1.

(a) Force on the right hand side:

$$F_R = \int_{z=0}^{z=h} \rho g z W dz = \rho g W \frac{h^2}{2}$$

where W is the width of the barrage (= 10 m) and z is the distance below water surface.

The moment about the water surface is:

$$M_R = \int_{z=0}^{z=h} \rho g z^2 W dz = \rho g W \frac{h^3}{3}$$

So the point of action is:

$$\frac{M_R}{F_R} = \frac{\rho g W \frac{h^3}{3}}{\rho g W \frac{h^2}{2}} = \frac{2}{3} h$$

(b) Force on the left hand side:

$$F_L = \rho_f g W \frac{h_f^{\ 2}}{2} = 1000 \times 9.81 \times 10 \times \frac{6 \times 6}{2} = 1765800 = 1765.8 \text{ kN}$$

Point of action is located at $\frac{2}{3} \times 6$ m below the freshwater surface, that is:

5 m below the hinge O.

(c) Force on the right hand side:

$$F_R = \rho_s g W \frac{h_s^{\ 2}}{2} = 1030 \times 9.81 \times 10 \times \frac{h_s^{\ 2}}{2} = 50521.5 \cdot h_s^{\ 2}\ N = 50.5215 \cdot h_s^{\ 2} \text{ kN}$$

Point of action is located at $\frac{2}{3} h_s$ below the seawater surface, that is:

$$7 - h_s + \frac{2}{3} h_s = 7 - \frac{h_s}{3} \text{ m below the hinge O.}$$

Taking moment about O:

$$F_L \times 5 = F_R \times \left(7 - \frac{h_s}{3}\right)$$

$$1765.8 \times 5 = 50.5215 \times h_s^{\ 2} \times \left(7 - \frac{h_s}{3}\right)$$

$$16.8405 h_s^{\ 3} - 353.6505 h_s^{\ 2} + 8829 = 0$$

5.89 m is a solution to the above equation.

D. Liang, 2010

2.

(a) According to continuity equation, the velocity at Section A is 1.25 m/s.
According to Bernoulli equation:

$$P_A + \frac{1}{2}\rho V_A^{\,2} + \rho g y_A = P_B + \frac{1}{2}\rho V_B^{\,2} + \rho g y_B$$

$$P_B = P_a,\ V_B = 5 \text{ m/s},\ y_B = 2 \text{ m},\ V_A = 1.25 \text{ m/s},\ y_A = 6 \text{ m}$$

$$P_A - P_a = \frac{1}{2}\rho\left(V_B^{\,2} - V_A^{\,2}\right) + \rho g\left(y_B - y_A\right)$$

$$= \frac{1}{2}\times 1000 \times \left(5^2 - 1.25^2\right) + 1000 \times 9.81 \times (2-6) = 11718.75 - 39240 = -27521.25$$

Gauge pressure is -27.5 kPa.

(b)

$$V_c \approx 0,\ y_c = 5 \text{ m}$$

$$P_c = P_a + \frac{1}{2}\rho\left(V_B^{\,2} - V_c^{\,2}\right) + \rho g\left(y_B - y_c\right)$$

$$= 101000 + \frac{1}{2}\times 1000 \times \left(5^2 - 0^2\right) + 1000 \times 9.81 \times (2-5) = 101000 + 12500 - 29430 = 84070$$

Absolute pressure is 84.07 kPa.

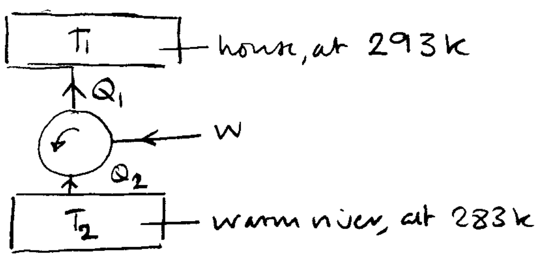
D. Liang, 2010

3. A heat engine extracts heat from a river at 283 K and rejects to another river at 277 K.

(a) Carnot efficiency (reversible) : $\eta = 1 - \frac{T_C}{T_H} = 1 - \frac{277}{283} = 0.0212$

max possible efficiency is 2.120% [3]

(b)



(max COP_p will be from warm river)

