YEAR	PART (of Tripos)	PAPER NUMBER & TITLE	NAME OF AUTHOR(S)
2010	IA	P4 – Mathematical Methods	Prof RS Cant

<u>Crib</u> Section A

1 $$x_{n+2} - 12x_{n+1} + 27x_n = 0 \qquad \begin{cases} x_0 = 0 \\ x_1 = 6 \end{cases}$$

let $x_n = \lambda^n$: $\lambda^2 - 12\lambda + 27 = 0$

$(\lambda-9)(\lambda-3) = 0$

$x_n = A.9^n + B.3^n$ general solution

let $0 = A.1 + B.1 \Rightarrow A = -B$

$6 = 9A + 3B \Rightarrow A = 1;\ B = -1$

$\therefore\ \underline{x_n = 9^n - 3^n}$

2 (a) $$\lim_{x\to 0} \frac{2\sin x - \sin 2x}{x - \sin x}$$

l'Hôpital: $$\lim_{x\to 0} \frac{2\cos x - 2\cos 2x}{1 - \cos x}$$

again: $$\lim_{x\to 0} \frac{-2\sin x + 4\sin 2x}{\sin x}$$

again: $$\lim_{x\to 0} \frac{-2\cos x + 8\cos 2x}{\cos x} = 6$$

(b) $\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right) = \lim_{x\to 0}\left(\frac{x^2 - \sin^2 x}{x^2 \sin^2 x}\right)$

power series : $\sin x = x - x^3/3! + \ldots$

$\sin^2 x = x^2 - 2x^4/3! + \ldots$

$x^2 \sin^2 x = x^4 - 2x^6/3! + \ldots$

$\therefore \lim_{x\to 0}\left(\frac{x^2 - x^2 + 2x^4/3! + \ldots}{x^4 - 2x^6/3!}\right) = \frac{2}{3!} = \underline{\frac{1}{3}}$

3 Eigenvectors $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$; $\begin{bmatrix} 3 \\ -2 \end{bmatrix}$

Eigenvalues 1 ; 4

$A = U\Lambda U^{-1}$; $U = \begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$; $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

$U^{-1} = -\frac{1}{13}\begin{bmatrix} -2 & -3 \\ -3 & 2 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$

$A = \frac{1}{13}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} 2 & 12 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$

ie $A = \frac{1}{13}\begin{bmatrix} 40 & -18 \\ -18 & 25 \end{bmatrix}$

4. (a) $$2\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 2x = 2t + 9$$

with initial conditions $x = 3$ and $\frac{dx}{dt} = -1$ at $t = 0$

characteristic eqn: $2\lambda^2 + 5\lambda + 2 = 0$

$$\lambda = -2 \text{ or } \lambda = -\tfrac{1}{2}$$

complementary function is $x = Ae^{-2t} + Be^{-t/2}$

particular integral: try $x = at + b$

$$\frac{dx}{dt} = a; \quad \frac{d^2x}{dt^2} = 0$$

$$\therefore \quad 2.0 + 5a + 2(at + b) = 2t + 9$$

$$\therefore a = 1;\ b = 2 \quad \Rightarrow \text{PI} \quad x = t + 2$$

General solution is $x = Ae^{-2t} + Be^{-t/2} + t + 2$

Initial conditions $\quad 3 = A + B + 2 \quad$ at $t = 0$

$$\frac{dx}{dt} = -2A - B/2 + 1 = -1 \quad \text{at } t = 0$$

$$\therefore \quad A = 1;\ B = 0$$

$$x = e^{-2t} + t + 2$$

(b) Minimum is set by $\frac{dx}{dt} = 0 = -2e^{-2t} + 1$

$\therefore e^{-2t} = 1/2 \Rightarrow t = \frac{1}{2}\ln 2$

$\therefore x = e^{-2(\frac{1}{2}\ln 2)} + \frac{1}{2}\ln 2 + 2 = \underline{\frac{1}{2}(5 + \ln 2)}$

