2011 Part 1A P2 Structures and Materials

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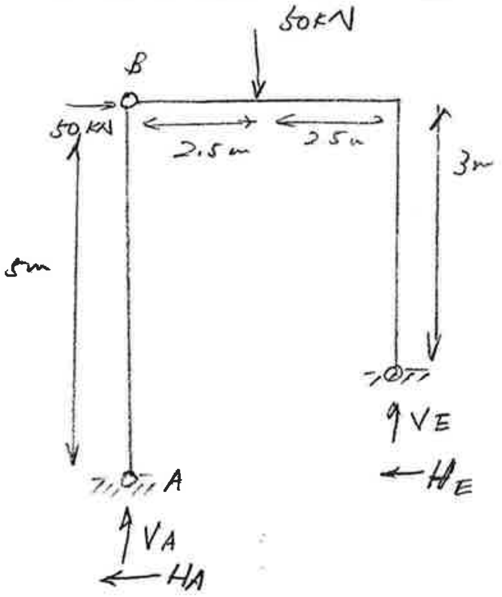
Dr G McShane

Q1



$\Sigma V = 0$

$V_A + V_E = 50$

$\Sigma H = 0$

$50 = H_A + H_E$

A cut at B

$\Sigma M_{cut} = H_A \times 5 = 0$

$\therefore H_A = 0$

$\therefore \underline{H_E = 50\ kN}$

Overall moment equilibrium at A

$\Sigma M = 0$

$50 \times 5 + 50 \times 2.5 - V_E \times 5 - \overset{50}{H_E} \times 2 = 0$

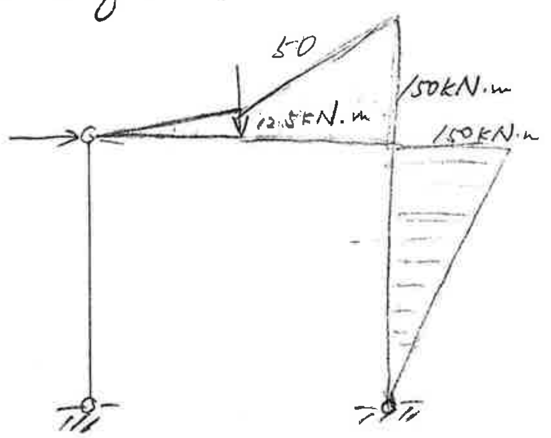
$5 V_E = 50 \times 5 + 50 \times 2.5 - 50 \times 2$

$V_E = 50 + 25 - 20 = \underline{55\ kN}$

$\underline{V_A = -5\ kN}$

(b)

Moment diagram.



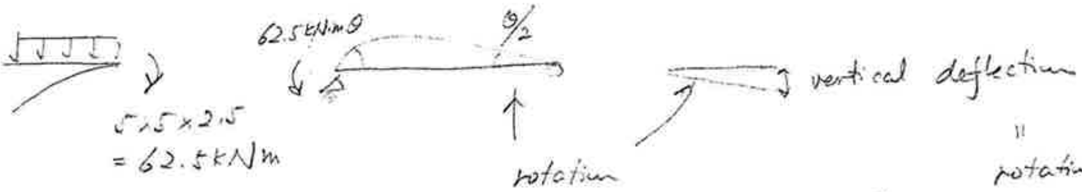
tension side positive

//

Q2

(i)

From the Data book

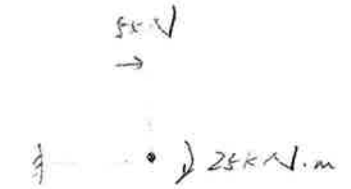


$= \frac{1}{2} \cdot \frac{62.5 \cdot 5}{3B} = \frac{52.08}{10.000} \quad = \frac{52.08}{10\,000} \cdot 5$

$= \frac{260.42}{10.000}$

$= 26\,mm$

(ii)



vertical displacement at D

$= \frac{25 \cdot 5^2}{2B} = \frac{312.5}{10000} = 31.25\,mm$

(iii)

