ENGINEERING TRIPOS   PART IA

Tuesday 10th June 2008   9 to 12

Paper 4

MATHEMATICAL METHODS

Answer all questions.

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 **(short)** Find an equation for the line of intersection of the two planes:

$$x + y + 2z = 3$$
$$x + 2y + z = 2$$

in the form $\underline{l} = \underline{a} + \lambda \underline{b}$.   [10]

2 **(short)** Find the general solution of the differential equation:

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 4y = 5e^x.$$

[10]

3 **(short)** A discrete model for the equilibrium heat distribution along a straight piece of wire can be represented by

$$x_n = \frac{x_{n-1} + x_{n+1}}{2}$$

a) Show that $x_n = n$ is a solution of this equation.   [5]

b) Find the solution for x_n when $x_0 = 0$ and $x_{10} = 21$.   [5]

4 (long)

    (a)   Find the locus of points which satisfy

$$|z-5|=6 \text{ where } z=x+iy$$

                                           [10]

    (b)   Find:

        (i) $\displaystyle \lim_{x \to \frac{1}{4}} \frac{\cos^3 2\pi x}{1-16x^2}$ [5]

        (ii) $\displaystyle \lim_{x \to 0} \frac{\sin x - x}{\tan x - x}$ [5]

    (c)   Given that $z = 2-i$ is a root of the equation:

$$z^4 - 2z^3 - 18z^2 + 70z - 75 = 0,$$

find the other roots.   [10]

5 (long)

(a)   Find the eigenvalues and the eigenvectors of the matrix A, where:

$$A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

[10]

(b)   Show that a matrix formed by taking the normalised eigenvectors as its columns is orthonormal. What transformation does your normalised eigenvector matrix represent?   [10]

(c)   Calculate A^{10}.   [10]

SECTION B

6 (short)   Solve using Laplace transforms (and no other method) the differential equation:

$$\frac{d^2x}{dt^2} + 6\frac{dx}{dt} + 5x = 0$$

subject to the initial conditions $x(0) = 2$, $\dot{x}(0) = 3$.   [10]

7 (short)   For the linear system described by the differential equation

$$\frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 14y = f(t)$$

determine the step response and the impulse response of the system.   [10]

8 (short)   A box contains 24 memory chips, 3 of which are defective. Three chips are picked successively at random from the box. Find the probability that two of the three chips are defective if:

(a)   each chip is replaced in the box before the next one is picked;

(b)   each chip is **not** replaced in the box before the next one is picked.   [10]

9 **(long)**   A sinusoidal signal is passed through a half-wave rectifier such that the negative portions are removed. The resulting voltage is represented by the function

$$V(t) = \begin{cases} 0 & -T/2 \le t < 0; \\ V_0 \sin \omega t & 0 \le t < T/2. \end{cases}$$

where T is the period.   Obtain the complete Fourier series of this function from first principles in terms of sines and cosines, evaluating all coefficients by integration.

[30]

10 **(long)**   For the function

$$f(x,y) = x^4 + y^4 + 4xy$$

(a)   Find the location and the nature of all of the stationary points.   [20]

(b)   Plot all the stationary points on the x-y plane, and make a rough sketch of the contours of the function within the domain $-2 \le x \le 2, -2 \le y \le 2$.   [10]

SECTION C

11 **(short)** A function is written in C++ to evaluate the factorial of a specified integer:

```
int factorial(int n)
{
    if(n==1) return 1;
    else return n*factorial(n-1);
}
```

The function is called with the C++ statement

```
                answer = factorial(4);
```

(a) With the help of a diagram, explain the sequence of calls to the function and the corresponding values returned. [8]

b) Explain what happens if the function is called with the C++ statement:

```
                answer = factorial(0);
```
[2]

12 (short)

    (a)   Describe the operation of the sorting algorithms

        (i)   Quicksort;
        (ii)   exchange sort

when used to sort a simple list of integer values.   [6]

    (b)   Which of these two algorithms would be more suitable for sorting a list of

        (i)   100 integers;
        (ii)   100000 integers?

In each case provide a brief explanation of your answer.   [4]

END OF PAPER