YEAR	PART (of Tripos)	PAPER NUMBER & TITLE	NAME OF AUTHOR(S)
2009	IA	P4 – Mathematical Methods	Prof RS Cant

Section A

Q1. $$\frac{\tan x}{(1+4x^2)^{1/2}} \simeq \left[x + \frac{x^3}{3} + O(x^5)\right]\left[1 - \frac{1}{2}\cdot 4x^2 + O(x^4)\right]$$

$$= x + x^3/3 - 2x^3 + O(x^4)$$

$$= \underline{x - 5/3\, x^3 + O(x^4)}$$

Q2. $$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 10y = 6$$

Aux eqn $\lambda^2 + 6\lambda + 10 = 0$; $\lambda = \dfrac{-6 \pm \sqrt{36 - 4.10}}{2}$

$$= -3 \pm i$$

$\therefore$ CF is $e^{-3x}(A\cos x + B\sin x)$

PI : try $y = \text{const}$; PI $= 3/5$

gen. soln. $y = 3/5 + e^{-3x}(A\cos x + B\sin x)$

$$\frac{dy}{dx} = -3e^{-3x}(A\cos x + B\sin x) + e^{-3x}(-A\sin x + B\cos x)$$

$$y(0) = 3/5 + A = 1 \quad \Rightarrow \quad A = 2/5$$

$$\frac{dy}{dx}(0) = -3A + B = 1 \quad \Rightarrow \quad B = 11/5$$

$$\therefore \quad \underline{y = 3/5 + 2/5\, e^{-3x}\cos x + 11/5\, e^{-3x}\sin x}$$

Q3. $y_{n+2} = 3y_{n+1} - 2y_n$

$\lambda^2 - 3\lambda + 2 = 0 \Rightarrow \lambda = 1$ or $\lambda = 2$

$\therefore \quad y_n = A.1^n + B.2^n$

initial values : $y_0 = A + B = 1$

$y_1 = A + 2B = 2(\sqrt{3}+1)$

$\therefore \quad B = 1 + 2\sqrt{3} \; ; \; A = -2\sqrt{3}$

$$\therefore \quad \underline{y_n = -2\sqrt{3} + (1+2\sqrt{3})\, 2^n}$$

Q4. (a) (i) $\lim\limits_{x\to 2} \dfrac{x^3-8}{2x^2-x-6} = \lim\limits_{x\to 2} \dfrac{3x^2}{4x-1} = \underline{\dfrac{12}{7}}$

l'Hopital's rule

(ii) $\lim\limits_{x\to 0} \dfrac{\ln(1+x) - x}{\sqrt{1+x^2} - 1} = \lim\limits_{x\to 0} \dfrac{x - x^2/2 + \dots - x}{1 + \frac{x^2}{2} + \dots - 1} = \underline{-1}$

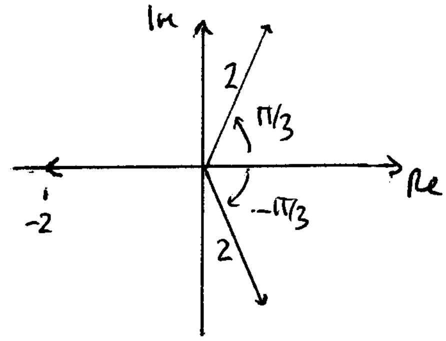
(iii) $\lim\limits_{x\to 0} \dfrac{\tan x - x}{x^2 \sin x} = \dfrac{x + x^3/3 + \dots - x}{x^2\left(x - \frac{x^3}{3} + \dots\right)} = \underline{\dfrac{1}{3}}$

(b) (i) $z^3 + 8 = 0 \; ; \; (re^{i\theta})^3 = -8$

$\therefore r = 2$ and $e^{3i\theta} = -1 = e^{in\pi}$ for n odd

$\therefore \quad \theta = \pi/3, \pi, 5\pi/3 \dots$

$\therefore \quad z = 2e^{in\pi}$, n odd



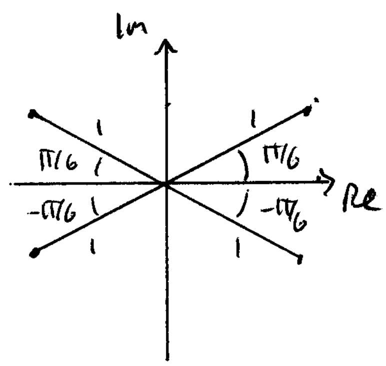
(b)(ii) $\quad z^4 - z^2 + 1 = 0$

set $u = z^2$; $\qquad u^2 - u + 1 = 0$

$$u = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad z^2 = e^{\pm i\pi/3} \quad \text{or} \quad e^{\pm i5\pi/3}$$

$$\therefore \quad z = e^{\pm i\pi/6} \quad \text{or} \quad e^{\pm i5\pi/6}$$



Q5 (a) $\quad \underline{r} = \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} 6 \\ 3 \\ 5 \end{bmatrix}$ and the point $(3,1,1)$ lie in the plane.

