x+iy-5| = 6$

$[(x-5) + iy][(x-5) - iy] = 6^2$

$(x-5)^2 + y^2 = 6^2$ cf. $x^2+y^2 = r^2$

Circle centred at $(5,0)$ with radius 6.

4) (b) (i) $\lim_{x\to 1/4} \dfrac{\cos^3 2\pi x}{1-16x^2}$ now $\cos^3 2\pi/4 = \cos^2 \pi/2 = 0$

and $1-16x^2 = 0$ for $x = 1/4$.

$\therefore$ by l'Hôpital's rule: $\left.\dfrac{dy}{dx}\right|_{\text{numerator}} = -3\cos^2 2\pi x \sin 2\pi x \cdot 2\pi$

$$\left.\frac{dy}{dx}\right|_{\text{denominator}} = -32x$$

$$\therefore \lim_{x\to 1/4} \frac{-6\pi \cos^2 2\pi x \sin 2\pi x}{-32x} \to \underline{0}$$

(ii) $\lim_{x\to 0} \dfrac{\sin x - x}{\tan x - x}$; Taylor series

$$\sin x \sim x - x^3/3! + x^5/5! + \ldots$$

$$\tan x \sim x + x^3/3 + 2/15\, x^5 + \ldots$$

$$\therefore \text{ have } \lim_{x\to 0} \frac{-x^3/3! + x^5/5! + \ldots}{x^3/3 + 2/15\, x^5 + \ldots} = \underline{-\frac{1}{2}}$$

(c) $z^4 - 2z^3 - 18z^2 + 70z - 75 = 0$

$z = 2-i$ is a root $\therefore$ so is $z = 2+i$

$\therefore [z-(2-i)][z+(2-i)] = z^2 - 4z + 5$ is a divisor

Divide: A (55)

$$\begin{array}{r|l} & z^2 + 2z - 15 \\ \hline z^2 - 4z + 5 & z^4 - 2z^3 - 18z^2 + 70z - 75 \\ & z^4 - 4z^3 + 5z^2 \\ \hline & \quad 2z^3 - 23z^2 \\ & \quad 2z^3 - 8z^2 + 10z \\ \hline & \qquad -15z^2 + 60z \\ & \qquad -15z^2 + 60z - 75 \\ \hline \end{array}$$

$\therefore \quad (z^2 + 2z - 15)(z^2 - 4z + 5)$ is the product.

$\therefore$ here $z = -5$ or 3 as extra roots

5) $$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

$$\det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 2 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & 1-\lambda \end{bmatrix} = 0$$

$$(1-\lambda)(1-\lambda)^2 - 0 + 2(-(1-\lambda).2) = 0$$

$$(1-\lambda)\left[(1-\lambda)^2 - 4\right] = 0$$

A (6)

$\therefore\ \lambda = 1 \quad \text{or} \quad \lambda^2 - 2\lambda - 3 = 0$

$\lambda = 1 \quad \text{or} \quad \lambda = 3 \quad \text{or} \quad \lambda = -1$

set $\lambda = 1$: $\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [0]$; $\quad \begin{matrix} 2x_3 = 0 \\ 2x_1 = 0 \end{matrix}$; $\begin{pmatrix} 0 \\ x_2 \\ 0 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} -2 & 0 & 2 \\ 0 & -2 & 0 \\ 2 & 0 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = [0]$; $\quad \begin{matrix} -2x_1 + 2x_3 = 0 \\ -2x_2 = 0 \\ 2x_1 - 2x_3 = 0 \end{matrix}$; $\begin{pmatrix} x_1 \\ 0 \\ x_3 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = (0)$; $\quad \begin{matrix} 2x_1 + 2x_3 = 0 \\ 2x_2 = 0 \\ 2x_1 + 2x_3 = 0 \end{matrix}$; $\begin{pmatrix} x_1 \\ 0 \\ -x_3 \end{pmatrix}$

$\therefore$ normalised eigenvectors are $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}, \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{pmatrix}$.

Matrix of normalised eigenvectors as columns

$$U = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \; ; \det U = -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = 1$$

$$UU^T = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore$ <u>matrix is orthonormal.</u>

Rotation matrix : e.g. $\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}$

represents a rotation about the y-axis through an angle of $-\pi/4$.