(c) $\frac{d^2y}{dx^2} - 4y = xe^x$; $y = 1$ at $x = 0$

$\frac{dy}{dx} = 2$ at $x = 0$

CF: $\lambda^2 - 4 = 0 \Rightarrow \lambda = 2$ or $\lambda = -2$

$y = Ae^{2x} + Be^{-2x}$

PI try $y = (ax+b)e^x$

$\frac{dy}{dx} = axe^x + ae^x + be^x$

$\frac{d^2y}{dx^2} = axe^x + 2ae^x + be^x$

$axe^x + 2ae^x + be^x - 4axe^x - 4be^x = xe^x$

xe^x : $a - 4a = 1$; $a = -1/3$

e^x : $2a + b - 4b = 0$; $b = -2/9$

general solution is $\underline{y = Ae^{2x} + Be^{-2x} - \frac{1}{3}xe^x - 2/9\, e^x}$

BCs: $y = 1$ at $x = 0$: $1 = A + B - 2/9$

$\frac{dy}{dx} = 2$ at $x = 0$: $2A - 2B - \frac{1}{3} - \frac{2}{9} = 2$

$A + B = 11/9$; $2A - 2B = 23/9$

$A = 5/4$; $B = -1/36$

$\therefore \quad \underline{y = 5/4\, e^{2x} - \frac{1}{36} e^{-2x} - \frac{1}{3} x e^{x} - 2/9\, e^{x}}$

5 (a) $\big| |z+2i| - |z-2i| \big| = 2+a$

$\big| |z+2i| - |z-2i| \big|^2 = (2+a)^2$

$|z+2i|^2 - 2|z+2i||z-2i| + |z-2i|^2 = (2+a)^2$

now $|z+2i|^2 = |x+iy+2i|^2 = x^2 + y^2 + 4y + 4$

$|z-2i|^2 = |x+iy-2i|^2 = x^2 + y^2 - 4y + 4$

$x^2 + y^2 \cancel{+4y} + 4 + x^2 + y^2 \cancel{-4y} + 4$

$-2\big[x^4 + x^2y^2 \cancel{- x^2 4y} + 4x^2 + x^2y^2 \cancel{+ x^2 4y} + 4x^2$

$+ y^4 - 8y^2 + 16\big]^{1/2} = (2+a)^2$

$$2(x^2+y^2+4) - (a+2)^2 = 2\left[(x^2+y^2+4)^2 - 16y^2\right]^{1/2}$$

Let $a=0$: $2(x^2+y^2)+4 = 2(x^2+y^2+2)$

$a=2$: $2(x^2+y^2)-8 = 2(x^2+y^2-4)$

(i) $a=0$: $(x^2+y^2+2)^2 = (x^2+y^2+4)^2 - 16y^2$

$\cancel{x^4} + \cancel{x^2y^2} + 2x^2 + \cancel{x^2y^2} + \cancel{y^4} + 2y^2 + 2x^2 + 2y^2 + 4$

$= \cancel{x^4} + \cancel{x^2y^2} + 4x^2 + \cancel{x^2y^2} + \cancel{y^4} + 4y^2 + 4x^2 + 4y^2 + 16$

$-16y^2$

$4x^2 + 4y^2 + 4 = 8x^2 + 8y^2 + 16 - 16y^2$

$4x^2 - 12y^2 + 12 = 0 \Rightarrow x^2 - 3y^2 + 3 = 0$

$\therefore 3y^2 = x^2 + 3$ or $y = \pm\sqrt{\frac{x^2}{3}+1}$

$a=2$: $(x^2+y^2-4)^2 = (x^2+y^2+4)^2 - 16y^2$

$\cancel{x^4} + \cancel{x^2y^2} - 4x^2 + \cancel{x^2y^2} + \cancel{y^4} - 4y^2 - 4x^2 - 4y^2 + \cancel{16}$

$= \cancel{x^4} + 2\cancel{x^2y^2} + 8x^2 + 8y^2 + \cancel{y^4} + \cancel{16} - 16y^2$

$0 = 16x^2 + 16y^2 - 16y^2 \Rightarrow x = 0$!