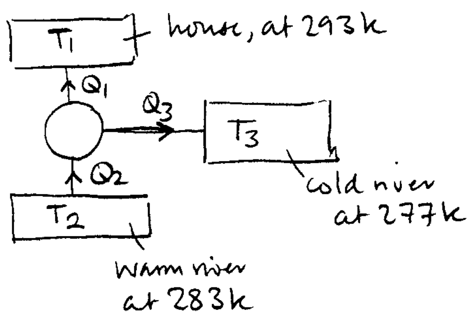
if reversible $\frac{Q_1}{T_1} = \frac{Q_2}{T_2}$

But also, we know that $Q_2 + W = Q_1$

$\Rightarrow COP_p \equiv \frac{Q_1}{W} = \frac{Q_1}{Q_1 - Q_2} = \frac{T_1}{T_1 - T_2}$

$COP_p\,(\text{max}) = \frac{293}{10} = 29.3$ [3]

(c)



considering overall system.

If reversible, $\frac{Q_1}{T_1} + \frac{Q_3}{T_3} = \frac{Q_2}{T_2}$ and also, $Q_2 = Q_1 + Q_3$

solve for $\frac{Q_2}{Q_1}$: $\frac{Q_2}{T_2} = \frac{Q_1}{T_1} + \frac{Q_2 - Q_1}{T_3}$

$\Rightarrow \frac{Q_2}{T_2} - \frac{Q_2}{T_3} = \frac{Q_1}{T_1} - \frac{Q_1}{T_3}$

$\Rightarrow \frac{Q_2}{Q_1} = \frac{\frac{1}{T_1} - \frac{1}{T_3}}{\frac{1}{T_2} - \frac{1}{T_3}} = \frac{\frac{1}{293} - \frac{1}{277}}{\frac{1}{283} - \frac{1}{277}} = 2.5756$

$\Rightarrow Q_2$ min $= 5\text{ kW} \times 2.5756 = 12.88\text{ kW}$ [4]

M. Juniper
2010

4. A cylinder contains $0.1\,m^3$ of CO_2, which is maintained at 298 K and at 2×10^5 Pa by a weighted piston. (Assume CO_2 is an ideal gas)

(a) $m = \rho V_{ol} = \dfrac{p V_{ol}}{RT}$; $R = \dfrac{8314}{RMM}$; CO_2 has RMM $\simeq 12+16+16 = 44$

$$\Rightarrow \quad m = \frac{2\times10^5 \times 0.1}{\frac{8314}{44} \times 298} = 0.3552 \text{ kg}$$ [3]

(b) $p = \rho R T \quad \Rightarrow \quad \dfrac{p}{R} = \rho T = \text{const}$

i) $$\Rightarrow \quad \frac{T_2}{T_1} = \frac{\rho_1}{\rho_2} = \frac{V_{ol2}}{V_{ol1}} = 2$$

$$\Rightarrow \quad T_2 = 298 \times 2 = 596 \text{ K}$$ [2]

ii) work done on piston by the CO_2 $= \displaystyle\int_{V_1}^{V_2} p \, dV_{ol} = p(V_2 - V_1)$ because p is constant

$$= 2\times10^5 (0.1) \quad \left(\frac{N}{m^2} \times m^3\right)$$

$$= 2\times10^4 \text{ Joules} \quad (\text{Nm units})$$ [2]

iii) $Q - W = \Delta E$; assume $\Delta K.E.$ and $\Delta P.E. \ll \Delta u$

$\Rightarrow \quad Q - W = \Delta u$

$\Delta u = m C_v \Delta T$ if we assume CO_2 is a perfect gas

$$\Rightarrow \quad Q = m C_v \Delta T + W = 0.3552 \times 630 \times 298 + 2\times10^4$$
$$= 86.7 \times 10^3 \text{ J}$$ [3]

M. Juniper
2010

5. (a) At sections 1 and 2, the streamlines are straight and parallel to the horizontal bed, so the fluid particles do not experience any vertical acceleration. The vertical force balance is between pressure differences and self weight, i.e. hydrostatic.
The fluid particles experience strong vertical movement cross the jump and close to the block, so the pressure distribution close to the block is not hydrostatic.