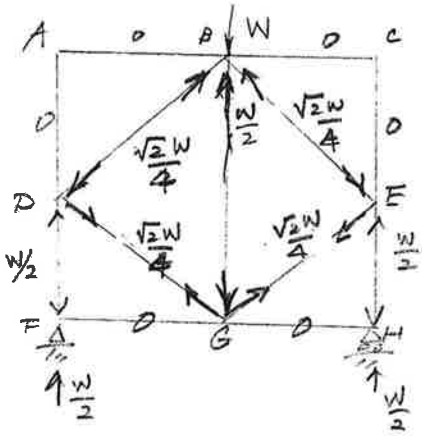
$\theta = \frac{ML}{3B} = \frac{25 \cdot 5}{3 \cdot B}$

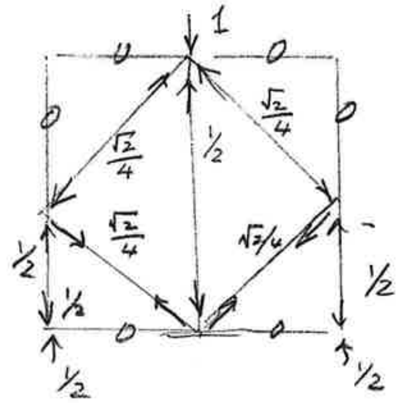
vertical displacement $\delta = \frac{25 \cdot 5}{3 \cdot B} \cdot 5 = \frac{208}{10.000}$

$= 20.8\,mm$

$\Sigma\delta = 26 + 31.25 + 20.8 = \underline{78.125\,mm}$ downwards

Q3





	Tension	Length	Extension e	Load 1 at B T^*	$T^* e$
AB	0	L	0	0	0
BC	0	L	0	0	0
AD	0	L	0	0	0
CE	0	L	0	0	0
DB	$-\frac{\sqrt{2}W}{4}$	$\sqrt{2}L$	$-\frac{1}{2}\frac{WL}{EA}$	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}\frac{WL}{EA}$
BE	$-\frac{\sqrt{2}W}{4}$	$\sqrt{2}L$	$-\frac{1}{2}\frac{WL}{EA}$	$-\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}\frac{WL}{EA}$
DG	$\frac{\sqrt{2}W}{4}$	$\sqrt{2}L$	$\frac{1}{2}\frac{WL}{EA}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}\frac{WL}{EA}$
GE	$\frac{\sqrt{2}W}{4}$	$\sqrt{2}L$	$\frac{1}{2}\frac{WL}{EA}$	$\frac{\sqrt{2}}{4}$	$\frac{\sqrt{2}}{8}\frac{WL}{EA}$
DF	$-W/2$	L	$-\frac{1}{2}\frac{WL}{EA}$	$-1/2$	$\frac{1}{4}\frac{WL}{EA}$
EH	$-W/2$	L	$-\frac{1}{2}\frac{WL}{EA}$	$-1/2$	$\frac{1}{4}\frac{WL}{EA}$
FG	0	L		0	0
GH	0	L		0	0
BG	$-W/2$	$2L$	$-\frac{WL}{EA}$	$-1/2$	$\frac{1}{2}\frac{WL}{EA}$

$$\Sigma T^* e = \left(1 + \frac{\sqrt{2}}{2}\right)\frac{WL}{EA}$$

Q4

(a) $(140 \times 8) \times 80 + (10 \times 150) \times 5 = (140 \times 8 + 10 \times 150)\,x$

$$97100 = 2620\,x$$

$x = 37$ mm from the bottom

(b) $$I = \left(\frac{8 \times 140^3}{12} + 8 \times 140 \times (80 - 37)^2\right)$$

$$+ \left(\frac{150 \times 10^3}{12} + 150 \times 10 \times (37 - 5)^2\right)$$

$$= 5.449 \times 10^6 \text{ mm}^4 \quad \text{or} \quad 5.449 \times 10^{-6} \text{ m}^4$$

$$B = EI = \underset{(kN/m^2)}{210 \times 10^6} \times \underset{(m^4)}{5.449 \times 10^{-6}} = \underline{1144 \text{ kN.m}^2}$$