$\therefore \; -(1,-5,4) + (3,1,1) = (2,6,-3)$
and $(6,3,5)$

lie in the plane

$\therefore$ plane is $\underline{r} = \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 6 \\ 3 \\ 5 \end{pmatrix} + \mu \begin{pmatrix} 2 \\ 6 \\ -3 \end{pmatrix}$

normal vector is $\perp$ to $(6,3,5)$ and $(2,6,-3)$

$\therefore$ $\underline{n}$ is $\parallel$ to $\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 6 & 3 & 5 \\ 2 & 6 & -3 \end{vmatrix} = \underline{i}(-39) + \underline{j}(28) + \underline{k}(30)$

plane is $\underline{r}.\underline{n}$ and $(3,1,1)$ lies in the plane

$\therefore$ $\underline{-39x + 28y + 30k = -117 + 28 + 30 = -59}$

(b) $\underline{r}_1 = \underline{a}_1 + \lambda \underline{b}_1$; $\underline{r}_2 = \underline{a}_2 + \lambda \underline{b}_2$

normal is $\dfrac{\underline{b}_1 \times \underline{b}_2}{|\underline{b}_1 \times \underline{b}_2|} = \underline{n}$; distance is $(\underline{a}_2 - \underline{a}_1).\underline{n} = \underline{s}$

$\underline{b}_1 \times \underline{b}_2 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 6 & 3 & 5 \\ 6 & -3 & 5 \end{vmatrix} = \underline{i}(30) + \underline{j}.0 + \underline{k}(-36)$

$= 30\underline{i} - 36\underline{k}$

$\therefore \underline{n} = \dfrac{5\underline{i} - 6\underline{k}}{\sqrt{61}}$

$\therefore \underline{s} = \left\{ \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ -5 \\ 4 \end{pmatrix} \right\} . \dfrac{5\underline{i} - 6\underline{k}}{\sqrt{61}} = 10\underline{j} . \dfrac{5\underline{i} - 6\underline{k}}{\sqrt{61}} = \underline{0}$

(c)

$$R = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{vmatrix} 0-\lambda & 0 & -1 \\ 1 & 0-\lambda & 0 \\ 0 & 1 & 0-\lambda \end{vmatrix} \Rightarrow -\lambda(\lambda^2-0) - 1 = 0$$

$$\lambda^3 = -1$$

$$\therefore \underline{\lambda = -1}$$

$$\therefore \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

for eigenvector

$x - z = 0$; $x + y = 0$; $y + z = 0$

$\therefore$ eigenvector is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Maths Sec B Final Cwb 2009

Q6
$$P(x,y) = \frac{a}{y} + \frac{1}{3x+2}$$

$$Q(x,y) = \frac{y}{y^2+9} - 2xy^b$$

Perfect differential $df = P\,dx + Q\,dy$

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$$\frac{\partial P}{\partial y} = -ay^{-2} \; ; \; \frac{\partial Q}{\partial x} = -2y^b$$

$$\therefore \quad \underline{a = 2, \; b = -2}$$

$$P = \frac{2}{y} + \frac{1}{3x+2} \; ; \; Q = \frac{y}{y^2+9} - \frac{2x}{y^2}$$

$$\int P\,dx = \frac{2x}{y} + \frac{1}{3}\ln(3x+2) + F(y)$$

$$\int Q\,dy = \frac{1}{2}\ln(y^2+9) + \frac{2x}{y} + G(x)$$

$$\therefore \; f(x,y) = \frac{2x}{y} + \frac{1}{3}\ln(3x+2) + \frac{1}{2}\ln(y^2+9) + \text{const.}$$

Q7

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = \sin 3t \, ;$$

$$x(0) = \dot{x}(0) = 0$$

$$L(x) = Y(s)$$

$$L(\dot{x}) = sY - x(0) = sY$$

$$L(\ddot{x}) = s^2Y - s\,x(0) - \dot{x}(0) = s^2 Y$$

$$L(\sin 3t) = \frac{3}{s^2+9}$$ Data Book p21

o.d.e.: $$s^2Y + 4sY + 3Y = \frac{3}{s^2+9}$$

$$(s+3)(s+1)\,Y = \frac{3}{s^2+9}$$

$$Y = \frac{3}{(s+3)(s+1)(s^2+9)} = \frac{A+Bs}{s^2+9} + \frac{C}{s+3} + \frac{D}{s+1}$$