$A = U \Lambda U^T$; Λ is matrix of eigenvalues :

Then $A^{10} = U \Lambda^{10} U^T$

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix} ; \quad \Lambda^n = \begin{bmatrix} 1^n & 0 & 0 \\ 0 & 3^n & 0 \\ 0 & 0 & (-1)^n \end{bmatrix}$$

$$\Lambda^{10} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A^{10} = \begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{3^{10}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \\ 0 & \frac{3^{10}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(3^{10}+1) & 0 & \frac{1}{2}(3^{10}-1) \\ 0 & 1 & 0 \\ \frac{1}{2}(3^{10}-1) & 0 & \frac{1}{2}(3^{10}+1) \end{bmatrix}$$

2008 Part IA Paper 4 Section B

Final crib

6)

$$x'' + 6x' + 5x = 0 \; ; \; x(0) = 2; \; \dot{x}(0) = 3$$

Laplace transforms

$$L(x) = Y(s)$$

$$L(x') = sY - x(0) = sY - 2$$

$$L(x'') = s^2Y - sx(0) - x'(0)$$

$$= s^2Y - 2s - 3$$

substitute in ode:

$$[s^2Y - 2s - 3] + 6[(sY - 2)] + 5Y = 0$$

$$s^2Y + 6sY + 5Y = 2s + 15$$

$$(s+5)(s+1)\,Y = 2s + 15$$

$$Y(s) = \frac{2s+15}{(s+5)(s+1)} = \frac{a}{s+5} + \frac{b}{s+1}$$

partial fractions

$$a(s+1) + b(s+5) = 2s + 15 \; ; \; \left.\begin{array}{l} a + b = 2 \\ a + 5b = 15 \end{array}\right\} \begin{array}{l} a = -5/4 \\ b = 13/4 \end{array}$$

$$\therefore \; Y(s) = \frac{-5/4}{s+5} + \frac{13/4}{s+1} \; ;$$

Maths Data Book p20 Inverse transforms

$$x(t) = -5/4\, e^{-5t} + 13/4\, e^{-t}$$

B (S10)

7) $$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 14y = f(t)$$

step at $t=0$: $\begin{cases} f(t) = 1 & t>0 \\ 0 & t\leq 0 \end{cases}$

solve the equation.

CF : auxiliary equation $\lambda^2 + 9\lambda + 14 = 0$

$\lambda = -7$ or $\lambda = -2$

CF is $Ae^{-7t} + Be^{-2t}$

PI : try $y = \alpha$: $14\alpha = 1$ for $t > 1$

$\alpha = 1/14$

General solution is $y = \frac{1}{14} + Ae^{-7t} + Be^{-2t}$

at $t=0$ $y = y' = 0$ for continuous function and first derivative

$\therefore$ $\begin{cases} 0 = \frac{1}{14} + A + B \\ 0 = -7A - 2B \end{cases}$ $\Rightarrow$ $A = \frac{1}{35}$; $B = -\frac{1}{10}$

$\therefore$ Step response is $y = \frac{1}{14} + \frac{1}{35}e^{-7t} - \frac{1}{10}e^{-2t}$ for $t>0$

Impulse response is $\frac{d}{dt}$ (step response)

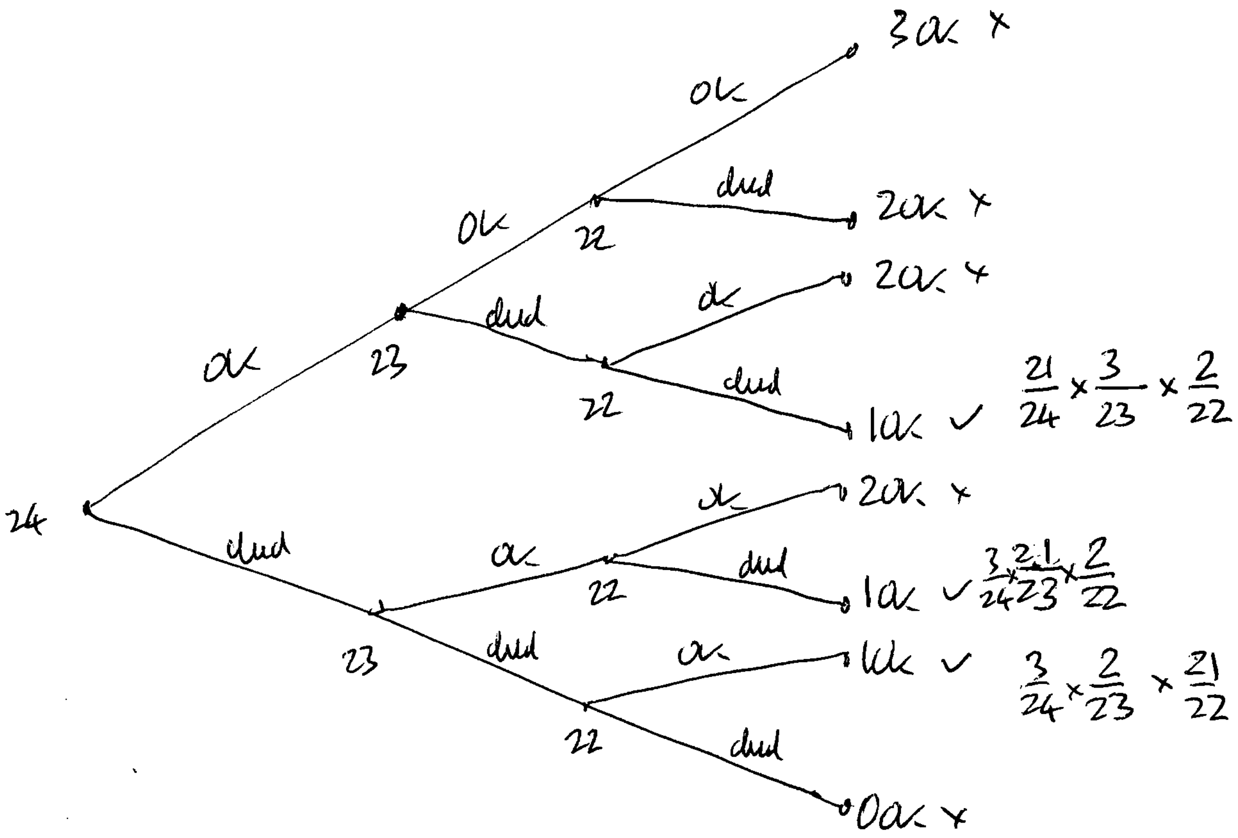
$= -\frac{1}{5}e^{-7t} + \frac{1}{5}e^{-2t}$ for $t>0$