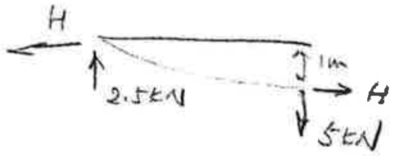
(c) $$\sigma = \frac{My}{I} = \frac{25 \times 10^3 \times (150 - 37)}{5.449 \times 10^6}$$

$$= 0.518 \text{ kN/mm}^2$$

$$= \underline{518 \text{ MPa} \quad \text{at the top}}$$

Q 5

(a) Vertical force from symmetry



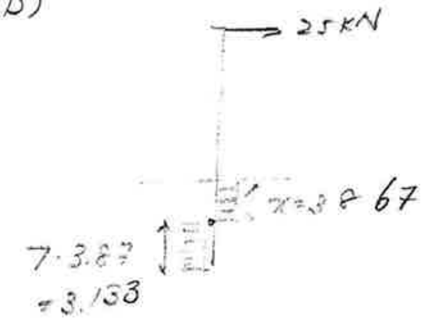
$Y = \underline{2.5\ kN}$ //

Horizontal force

$H \cdot 1 = 2.5 \times 10$

$\underline{H = 25\ kN}$ //

(b)



Horizontal eq.

$25 + \omega \times 3.133 = \omega \times 3.867$

$\omega \times 0.734 = 25$

$\underline{\omega = 34.06\ kN/m}$ //

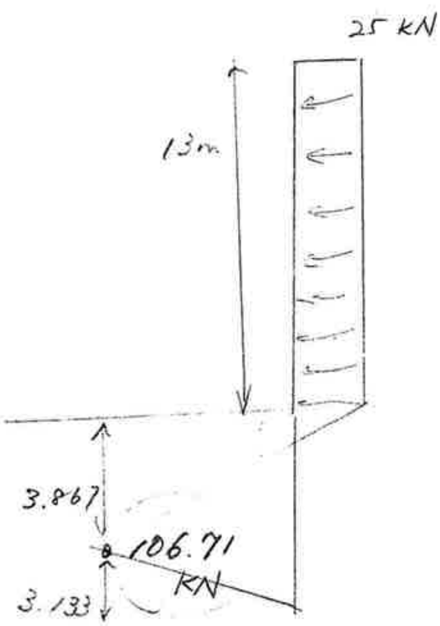
Moment eq.

$$25 \times (13 + 3.867) = 34.06 \times 3.867 \times \frac{3.867}{2} + 34.06 \times 3.133 \times \frac{3.133}{2}$$

$$421.675 = 254.66 + 167.16$$

$$= 421.82 \qquad \underline{OK}$$

(C)



(d)

$$I = \frac{300 \times (400)^3}{12} - \frac{200 \times (300)^3}{12}$$

$$= 11.5 \times 10^8 \text{ mm}^4$$

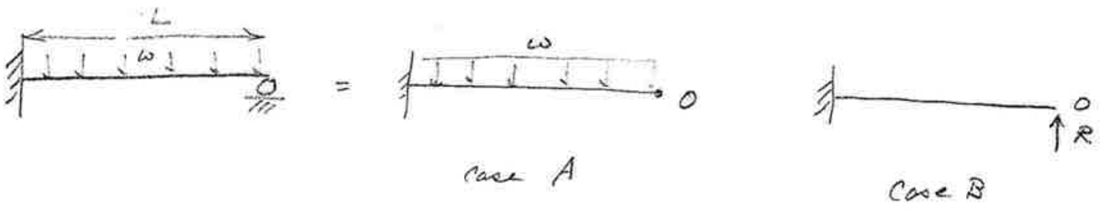
$$\tau = \frac{SA\bar{y}}{I \cdot t} = \frac{106.71 \times (0.3 \times 0.05) \times (0.2 - 0.025)}{11.5 \times 10^8 \times 10^{-12} \times (2 \times 0.05)}$$

$$= 0.244 \times 10^4 \text{ kN/m}^2$$

$$= \underline{2.44 \text{MPa}} \;//$$

Q6

(a)



The sum of the vertical displacements of the two cases becomes zero.