Need to find limits of y

Return to original equation : $z = iy$ (purely imaginary)

Thus $\left| |i(y+2)| - i|y-2| \right| = 4$

ie $\left| |y+2| - |y-2| \right| = 4$

$$\left(|y+2| - |y-2| \right)^2 = 16$$

$$(y+2)^2 - 2|y+2||y-2| + (y-2)^2 = 16$$

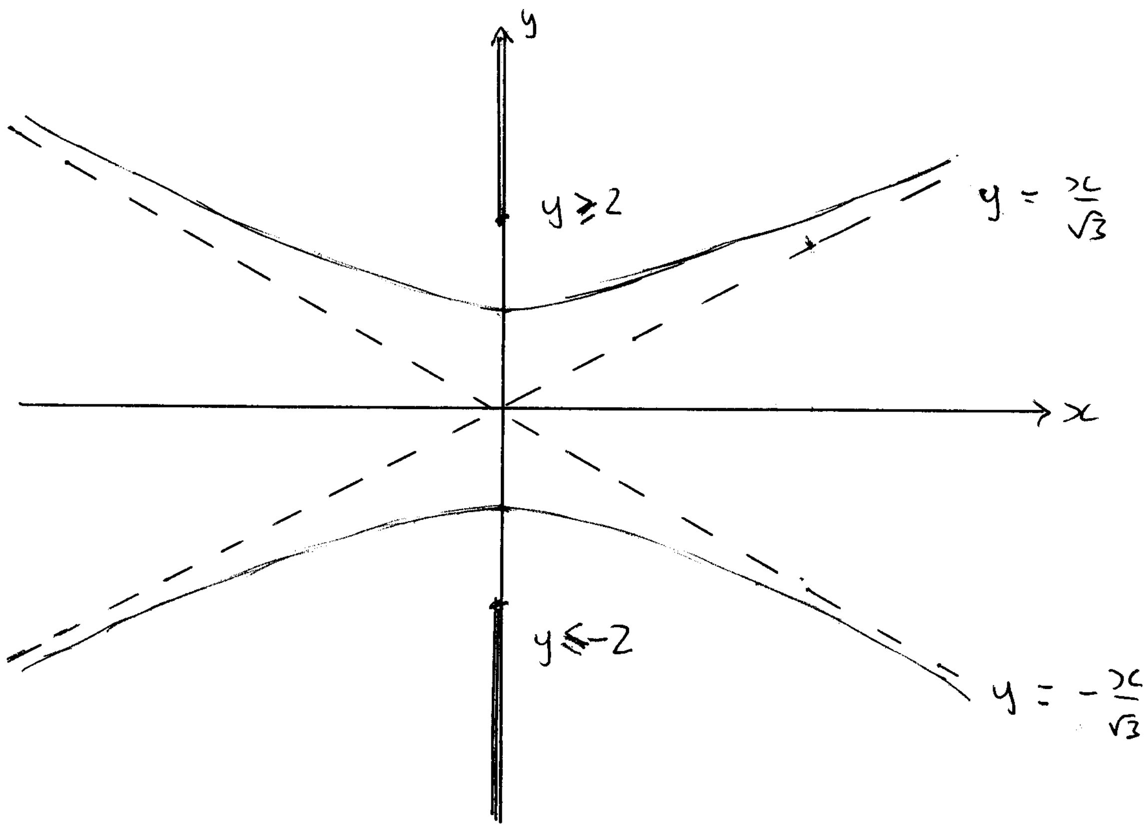
$$y^2 + \cancel{4y} + 4 - 2|y+2||y-2| + y^2 - \cancel{4y} + 4 = 16$$

$$2y^2 - 2\sqrt{y^2-4} + 8 = 16$$

$$y^2 - \sqrt{y^2-4} = 4$$

For real solutions (recall that y is purely real) $\underline{y^2 \geq 4}$

ii)



b) $||z+2i| - |z-2i|| = 4$

$|z-b| = |z+i|$; b real

$|z-b|^2 = |z+i|^2$

$(x-b)^2 + y^2 = x^2 + (y+1)^2$

$\cancel{x^2} - 2bx + b^2 + \cancel{y^2} = \cancel{x^2} + \cancel{y^2} + 2y + 1$

$2y = (b^2-1) - 2bx$

$y = -bx + \frac{1}{2}(b^2-1)$

From part (a) with $a=2$ have $|y| \geq 2$; $x=0$

$\therefore$ minimum value of b occurs for

$2 = -b \times 0 + \frac{1}{2}(b^2-1)$

$\therefore b^2 = 5$

$\therefore \underline{b = \sqrt{5}}$.