(b) Use momentum equation:

$$\sum F_x = \dot{m}(V_2 - V_1)$$

$$\frac{1}{2}\rho g W h_1^2 - \frac{1}{2}\rho g W h_2^2 - F = \dot{m}(V_2 - V_1)$$

$$F = \frac{1}{2}\rho g W h_1^2 - \frac{1}{2}\rho g W h_2^2 - \rho\dot{m}(V_2 - V_1)$$

$$F = \frac{1}{2}\rho g W\left(h_1^2 - h_2^2\right) - \rho\dot{m}\left(\frac{\dot{m}}{\rho W h_2} - \frac{\dot{m}}{\rho W h_1}\right)$$

$$F = \frac{1}{2}\rho g W\left(h_1^2 - h_2^2\right) - \frac{\dot{m}^2}{\rho}\left(\frac{1}{W h_2} - \frac{1}{W h_1}\right)$$

$$F = \frac{1}{2}\rho g W\left(h_1^2 - h_2^2\right) + \frac{\dot{m}^2}{\rho}\left(\frac{1}{W h_1} - \frac{1}{W h_2}\right)$$

This is the force acting on the water by the block. According to the Newton's 3rd law, the force on the block has the same amplitude, but points downstream.

(c)

$$F = \frac{1}{2}\times 1000\times 9.81\times 15\times\left(1.15^2 - 7.35^2\right) + 1000\times 310^2\times\left(\frac{1}{15\times 1.15} - \frac{1}{15\times 7.35}\right)$$

$$= -3{,}877{,}402.5 + 4{,}699{,}359 = 821{,}959.5 \text{ N} \approx 822 \text{ kN}$$

The force on the block is toward downstream.

(d)

At section 1: $h_1 = 1.15$ m, $V_1 = \dfrac{310}{15\times 1.15} = 17.97$ m/s.

At section 2: $h_1 = 7.35$ m, $V_1 = \dfrac{310}{15\times 7.35} = 2.81$ m/s.

$$C_1 = \rho g h_1 + \frac{1}{2}\rho V_1^2 = 1000\times 9.81\times 1.15 + \frac{1}{2}\times 1000\times 17.97^2 = 11281.5 + 161460.45$$

$$= 172741.95 \text{ Pa} = 172.74 \text{ kPa}$$

$$C_2 = \rho g h_2 + \frac{1}{2}\rho V_2^2 = 1000\times 9.81\times 7.35 + \frac{1}{2}\times 1000\times 2.81^2 = 72103.5 + 3948.05$$

$$= 76051.55 \text{ Pa} = 76.05 \text{ kPa}$$

They should be different, as there is mechanical energy loss in the hydraulic jump.

(e) $$\Delta C = \rho g h_1 + \frac{1}{2}\rho V_1^2 - \left(\rho g h_2 + \frac{1}{2}\rho V_2^2\right) = 172.74 - 76.05 = 96.69 \text{ kPa}$$

$$\text{Power} = \frac{\dot{m}}{\rho}\cdot\Delta C = 310\times 96.69\times 10^3 = 29{,}973{,}900 \text{ W} \approx 29.97 \text{ MW}$$

D. Liang

6 (a) (i) First law of thermodynamics: $Q - W = \Delta E$

$\Delta E = \Delta KE + \Delta PE + \Delta U$; Assume $\Delta KE = \Delta PE = 0$

$$\Rightarrow \quad Q - W = \Delta U$$

Consider a simple compressible system undergoing an infinitesimal change of state by heat and work interactions with the surroundings. The first law gives:

$$\delta Q - \delta W = dU$$

From the definition of entropy, $\delta Q = TdS$ and for fully-resisted compression, $\delta W = pdV$ so, for a reversible process.

$$TdS - pdV = dU$$

Examiner's note: justifications were required for full marks. It was not sufficient to write the equations without stating the assumptions

dividing by the mass of the system and re-arranging.

$$Tds = du + pdv$$ [4]

(ii) $h = u + pv$ (Definition of enthalpy)

$$\Rightarrow dh = du + pdv + vdp$$

$$\Rightarrow du + pdv = dh - vdp \quad \Rightarrow \quad Tds = dh - vdp.$$ [2]

iii) For a perfect gas changing reversibly from state 1 to state 2, the above equations apply and also C_p = const and C_v = const

$$\int_{s_1}^{s_2} ds = \int_{T_1}^{T_2} \frac{dh}{T} - \int_{p_1}^{p_2} \frac{vdp}{T}$$

but, by definition, $dh = C_p dT$, where C_p = const for a perfect gas

and, since $pv = RT$ for a perfect gas, $\frac{v}{T} = \frac{R}{p}$ and therefore:

$$\int_{s_1}^{s_2} ds = C_p \int_{T_1}^{T_2} \frac{dT}{T} - R \int_{p_1}^{p_2} \frac{dp}{p}$$

$$\Rightarrow s_2 - s_1 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$ [6]

iv) But, because s, T and p are thermodynamic properties, this relation also applies to irreversible processes. [2]