8) 24 chips, 3 are defective.

a) Sampling with replacement : probability of picking a dud $= \frac{3}{24} = \frac{1}{8}$

$\therefore$ prob. of picking 2 dud out of 3 $= 3\left(\frac{1}{8}\right)^2 \cdot \frac{7}{8} = \underline{0.0410}$

b) Sampling without replacement : need a tree diagram



$\therefore$ prob. of picking 2 dud out of 3

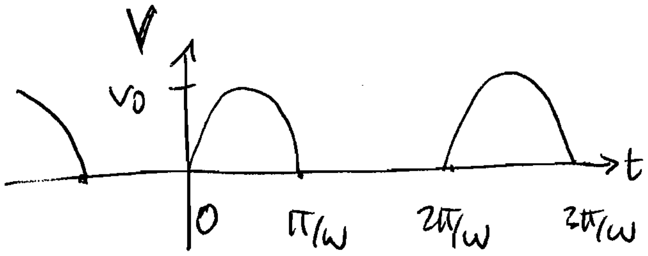
$$= \frac{21}{24} \times \frac{3}{23} \times \frac{2}{22} + \frac{3}{24} \times \frac{21}{23} \times \frac{2}{22} + \frac{3}{24} \times \frac{2}{23} \times \frac{21}{22}$$

$$= 3 \times \frac{126}{12144} = \underline{0.0311}$$

B (S12)

9)

$$V = \begin{cases} V_0 \sin \omega t & 0 < t < T/2 \\ 0 & -T/2 < t < 0 \end{cases} \qquad T = 2\pi/\omega$$



Note: neither odd nor even.

Maths Data Book p17

$$V(t) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{2\pi n t}{T} + b_n \sin \frac{2\pi n t}{T} \right)$$

with

$$\left. \begin{aligned} a_n &= 2/T \int_0^T f(t) \cos \frac{2\pi n t}{T} \, dt \\ b_n &= 2/T \int_0^T f(t) \sin \frac{2\pi n t}{T} \, dt \end{aligned} \right\}$$

First find a_0: $\quad a_0 = \frac{2}{T} \int_0^{T/2} V_0 \sin \omega t \, dt = \frac{2}{T} \left[-\frac{V_0}{\omega} \cos \omega t \right]_0^{T/2}$

$$= \frac{V_0}{\pi} \quad \text{since } T/2 = \pi/\omega$$

Then $\quad a_n = \frac{\omega}{\pi} \int_0^{\pi/\omega} V_0 \sin \omega t \cos n\omega t \, dt$

B(S13)

ie $a_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} \frac{V_0}{2}\left[\sin\omega(n+1)t - \sin\omega(n-1)t\right]dt$

Maths Data Book p 2

For $n=1$; $\int_0^{\pi/\omega} \sin\omega t\cos\omega t\,dt = \int_0^{\pi/\omega}\frac{1}{2}\sin 2\omega t\,dt = -\frac{1}{4\omega}\left[\cos 2\omega t\right]_0^{\pi/\omega}$