Part IA Paper 4 Mathematical Methods 2010

Crib Section B

6. $g(t) = e^{-\alpha t}$; $t \geq 0$; $y(t) = 0$ at $t = 0$.

Impulse response.

a) Step response $\int g(t)\,dt = -\frac{1}{\alpha} e^{-\alpha t} + \text{const} = y(t)$

$y(0) = 0 \therefore \text{const} = 1/\alpha$

$$\therefore \underline{y(t) = \frac{1}{\alpha}(1 - e^{-\alpha t})}$$

b) Input $x(t) = \cos\omega t = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t})$

$$y(t) = \int_0^t e^{-\alpha(t-\tau)} \cdot \frac{1}{2}(e^{i\omega\tau} + e^{-i\omega\tau})\,d\tau$$

convolution

$$= \frac{e^{-\alpha t}}{2} \int_0^t \left(e^{(i\omega + \alpha)\tau} + e^{(-i\omega + \alpha)\tau}\right) d\tau$$

$$= \frac{e^{-\alpha t}}{2} \left[\frac{1}{i\omega + \alpha} e^{(i\omega + \alpha)\tau} + \frac{1}{(-i\omega + \alpha)} e^{(-i\omega + \alpha)\tau} \right]_0^t$$

$$= \frac{e^{-\alpha t}}{2} \left[\frac{(\alpha - i\omega)(e^{(\alpha + i\omega)t} - 1) + (\alpha + i\omega)(e^{(\alpha - i\omega)t} - 1)}{\alpha^2 + \omega^2} \right]$$

$$= \frac{e^{-\alpha t}}{2(\alpha^2+\omega^2)}\Big[\alpha e^{\alpha t} e^{i\omega t} - i\omega e^{\alpha t} e^{i\omega t} - \alpha + i\omega$$
$$+ \alpha e^{\alpha t} e^{-i\omega t} + i\omega e^{\alpha t} e^{-i\omega t} - \alpha - i\omega\Big]$$

$$= \frac{e^{-\alpha t}}{2(\alpha^2+\omega^2)}\Big[\alpha e^{\alpha t}\left(e^{i\omega t} + e^{-i\omega t}\right) - i\omega e^{\alpha t}\left(e^{i\omega t} - e^{-i\omega t}\right)$$
$$-2\alpha\Big]$$

ie $$y(t) = \frac{\alpha}{\alpha^2+\omega^2}\cos\omega t + \frac{\omega}{\alpha^2+\omega^2}\sin\omega t - \frac{\alpha e^{-\alpha t}}{\alpha^2+\omega^2}$$

7 $\quad x^2 - 3y^2 + z^3 = 5$

a) let $f(x,y,z) = x^2 - 3y^2 + z^3 - 5$

$$\nabla f = 2x\underline{i} - 6y\underline{j} + 3z^2\underline{k}$$

direction $\underline{a} = 2\underline{i} + \underline{j} - 3\underline{k}$

$$|\underline{a}|^2 = 4 + 1 + 9 = 14$$

$\therefore \underline{b} = \dfrac{2\underline{i} + \underline{j} - 3\underline{k}}{\sqrt{14}}$ unit vector in required direction.

∇f at $(3,2,2)$ is $6\underline{i} - 12\underline{j} + 12\underline{k}$

$$\underline{b}.\nabla f = \frac{1}{\sqrt{14}}(12 - 12 - 36) = \underline{-\frac{36}{\sqrt{14}}}$$

b) $|\nabla f|^2 = 4x^2 + 36y^2 + 9z^4$

At $(4,1,-2)$ have $|\nabla f|^2 = 64 + 36 + 144 = 244$

$$\therefore |\nabla f| = \sqrt{244} = 2\sqrt{61}$$

$\therefore$ unit normal vector is $\dfrac{\nabla f}{|\nabla f|} = \dfrac{1}{2\sqrt{61}}(2x\underline{i} - 6y\underline{j} + 3z^2\underline{k})$

$$= \frac{1}{2\sqrt{61}}(8\underline{i} - 6\underline{j} + 12\underline{k})$$

$$= \underline{\frac{1}{\sqrt{61}}(4\underline{i} - 3\underline{j} + 6\underline{k})}$$