(b) The whole system is insulated and the process is reversible so the process is isentropic. From the previous expression,

$$0 = C_p \ln\left(\frac{T_2}{T_1}\right) - R \ln\left(\frac{p_2}{p_1}\right)$$

$$\Rightarrow \quad \ln\left(\frac{T_2}{T_1}\right) = \frac{R}{C_p} \ln\left(\frac{p_2}{p_1}\right)$$

$$\Rightarrow \quad \frac{T_2}{T_1} = \left(\frac{p_2}{p_1}\right)^{R/C_p} \quad \leftarrow \text{but } \frac{R}{C_p} = \frac{C_p - C_v}{C_p} = 1 - \frac{1}{\gamma} = \frac{\gamma - 1}{\gamma}$$

$$\Rightarrow \quad \frac{T}{p^{(\gamma-1/\gamma)}} = \text{constant}$$

[4]

(c) $p_1 = 10^5$ Pa, $T_1 = 290$ K $\qquad T_2 = T_1 \left(15\right)^{\frac{0.4}{1.4}} = 628.67$ K ← because it's an isentropic process

$p_2 = 15 \times 10^5$ Pa, $T_2 = \ldots$

$$\text{mass of air} = V\rho = \frac{pV}{RT} = \frac{15\times10^5 \times 15}{287 \times 628.67} = 124.7 \text{ kg} = m_2$$

$$m_1 = \frac{10^5 \times 15}{287 \times 290} = 18.022 \text{ kg}$$

[4]

(d) place a control volume around the compressor and vessel. The non-steady flow energy equation is:

$$\dot{Q} - \dot{W}_x = \frac{dE_{cv}}{dt} - \dot{m}_{in}\left(h_{in} + \tfrac{1}{2}V_{in}^2 + gz_{in}\right)$$

zero, because adiabatic

← Assume v. small relative to h_{in}

$$\Rightarrow \quad \text{total work} = E_{cv_{end}} - E_{cv_{start}} - (m_2 - m_1)h_{in}$$

$$= m_2 C_v T_2 - m_1 C_v T_1 - (m_2 - m_1) C_p T_1$$

$$= 124.7 \times 718 \times 628.67 - 18.022 \times 718 \times 290 \ldots$$

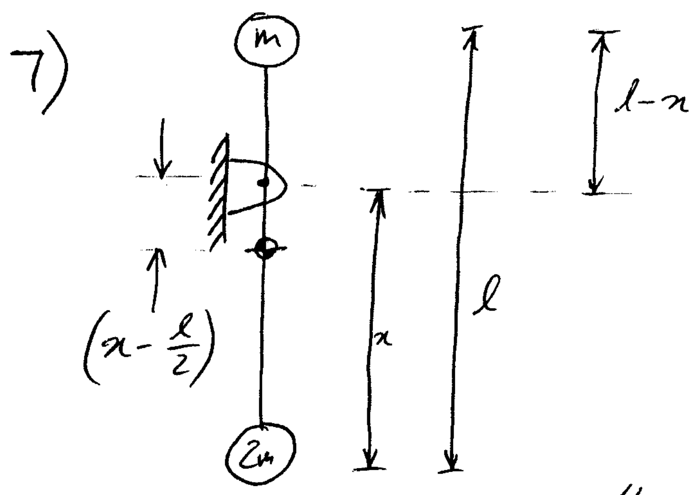
$$\ldots - (124.7 - 18.022) \times 1005 \times 290$$

$$= 21.44 \times 10^6 \text{ J}$$

[8]

M. Juniper 2010

7)



