ENGINEERING TRIPOS PART IA

Tuesday 8 June 2010 9 to 12

Paper 4

MATHEMATICAL METHODS

Answer ***all*** *questions.*

The ***approximate*** *number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 **(short)** Solve the difference equation

$$x_{n+2} - 12x_{n+1} + 27x_n = 0$$

with initial values $x_0 = 0$ and $x_1 = 6$. [10]

2 **(short)**

(a) Using l'Hôpital's rule find:

$$\lim_{x\to 0}\left(\frac{2\sin x - \sin 2x}{x - \sin x}\right).$$ [5]

(b) Using a power series expansion find:

$$\lim_{x\to 0}\left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$$ [5]

3 **(short)** Find the 2 by 2 matrix which has eigenvectors $[2\ \ 3]^T$ and $[3\ \ -2]^T$ with corresponding eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

[10]

4 **(long)**

(a) Find the solution of the differential equation

$$2\frac{d^2x}{dt^2}+5\frac{dx}{dt}+2x=2t+9$$

subject to the initial conditions $x=3$ and $\frac{dx}{dt}=-1$ when $t=0$. [12]

(b) The equation is used to model the motion of a particle P on the x-axis. At time t seconds ($t \geq 0$), P is x metres from the origin O. Show that the minimum distance between O and P is

$$\frac{1}{2}\left(5+\ln(2)\right)$$ [6]

(c) Find the solution of the differential equation

$$\frac{d^2y}{dx^2}-4y=xe^x$$

which satisfies the boundary conditions $y=1$ at $x=0$ and $\frac{dy}{dx}=2$ at $x=0$. [12]

5 **(long)**

(a) A variable complex number z satisfies the equation

$$\left||z+2i|-|z-2i|\right|=2+a$$

(i) Find the equations of the locii of z on an Argand diagram for $a=0$ and $a=2$. [15]

(ii) Plot the solutions of the equation on an Argand diagram. [7]

(b) Consider the pair of simultaneous equations

$$\left||z+2i|-|z-2i|\right|=4; \qquad |z-b|=|z+i|, \quad \text{where } b \text{ is real.}$$

What is the minimum value of b for which there are solutions? [8]

SECTION B

6 **(short)** A linear system has impulse response $g(t) = e^{-\alpha t}$ for $t \geq 0$. Initially the output is zero.

(a) Determine the step response of the system. [3]

(b) Find the response of the system to the input $x(t) = \cos \omega t$ for $t \geq 0$. [7]

7 **(short)** A function is defined as $\phi(x, y, z) = x^2 - 3y^2 + z^3 - 5$.

(a) Find the directional derivative of $\phi(x, y, z)$ in the direction $2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ at the point (3, 2, 2). [6]

(b) Find the unit normal vector to the surface $\phi(x, y, z) = 0$ at the point $(4,\ 1, -2)$. [4]

8 **(short)** A factory produces batches of 50 metal components. In one particular batch, six components are scratched, three are dented, and two are both scratched and dented. Three components are chosen all together at random from the batch for inspection.

(a) What is the probability that none of the chosen components is either scratched or dented? [3]

(b) Of the three components, what is the probability that two are scratched and one dented? [4]

(c) One component is found to be scratched. What is the probability that it is also dented? [3]

9 **(long)**

(a) Sketch the function $f(t) = \sinh t$ within the range $-\pi \le t \le \pi$. [5]

(b) Obtain the complete Fourier series for this function within this range in <u>real</u> form, evaluating all coefficients by integration.

[10]

(c) Obtain the complete Fourier series for this function within this range in <u>complex</u> form, evaluating all coefficients by integration.

[10]

(d) Show that the real and complex Fourier series are equivalent. [5]

10 **(long)**

(a) Use Laplace transforms to solve the differential equations

$$\frac{d^2 y_1}{dt^2} - y_1 = 3y_2$$

$$\frac{d^2 y_2}{dt^2} = 4y_1 - 4e^t$$

subject to the initial conditions $y_1(0) = 2$, $\dot{y}_1(0) = 3$, $y_2(0) = 1$ and $\dot{y}_2(0) = 2$. [20]

(b) Show clearly that your solution satisfies the initial conditions. [10]

(TURN OVER

SECTION C

11 **(short)** Determine what output will be produced by the C++ program shown below:

```
int main ()
{
    string s1 = "aabcdee";
    string s2 = "abc";
    string s3 = "bad";
    string s4 = "b*d";

    cout << func (s2, s1) << endl;
    cout << func (s3, s1) << endl;
    cout << func (s4, s1) << endl;
}

int func (string t, string s)
{
    int i, j;

    i = 0;
    while (i < s.length()) {
      j = 0;
      while (j < t.length() && (t[j] == s[i+j] || t[j] == '*')) {
          j++;
      }
      i++;
      if (j >= t.length())
          return (i);
    }
    return (0);
}
```

Hence describe the purpose of the function `func().` [10]

12 **(short)** Explain what is meant by algorithmic complexity, illustrating your answer by reference to the Quicksort and Exchange Sort algorithms. On a particular computer, it takes approximately the same time to sort 25,000 items using Exchange Sort as it does to sort one million using Quicksort. What is the greatest number of items for which it would be quicker to use Exchange Sort?

[10]

END OF PAPER

Engineering Tripos Part IA 2010

Paper 4: Mathematical Methods

Short Answers

Section A

Q1: $x_n = 9^n - 3^n$

Q2: (a) 6; (b) $\frac{1}{3}$

Q3: $\frac{1}{13}\begin{pmatrix} 40 & -18 \\ -18 & 25 \end{pmatrix}$

Q4: (a) $x(t) = \exp(-2t) + t + 2$

(c) $y(x) = \frac{5}{4}\exp(2x) - \frac{1}{36}\exp(-2x) - \frac{1}{3}x\exp(x) - \frac{2}{9}\exp(x)$

Q5: (a) for $a = 0$: $3y^2 = x^2 + 3$ (hyperbolae); for $a = 2$: $x = 0$, $y^2 \geq 4$ (part of the y-axis).

(b) $b \geq \sqrt{5}$

Section B

Q6: (a) $y(t) = \frac{1}{\alpha}(1 - \exp(-\alpha t))$

(b) $y(t) = \frac{1}{\alpha^2 + \omega^2}[\alpha\cos(\omega t) + \omega\sin(\omega t) - \alpha\exp(-\alpha t)]$

Q7: (a) $-\frac{36}{\sqrt{14}}$; (b) $\frac{1}{\sqrt{61}}(4\mathbf{i} - 3\mathbf{j} + 6\mathbf{k})$

Q8: (a) 0.466; (b) 0.00115; (c) 0.25

Q9: (b) $\sinh t = \sum_{n=1}^{\infty} \left[\frac{-2n(-1)^n \sinh \pi}{\pi(1+n^2)} \right] \sin nt$

(c) $\sinh t = \sum_{n=-\infty}^{\infty} \left[\frac{(-1)^n in \sinh \pi}{\pi(1+n^2)} \right] \exp(-int)$

Q10: (a) $y_1(t) = \exp(t) + \exp(2t), \quad y_2(t) = \exp(2t)$

Section C

Q11: Output is: 2
0
3

Q12: 249