8 50 objects, 6 scratched, 3 dented
2 scratched and dented

a) sampling without replacement

First pick : $\frac{39}{50}$ = prob. of undamaged component

Second pick : $\frac{38}{49}$

Third pick : $\frac{37}{48}$

Prob. that all 3 are undamaged $= \frac{39}{50} \times \frac{38}{49} \times \frac{37}{48} = \underline{0.466}$

b) Prob. of 2 scratched and 1 dented

2 ways to pick S from 6 $= {}^6C_2 = \frac{6!}{2!4!}$

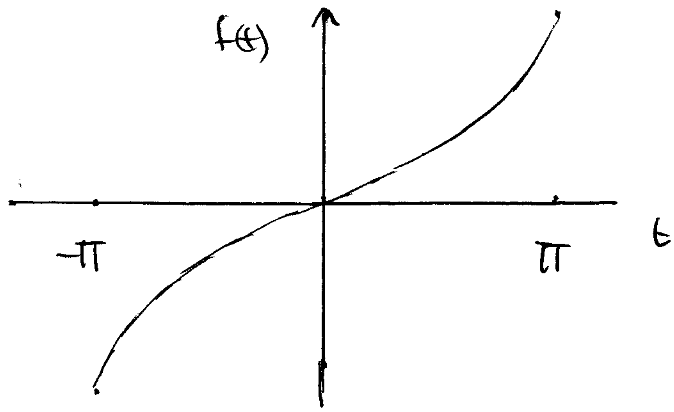
1 " " " D " 3 $= {}^3C_1 = \frac{3!}{1!2!}$

In all, need 3 from 50 $= {}^{50}C_3 = \frac{50!}{3!47!}$

$\therefore$ Prob. is $\frac{{}^6C_2 \times {}^3C_1}{{}^{50}C_3} = \frac{15 \times 3}{19600} = \underline{0.0023}$

c) $P(D|S) = \frac{P(D \cap S)}{P(S)} = \frac{2/50}{8/50} = \underline{1/4}$

9 a) $f(t) = \sinh t \quad ; -\pi \text{ to } \pi$



$\sinh \pi \approx 11.55$

odd function

b) Real series : $d = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sinh t \, dt$

$T = 2\pi, \; \omega = \frac{2\pi}{T} = 1$

$$= \frac{1}{2\pi}\left[\cosh t\right]_{-\pi}^{\pi} \equiv 0$$

$a_n = 0 \;\; \forall n$ since function is odd

or $a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} \sinh t \cos \frac{2\pi n t}{T} \, dt$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{dV}{\sinh t} \, \overset{u}{\cos nt} \, dt$$

$$= \frac{1}{\pi}\left[\cosh t \overset{0}{\cancel{\cos nt}}\right]_{-\pi}^{\pi} + \frac{n}{\pi}\int_{-\pi}^{\pi} \overset{dv}{\cosh t} \, \overset{u}{\sin nt} \, dt$$

$$= \frac{n}{\pi}\left[\sinh t \sin nt\right]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} \sinh t \cos nt \, dt$$

$$\therefore (1+n^2) a_n = \frac{n}{\pi}\left[\sinh \pi \sin n\pi - \sinh(-\pi)\sin(-n\pi)\right]$$

$$\equiv 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \overset{dv}{\sinh t}\, \overset{u}{\sin nt}\, dt$$

$$= \frac{1}{\pi}\left[\cosh t \,\overset{0}{\cancel{\sin nt}}\right]_{-\pi}^{\pi} - \frac{n}{\pi}\int_{-\pi}^{\pi} \overset{dv}{\cosh t}\, \overset{u}{\cos nt}\, dt$$

$$= -\frac{n}{\pi}\left[\sinh t \cos nt\right]_{-\pi}^{\pi} - \frac{n^2}{\pi}\int_{-\pi}^{\pi} \sinh t \sin nt\, dt$$

$$(1+n^2)\, b_n = -\frac{n}{\pi}\left[\sinh \pi \cos(n\pi) - \sinh(-\pi)\cos(-n\pi)\right]$$

$$= -\frac{2n}{\pi} \sinh \pi \times (-1)^n$$

$\cos(n\pi)$ or $\cos(-n\pi)$

$$\therefore b_n = \frac{-2n(-1)^n \sinh \pi}{\pi(1+n^2)}$$

$$\therefore f(t) = \sum_{n=1}^{\infty}\left[\frac{-2n(-1)^n \sinh \pi}{\pi(1+n^2)}\right] \sin nt$$

c)