[CG symbol] INDICATES CG OF ROD

a) $$I_c = \underbrace{\frac{ml^2}{12}}_{\text{ROD}} + \underbrace{m\left(x-\frac{l}{2}\right)^2}_{\text{// AXIS}} + \underbrace{2mx^2}_{2m} + \underbrace{m(l-x)^2}_{m}$$

$$= m\left(4x^2 - 3lx + 4\frac{l^2}{3}\right)$$

b) $$\frac{dI_c}{dx} = m(8x - 3l) = 0 \Rightarrow x = 3l/8$$

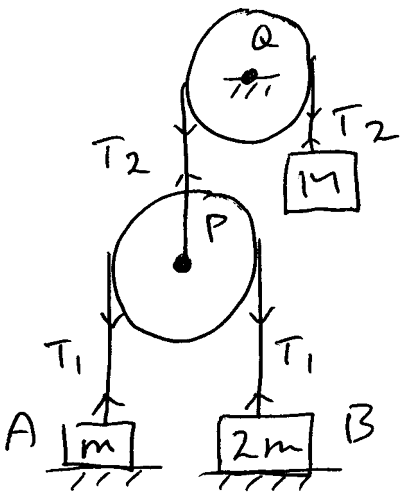
$$\Rightarrow \text{MINIMUM } I_c = m\left(4\left(3l/8\right)^2 - 3l\left(3l/8\right) + 4l^2/3\right)$$

$$= \frac{37}{48} ml^2$$

c) SAME AS (b). OR TAKE MOMENTS TO FIND $x = 3l/8$

A. Johnson
N. Crilly 2010

8 a)



For equilibrium

$T_1 \leq mg$ or A will lift

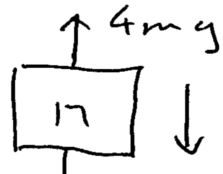
$\therefore T_2 \leq 2mg$

$\therefore \underline{M \leq 2m}$

b) A and B will both lift when $T_1 \geq 2mg$.

If $T_1 = 2mg$, A will rise with acceleration of g, and B does not move. So pulley P rises, and M falls, with acceleration $g/2$.

Also, $T_2 = 4mg$, so for mass M:



$$Mg - 4mg = M(g/2)$$

$$\therefore \underline{M = 8m}$$

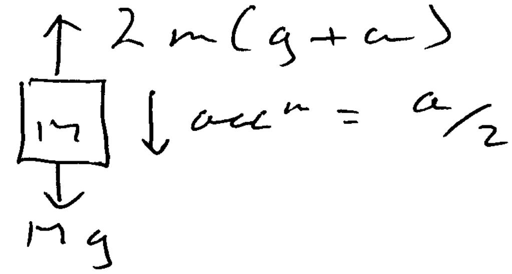
$\therefore$ Both masses lift if $M > 8m$

8 c) If $M < 8m$, B does not move

Let upwards acceleration of A be a

Then $T_1 = m(g+a)$

So for M:



$$Mg - 2m(g-a) = M\,a/2$$

$\therefore\ 2mg - 2ma = 2ma$ since $M = 4m$

$\therefore\ \underline{a = g/2}$

A. Johnson
N. Crilly 2010

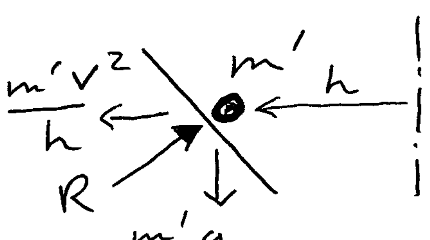
9 a) M of M conserved $\rightarrow V_2 = 2V_1$

Energy conserved $\rightarrow \frac{1}{2}m(V_2^2 - V_1^2) = mgh$

So $\frac{3}{2}V_1^2 = gh$

$\rightarrow \underline{V_1 = \sqrt{\frac{2gh}{3}}}$ $\quad \underline{V_2 = \sqrt{\frac{8gh}{3}}}$

b) For circular motion at height h



$V^2/h = g$ so

$V = \sqrt{gh}$

$mV_2 = m'V$, so $\frac{m'}{m} = \sqrt{8/3}$

(Momentum conserved)

So mass of 2nd particle $= \underline{m(\sqrt{8/3} - 1) = 0.633m}$

A. Johnson, N. Crilly 2010

10a) $J\ddot{\theta}_1 = k(\theta_2 - \theta_1) \rightarrow J\ddot{\theta}_1 + k\theta_1 - k\theta_2 = 0$

$2J\ddot{\theta}_2 = k(\theta_1 - \theta_2) + k(\theta_3 - \theta_2)$

$\rightarrow 2J\ddot{\theta}_2 + 2k\theta_2 - k\theta_1 - k\theta_3 = 0$

$J\ddot{\theta}_3 = k(\theta_2 - \theta_3) \rightarrow J\ddot{\theta}_3 + k\theta_3 - k\theta_2 = 0$