$= 0$

For $n>1$; $a_n = \frac{\omega V_0}{2\pi}\left[-\frac{\cos\omega(n+1)t}{\omega(n+1)} + \frac{\cos\omega(n-1)t}{\omega(n-1)}\right]_0^{\pi/\omega}$

$= \frac{\omega V_0}{2\pi}\left[-\frac{\cos(n+1)\pi}{\omega(n+1)} + \frac{\cos(n-1)\pi}{\omega(n-1)} + \frac{1}{\omega(n+1)} - \frac{1}{\omega(n-1)}\right]$

For n odd $\cos(n+1)\pi = \cos(n-1)\pi = 1$

n even $\cos(n+1)\pi = \cos(n-1)\pi = -1$

$\therefore\ a_n = \frac{\omega V_0}{2\pi}\left[\frac{1-\cos(n+1)\pi}{\omega(n+1)} - \frac{1-\cos(n-1)\pi}{\omega(n-1)}\right] = 0$ for n odd

$a_n = \frac{\omega V_0}{2\pi}\left[\frac{2}{\omega(n+1)} - \frac{2}{\omega(n-1)}\right] = -\frac{V_0}{\pi}\frac{2}{(n+1)(n-1)}$ for n even

Now evaluate $b_n = \frac{\omega}{\pi}\int_0^{\pi/\omega} V_0 \sin\omega t \sin n\omega t \, dt$

$$= \frac{\omega V_0}{2\pi}\int_0^{\pi/\omega}\left[\cos\omega(n-1)t - \cos(n+1)\omega t\right]dt$$

Maths Data Book p2.

For $n=1$ have $b_1 = \frac{\omega V_0}{2\pi}\int_0^{\pi/\omega}(1-\cos 2\omega t)\,dt$

$$= \frac{\omega V_0}{2\pi}\left[t + \frac{\sin 2\omega t}{2\omega}\right]_0^{\pi/\omega}$$

$$= \frac{\omega V_0}{2\pi}\left[\pi/\omega + 0\right] = \frac{V_0}{2}$$

For $n>1$ have $b_n = \frac{\omega V_0}{2\pi}\left[-\frac{\sin\omega(n-1)t}{\omega(n-1)} + \frac{\sin\omega(n+1)t}{\omega(n+1)}\right]_0^{\pi/\omega}$

$= 0$ for all n.

$\therefore$ Fourier series is

$$V(t) = \frac{V_0}{\pi} + \frac{V_0}{2}\sin\omega t - \frac{V_0}{\pi}\cdot\frac{2}{3\cdot 1}\cos 2\omega t - \frac{V_0}{\pi}\frac{2}{5\cdot 3}\cos 4\omega t + \dots$$

$$= \frac{V_0}{\pi} + \frac{V_0}{2}\sin\omega t - \frac{V_0}{\pi}\sum_{n=1}^{\infty}\frac{2}{(2n+1)(2n-1)}\cos 2n\omega t$$

10) (a) $f(x,y) = x^4+y^4+4xy \quad = z,$ say

$$\frac{\partial z}{\partial x} = 4x^3+4y \; ; \; \frac{\partial z}{\partial y} = 4y^3+4x$$

solve simultaneously: $\left.\begin{aligned} x^3+y &= 0 \\ y^3+x &= 0 \end{aligned}\right\}$ for stationary points

Have $-x^9+x=0$; $x(1-x^8) = 0 \Rightarrow x=0$ or $x=\pm 1$

similarly $-y^9+y=0$; $y(1-y^8) = 0 \Rightarrow y=0$ or $y=\pm 1$

check: $(0,0)$ is a pair satisfying the equations

set $x=1$: $\left.\begin{aligned} 1+y &= 0 \Rightarrow y=-1 \\ y^3+1 &= 0 \Rightarrow y=-1 \end{aligned}\right\}$ consistent

set $x=-1$: $\left.\begin{aligned} -1+y &= 0 \Rightarrow y=+1 \\ y^3-1 &= 0 \Rightarrow y=+1 \end{aligned}\right\}$ consistent

$\therefore$ Stationary points are $(0,0)$, $(1,-1)$, $(-1,1)$

Second derivatives: $\dfrac{\partial^2 z}{\partial x^2} = 12x^2$; $\dfrac{\partial^2 z}{\partial y^2} = 12y^2$

$$\frac{\partial^2 z}{\partial y \partial x} = 4 \; ; \; \frac{\partial^2 z}{\partial x \partial y} = 4 \quad \text{(check)}.$$

B (S16)

Maths Data Book p5: $\Delta = \frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x \partial y}\right)^2$

$= 144x^2y^2 - 4^2$

point	$\frac{\partial^2 z}{\partial x^2}$	$\frac{\partial^2 z}{\partial y^2}$	$\frac{\partial^2 z}{\partial x \partial y}$	Δ	type
(0,0)	0	0	4	-16	saddle
(1,-1)	12	12	4	128	minimum
(-1,1)	12	12	4	128	minimum.