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} \sinh t \, e^{-int}\, dt$$ complex series

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left(e^{t} - e^{-t}\right) e^{-int}\, dt$$

$$= \frac{1}{4\pi}\int_{-\pi}^{\pi} \left[e^{t(1-in)} - e^{-t(1+in)}\right] dt$$

$$= \frac{1}{4\pi}\left[\frac{1}{1-in}\, e^{t(1-in)} + \frac{1}{1+in}\, e^{-t(1+in)}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{4\pi(1+n^2)}\Big[(1+in)\left(e^{\pi(1-in)} - e^{-\pi(1-in)}\right)$$
$$+ (1-in)\left(e^{-\pi(1+in)} - e^{\pi(1+in)}\right)\Big]$$

$$= \frac{1}{4\pi(1+n^2)}\Big[(1+in)\left(e^{\pi}e^{-in\pi} - e^{-\pi}e^{in\pi}\right)$$
$$+ (1-in)\left(e^{-\pi}e^{-in\pi} - e^{\pi}e^{in\pi}\right)\Big]$$

$$= \frac{1}{4\pi(1+n^2)}\Big[e^{-in\pi}\left\{(1+in)e^{\pi} + (1-in)e^{-\pi}\right\}$$
$$- e^{+in\pi}\left\{(1+in)e^{-\pi} + (1-in)e^{\pi}\right\}\Big]$$

$$= \frac{1}{4\pi(1+n^2)}\Big[e^{-in\pi}\left(2\cosh\pi + in\, 2\sinh\pi\right)$$
$$- e^{in\pi}\left(2\cosh\pi - in 2\sinh\pi\right)\Big]$$

$$e^{in\pi} = \cos n\pi + i\cancel{\sin n\pi}^{\,0} = (-1)^n$$
$$e^{-in\pi} = \cos n\pi - i\cancel{\sin n\pi}^{\,0} = (-1)^n$$

$\therefore \; c_n = \dfrac{(-1)^n}{\pi(1+n^2)} \, in \sinh \pi$

$\therefore$ Series is

$$f(t) = \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n}{\pi(1+n^2)} \, in \sinh \pi \right] e^{-int}$$

d) Real coefficients : $a_n = 2\mathrm{Re}(c_n) = 0$

$b_n = -2\,\mathrm{Im}(c_n)$

$= \dfrac{-2n(-1)^n \sinh \pi}{\pi(1+n^2)}$

as required.

when $n=0$ $\quad c_n = 0 = d$, as required.