Hence

$$\underline{\begin{bmatrix} J & 0 & 0 \\ 0 & 2J & 0 \\ 0 & 0 & J \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{bmatrix} + \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{bmatrix} = 0}$$

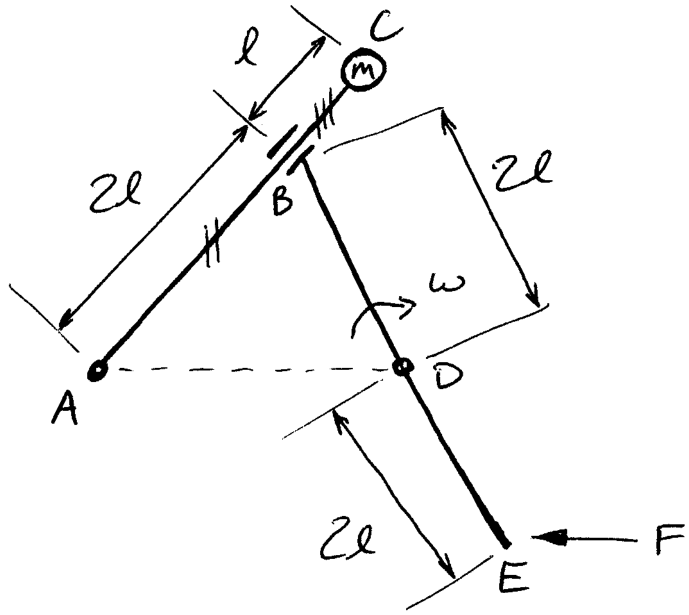
b) Mode 1: Solid body rotation, $\underline{\omega = 0}$

Mode 2: Node at J_2, outer rotors moving in opposite directions, $\underline{\omega = \sqrt{\frac{k}{J}}}$

Mode 3: Nodes at midpoints of shafts, $\underline{\omega = \sqrt{\frac{2k}{J}}}$

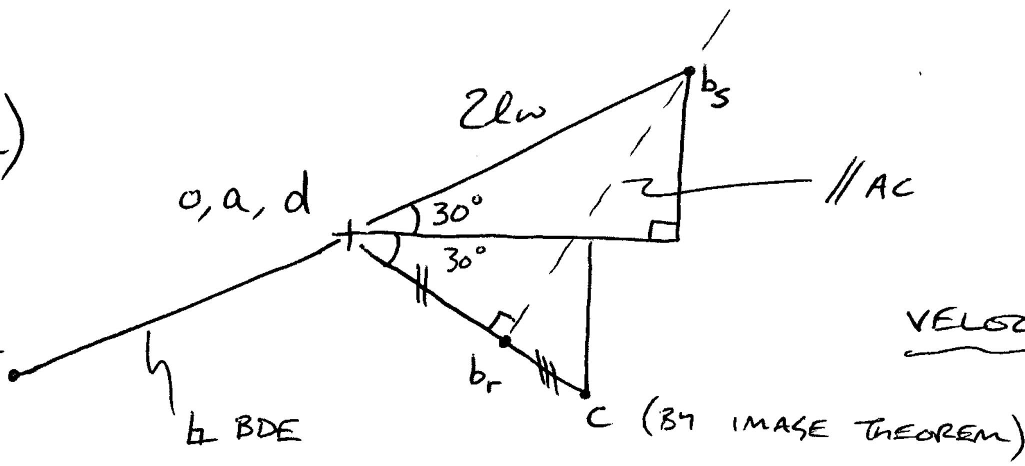
A. Johnson
N. Crilly 2010

11)



SPACE DIAGRAM

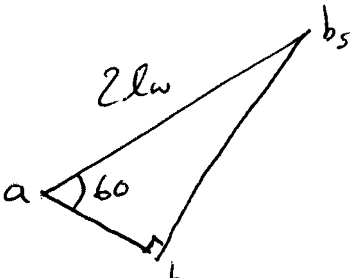
a)



VELOCITY DIAGRAM

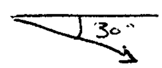
i)

ii)