D: set $s=-1$; $D = \frac{3}{2 \cdot 10} = \frac{3}{20}$

C: set $s=-3$; $C = \frac{3}{(-2) \cdot 18} = -\frac{1}{12}$

A, B: set $s = 3i$; $A + 3iB = \frac{3}{(3+3i)(1+3i)} = \frac{3}{12i-6}$

$$A + 3iB = \frac{3(-12i-6)}{(12i-6)(-12i-6)} = -\frac{1}{5}i - \frac{1}{10}$$

$$A = -1/10 \; ; \quad B = -1/15 .$$

$$\therefore Y = -\frac{1}{10}\left(\frac{1}{s^2+9}\right) - \frac{1}{15}\left(\frac{s}{s^2+9}\right) - \frac{1}{12}\left(\frac{1}{s+3}\right) + \frac{3}{20}\left(\frac{1}{s+1}\right)$$

$$\therefore x(t) = -\frac{1}{30}\sin 3t - \frac{1}{15}\cos 3t - \frac{1}{12}e^{-3t} + \frac{3}{20}e^{-t}$$

Q8 (a) Gender balance irrelevant: 12 men + 4 women = 16

Require 7 from 16 = ${}^{16}C_7$ = 11440 ways to pick.

(b) Require 5 men + 2 women
or 4 men + 3 women
or 3 men + 4 women,

ie $^{12}C_5 \times {}^4C_2 + {}^{12}C_4 \times {}^4C_3 + {}^{12}C_3 \times {}^4C_4$

$792 \times 6 + 495 \times 4 + 220 \times 1 = \underline{6952 \text{ ways}}$

Q9

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 4y = f(t)$$

Step response $\quad f(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t \geq 0 \end{cases} \quad y = \frac{dy}{dt} = 0 \text{ at } t=0$

CF: try $y = e^{\lambda t}$

$$\lambda^2 + 5\lambda + 4 = 0 = (\lambda+4)(\lambda+1) \Rightarrow \lambda = -4, -1$$

CF is $\quad Ae^{-4t} + Be^{-t}$

PI try $y = \alpha$; $\quad 4\alpha = 1 \Rightarrow \alpha = 1/4$

$\therefore$ General solution is $\quad y = \frac{1}{4} + Ae^{-4t} + Be^{-t}$

At $t=0$, $y = \dot{y} = 0$: $\quad \frac{1}{4} + A + B = 0$

$$-4A - B = 0$$

$\therefore \quad A = \frac{1}{12}, \quad B = -1/3$

Step response is $\quad \underline{y = \frac{1}{4} + \frac{1}{12}e^{-4t} - \frac{1}{3}e^{-t} \quad \text{for } t > 0}$

Impulse response is time derivative of step response:

$$\underline{g(t) = -\frac{1}{3}e^{-4t} + \frac{1}{3}e^{-t} \quad \text{for } t > 0}$$

Now impose an input $x(t) = \begin{cases} 0 & \text{for } t<0 \\ e^{-\alpha t} & \text{for } t \geq 0 \end{cases}$

output is $y(t) = \int_{-\infty}^{t} g(t-\tau)\, f(\tau)\, d\tau$

$$= \int_{-\infty}^{t} \left[-\frac{1}{3} e^{-4(t-\tau)} + \frac{1}{3} e^{-(t-\tau)} \right] e^{-\alpha\tau}\, d\tau$$

$$= -\frac{e^{-4t}}{3} \int_{-\infty}^{t} e^{(4-\alpha)\tau} d\tau + \frac{e^{-t}}{3} \int_{-\infty}^{t} e^{(1-\alpha)\tau} d\tau$$

$$= -\frac{e^{-4t}}{3(4-\alpha)} \left[e^{(4-\alpha)\tau} \right]_0^t + \frac{e^{-t}}{3(1-\alpha)} \left[e^{(1-\alpha)\tau} \right]_0^t$$

$$= -\frac{e^{-4t}}{3(4-\alpha)} \left[e^{(4-\alpha)t} - 1 \right] + \frac{e^{-t}}{3(1-\alpha)} \left[e^{(1-\alpha)t} - 1 \right]$$

ie $$\underline{y(t) = -\frac{1}{3(4-\alpha)} \left(e^{-\alpha t} - e^{-4t} \right) + \frac{1}{3(1-\alpha)} \left(e^{-\alpha t} - e^{-t} \right)}$$