10 a) $\frac{d^2y_1}{dt^2} = y_1 + 3y_2$; $\frac{d^2y_2}{dt^2} = 4y_1 - 4e^t$

$y_1(0) = 2$; $\dot{y}_1(0) = 3$; $y_2(0) = 1$; $\dot{y}_2(0) = 2$

$$\mathcal{L}\,\ddot{y}_1 = s^2Y_1 - sy_1(0) - \dot{y}_1(0) = s^2Y_1 - 2s - 3$$

$$\mathcal{L}\,\ddot{y}_2 = s^2Y_2 - sy_2(0) - \dot{y}_2(0) = s^2Y_2 - s - 2$$

equations are

$$s^2Y_1 - 2s - 3 = Y_1 + 3Y_2$$

$$s^2Y_2 - s - 2 = 4Y_1 - \frac{4}{s-1}$$

$$(s^2-1)Y_1 - 3Y_2 = 2s+3 \qquad — \quad (1)$$

$$s^2Y_2 - 4Y_1 = s+2 - \frac{4}{s-1}$$

$$= \frac{s^2+s-6}{s-1} \qquad — \quad (2)$$

$(1) \times 4$ $\qquad 4(s^2-1)Y_1 - 12Y_2 = 4(2s+3)$

$(2) \times s^2-1$ $\qquad (s^2-1)s^2Y_2 - 4(s^2-1)Y_1 = (s+1)(s^2+s-6)$

Add $\qquad (s^4-s^2-12)Y_2 = s^3+2s^2+3s+6$

$$Y_2 = \frac{s^2+2s^2+3s+6}{(s^2-4)(s^2+3)}$$

ie $Y_2 = \frac{A+Bs}{s^2-4} + \frac{C+Ds}{s^2+3}$

$$(A+Bs)(s^2+3) + (C+Ds)(s^2-4) = s^3+2s^2+3s+6$$

$$= As^2+3A+Bs^3+3Bs+Cs^2+Ds^3-4C-4Ds$$

s^3: $B+D=1$ $\qquad$ $4B+4D=4$

s^2: $A+C=2$ $\qquad$ $4A+4C=8$

s: $3B-4D=3$ $\qquad$ $7B=7$; $B=1$; $D=0$

1: $3A-4C=6$ $\qquad$ $7A=14$; $A=2$; $C=0$

$$\therefore Y_2 = \frac{s+2}{s^2-4}$$ and s^2+3 is a factor of the RHS

$$= \frac{1}{s-2}$$

$$\therefore Y_1 = \frac{2s+3}{s^2-1} + \frac{3}{(s^2-1)(s-2)}$$ from (1)

Now $$\frac{3}{(s^2-1)(s-2)} = \frac{E+Fs}{s^2-1} + \frac{G}{s-2}$$

$$3 = (E+Fs)(s-2) + G(s^2-1)$$

$$= Es-2E+Fs^2-2Fs+Gs^2-G$$

s^2: $0 = F+G \Rightarrow F=-G$

s: $0 = E-2F \Rightarrow E+2G=0$

1: $3 = -2E-G \Rightarrow G+2E-3$

$\therefore G=1$; $E=-2$; $F=-1$

$$Y_1 = \frac{2s+3}{s^2-1} - \frac{2+s}{s^2-1} + \frac{1}{s-2}$$

$$= \frac{s+1}{s^2-1} + \frac{1}{s-2} = \frac{1}{s-1} + \frac{1}{s-2}$$

$$\therefore \; y_1(t) = e^t + e^{2t}$$

$$y_2(t) = e^{2t}$$

b) at $t=0$ have $y_1(0) = 1+1 = 2$

$$y_2(0) = 1$$

$$\dot{y}_1(t) = e^t + 2e^{2t} \Rightarrow \dot{y}_1(0) = 3$$

$$\dot{y}_2(t) = 2e^{2t} \Rightarrow \dot{y}_2(0) = 2$$

$$\ddot{y}_1(t) = e^t + 4e^{2t} = \underbrace{e^t + e^{2t}} + 3e^{2t}$$

$$\ddot{y}_2(t) = \underbrace{4e^t + 4e^{2t}} - 4e^t = 4e^{2t}$$

Crib Section C

11) The program will return the output 2
0
3

The function "func" takes two string arguments. It searches the second string for the first occurrence of the first string. If it is found, the function returns the location in the second string of the first element of the first string. If not, the function returns zero. A wild card character of "*" is allowed in the first string and will match any character in the second string.

12) Algorithmic complexity describes how the number of operations scales with the size of the problem.

For exchange sort, the number of operations scales as n^2 where n is the number of items to sort. It has complexity $O(n^2)$. For Quicksort, the complexity is $O(n\log_2 n)$

Exchange sort $t_e = k_e n_e^2$

Quicksort $t_q = k_q n_q \log_2 n_q$

$t_e = t_q$ when $n_e = 25\,000$ and $n_q = 10^6$

$$\therefore k_e . (2.5 \times 10^4)^2 = k_q 10^6 \log_2 10^6$$

$$k_e / k_q = \frac{10^6 \log_2 10^6}{(2.5 \times 10^4)^2} = \frac{10^6 \times 19.93}{6.25 \times 10^8} = 0.03189$$

Now $t_e \leq t_q$ when $k_e n^2 \leqslant k_q n \log_2 n$

$$\log_2 n \geqslant n \frac{k_e}{k_q}$$

n	$\log_2 n$	$n \frac{k_e}{k_q}$
2	1	0.06378
8	3	0.2552
16	4	0.5
32	5	1.02
64	6	2.04096
128	7	4.08192
256	8	8.1638
200	7.6446	6.378
250	7.966	7.9725
249	7.961	7.94

→ 249 items