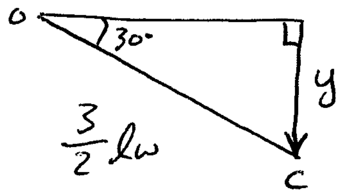
$$ab_r = 2l\omega \cos 60 = l\omega$$

$$\Rightarrow ac = oc = \underline{\underline{3l\omega/2}}$$

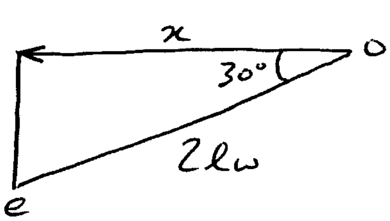


$$\omega_{AC} = \frac{ac}{AC} = \frac{3l\omega}{2\,3l} = \underline{\underline{\frac{\omega}{2}}}$$

b)



$$y = \frac{3}{2}l\omega \sin 30 = \frac{3}{4}l\omega$$

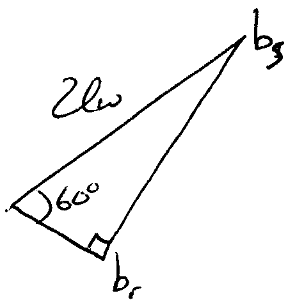


$$x = 2l\omega \cos 30 = \frac{\sqrt{3}}{2}2l\omega = \sqrt{3}\,l\omega$$

BY VIRTUAL WORK :

i) $$\frac{-3mg}{4} l\omega = F\sqrt{3}\, l\omega \quad \Rightarrow F = -\frac{\sqrt{3}}{4} mg$$

ii)



$$b_s b_r = 2l\omega \sin 60$$
$$= \sqrt{3}\, l\omega$$

$\Rightarrow$ LINEAR SLIDING FRICTION $= R\sqrt{3}\, l\omega$

$$\omega_A = \omega/2$$
$$\omega_D = \omega$$
$$\omega_B = \omega_D - \omega_A = \omega/2$$

$\Rightarrow$ ROTARY FRICTION $= T(\omega + \omega/2 + \omega/2) = 2T\omega$

BY VIRTUAL WORK :

$$F\sqrt{3}\, l\omega = R\sqrt{3}\, l\omega + 2T\omega - \frac{3}{4} l\omega\, mg$$

$$\Rightarrow F = \frac{2T}{l\sqrt{3}} - \frac{mg\sqrt{3}}{4} + R$$

A. Johnson
N. Crilly 2010

12 a) If charge is flowing round the circuit at rate $\dot{q}$, then

$$R\dot{q} + L\ddot{q} + \tfrac{1}{C}q = e \qquad (1)$$

Also $L\ddot{q} = V \qquad (2)$

So $\ddot{q} = V/L$, $\dddot{q} = \frac{\dot{V}}{L}$, $\ddddot{q} = \frac{\ddot{V}}{L}$

Differentiate (1) twice:

$$L\ddddot{q} + R\dddot{q} + \tfrac{1}{C}\ddot{q} = \ddot{e}$$

$$\ddot{V} + \frac{R\dot{V}}{L} + \frac{V}{LC} = \ddot{e}$$

$$\underline{LC\ddot{V} + RC\dot{V} + V = LC\ddot{e}}$$

From D.B. p10,

$$\frac{1}{\omega_n^2} = LC \longrightarrow \underline{\omega_n = \sqrt{\frac{1}{LC}}}$$

$$\frac{2\zeta}{\omega_n} = RC \longrightarrow \zeta = \frac{RC}{2\sqrt{LC}} = \underline{\frac{R}{2}\sqrt{\frac{C}{L}}}$$

b) From graph on D.B. p 11

$Y/X = 2$ when $\frac{\omega}{\omega_n} = \underline{0.90 \text{ and } 1.28}$

Corresponding ϕ is $\underline{-62° \text{ and } -142°}$

A.J., N.C.

12 b) (cont) This suggests that the output is lagging the input, but in fact it is leading the input, because the 'standard equation' has a minus sign on the RHS, and ours does not.

So phases are 62° and 142° Leading

c) ω_n is natural frequency for the system with no damping ($R=0$); resonant frequency is the frequency at which the response of the actual circuit is a maximum.

When $\zeta = 0.2$, $\frac{\omega_r}{\omega_n} = \frac{1}{\sqrt{1-0.008}} = 1.043$

For $\frac{\omega_r}{\omega_n} = 1.5$,

$$1 - 2\zeta^2 = \frac{1}{1.5^2} = 0.444$$

$$\therefore \zeta = \sqrt{\frac{1-0.444}{2}} = 0.527$$

ζ is proportional to R, so increase R by a factor of

$$\frac{0.527}{0.2} = 2.64$$

A. Johnson
N. Crilly 2010