(b) Some values of $z=f(x,y)$:

$z = x^4 + y^4 + 4xy$

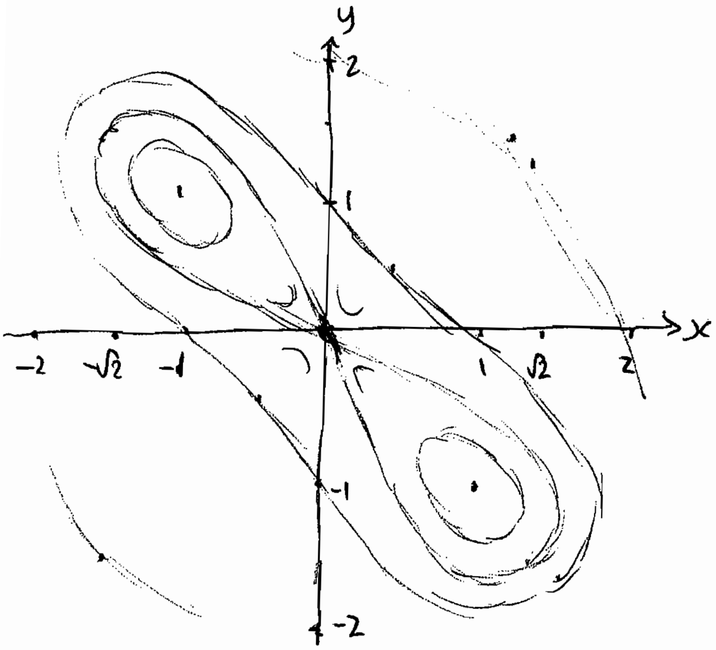
x	y	z
0	0	0
1	0	1
2	0	16
0	1	1
0	2	16
1	1	6
1	-1	-2
-1	1	-2

- For small x,y: $z \sim 4xy$ hyperbolae.
- Stationary points lie on the line $y = -x$
- set $x^4+y^4+4xy = 0$ for $z=0$

set $x=-y$:

$2x^4 - 4x^2 = 0 \quad = x^2(x^2-2) = 0 \Rightarrow x = 0$ or $x = \pm\sqrt{2}$

B (S17)



set $z=-1$, $x=-y$

$$2x^4 - 4x^2 = -1$$

$$2x^4 - 4x^2 + 1 = 0$$

$$x^2 = \frac{4 \pm \sqrt{16-8}}{4}$$

$\therefore x^2 = 1 \pm 1/\sqrt{2}$ four roots on $y=-x$

Now look at $x=y$: $z = 1 = 2x^4 + 4x^2$

$$x^2 = \frac{-4 \pm \sqrt{16+8}}{4} = -1 \pm \tfrac{1}{2}\sqrt{6}$$

$-$ve root leads to imaginary x

$+$ve root $x = \pm\sqrt{\sqrt{6}/2 - 1}$

$\sim \pm 0.47$

Also $x=-y$; $z=1 = 2x^4 - 4x^2$

$$x^2 = \frac{4 \pm \sqrt{16+8}}{4} = 1 \pm \tfrac{1}{2}\sqrt{6}$$

$-$ve root $\Rightarrow$ imag x

$+$ve root $x = \pm\sqrt{1 + \sqrt{6}/2}$

~ 1.49.

Finally $x=y$; $z = 16 = 2x^4 + 4x^2$

roots are $x = y = \sqrt{2}$

2008 Part IA Paper 4 Section C

Final Crib

11) (a) Factorial 4
return 4x? → Factorial 3
return 3x? → Factorial 2
return 2x? → Factorial (1)
return 1;

return 2x1

return 6

return 24.

(b) Factorial (0) causes an infinite loop and a seg. fault.

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12) (a) As lecture notes. Note algorithmic complexity.

(b) Quicksort scales as $k_q n \log_2 n$

Exchange sort scales as $k_e n^2$

Quicksort requires more operations per iteration $\therefore k_q > k_e$

Thus Exchange sort might be faster for 100 items.

For large numbers of items, $n \log_2 n << n^2$

$\therefore$ Quicksort will be faster for 100000 items.