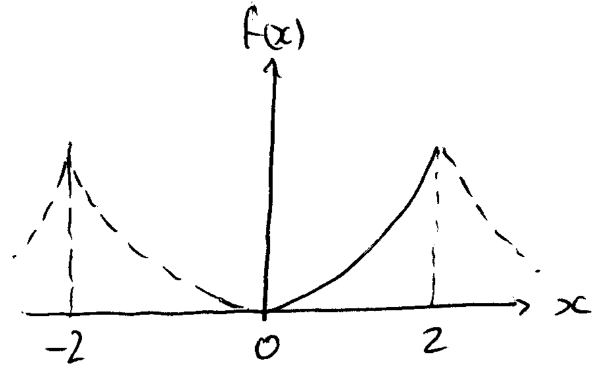
As $\alpha \to 0$ $$y(t) \to -\frac{1}{12}\left(1 - e^{-4t}\right) + \frac{1}{3}\left(1 - e^{-t}\right)$$

$$= \frac{1}{4} + \frac{1}{12} e^{-4t} - \frac{1}{3} e^{-t}$$

which is the step response.

B6

Q10. a) Extend the range



Even function of period 4

cosine series

$$f(x) = f(-x)$$

$$f(x+4n) = f(x) \text{ for integer } n$$

Coefficients from Data Book p17

$$d = \frac{1}{T}\int_0^T f(t)\,dt$$

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\frac{2\pi n t}{T}\,dt; \quad b_n = \frac{2}{T}\int_0^T f(t)\sin\frac{2\pi n t}{T}\,dt$$

Here have $d = \frac{1}{4}\int_{-2}^{2} x^2\,dx = \frac{1}{4}\left[\frac{x^3}{3}\right]_{-2}^{2} = 4/3$

$$a_n = \frac{2}{4}\int_{-2}^{2} x^2 \cos\frac{2\pi n x}{4}\,dx = \int_0^2 x^2\cos\frac{\cancel{2}\pi n x}{\cancel{4}2}\,dx$$

even function

Integrate by parts:

$$a_n = \left[x^2\,\frac{2}{\pi n}\sin\frac{\pi n x}{2}\right]_0^2 - \frac{2}{\pi n}\int_0^2 2x\sin\frac{\pi n x}{2}\,dx$$

$$= 0 + \frac{4}{\pi n}\frac{2}{\pi n}\left[x\cos\frac{\pi n x}{2}\right]_0^2 - \frac{4}{\pi n}\frac{2}{\pi n}\int_0^2 \cos\frac{\pi n x}{2}\,dx$$

$$= \frac{8}{\pi^2 n^2}\cdot 2\cos \pi n \quad - \frac{8}{\pi^2 n^2}\left[\frac{2}{\pi n}\sin\frac{\pi n x}{2}\right]_0^2$$

$$= \frac{16}{\pi^2 n^2}(-1)^n$$

Note all $b_n \equiv 0$ (even function)

$\therefore$ Have Fourier series

$$x^2 = 4/3 + 16\sum_{n=1}^{\infty}\frac{(-1)^n}{\pi^2 n^2}\cos\frac{\pi n x}{2} \; ; \; 0 < x \leq 2$$

b) Set $x=2$: $\cos\frac{\pi n 2}{2} = \cos \pi n = (-1)^n$

$$4 = 4/3 + 16\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{\pi^2 n^2} = 4/3 + \frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n^2} = \left(4 - 4/3\right)\frac{\pi^2}{16} = \pi^2/6$$

Maths Sec C Final Crib 2009

Q11 $-1.1825 \times 10^2 = -118.25 = -1.84765625 \times 2^6$

mantissa is 1.84765625; exponent is $133 = 128 + 4 + 1$

mantissa is $2^0 + 2^{-1} + 2^{-2} + 2^{-4} + 2^{-5} + 2^{-8}$

IEEE format is

1	10000101	11011001000000000000000

If the last digit is 1 instead of 0, error is $2^{-23} \times 2^6$

Fractional error is $\dfrac{2^{-23} \times 2^6}{118.25} = 6.45 \times 10^{-8}$ or $\underline{6.45 \times 10^{-6}\ \%}$

Q12 pinjoint contains 3 floats at 4 bytes each

= 12 bytes

member contains 2 pinjoints

= 24 bytes

framework contains 80 members

= $\underline{1920 \text{ bytes}}$ for building

y_local = building.members[16].joints[1].y_coord;