case (A) $\delta = \dfrac{\omega L^4}{8EI}$

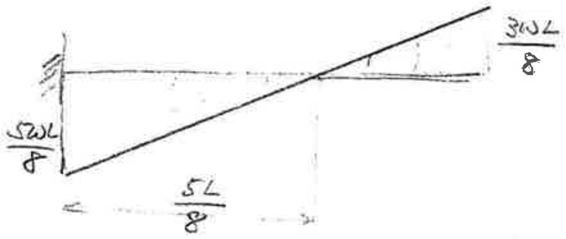
case (B) $\delta = \dfrac{RL^3}{3EI}$

$$\frac{RL^3}{3EI} = \frac{\omega L^4}{8EI} \qquad \underline{R = \frac{3\omega L}{8}}$$

(b) Bending moment at the root

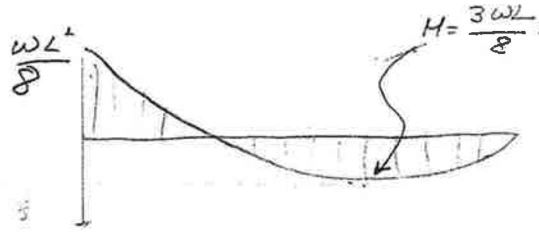
$$\omega L \cdot \frac{L}{2} - \frac{3\omega L}{8}L = \frac{\omega L^2}{8}$$

Shear force



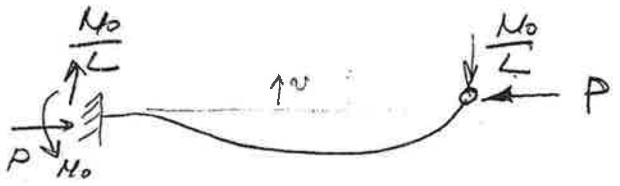
Bending moment

$$M = \frac{3\omega L}{8} \cdot \frac{3}{8}L - \frac{3\omega L}{8} \cdot \frac{3}{16}L = \frac{9\omega L^2}{128}$$

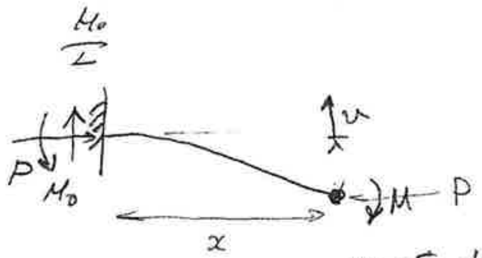


(C)

(i)



(ii)



opposite direction to x

$$M + P(-v) + \frac{M_0}{L}\cdot x = M_0$$

$$M = -EI\,\frac{d^2v}{dx^2}$$

$$EI\cdot\frac{d^2v}{dx^2} + Pv = M_0\left(\frac{x}{L}-1\right)$$

(iii) General solution $v = A\sin\alpha x + B\cos\alpha x + \frac{M_0}{P}\left(\frac{x}{L}-1\right)$

$\alpha^2 = P/EI$

At $x=0$ $v=0$ $\rightarrow$ $0 = B - \frac{M_0}{P}$ $\therefore$ $B = \frac{M_0}{P}$

At $x=0$ $\frac{dv}{dx}=0 \rightarrow 0 = A\alpha + \frac{M_0}{PL}$ $A = -\frac{M_0}{PL}$

At $x=l$ $v=0$ $\rightarrow$ $0 = -\frac{M_0}{PL}\sin\alpha L + \frac{M_0}{P}\cos\alpha L$

$$0 = -\tan\alpha L + \alpha L$$

$$\underline{\tan\alpha L = \alpha L}$$

iv) The deformed shape will act as an imperfection that grows as axial load is applied, with failure by yielding at a load below the bifurcation load found from (iii)

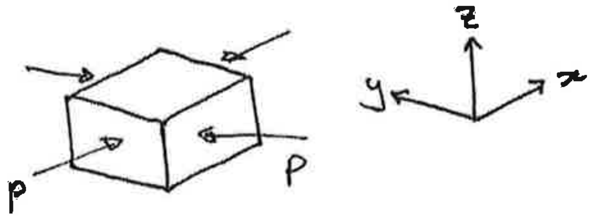
Engineering Tripos Part IA 2011

Solutions: Paper 2, Section B

Examiner: Dr GJ McShane

Q7 (SHORT)

(a)



$$\varepsilon_{xx} = \varepsilon_{yy} = -\frac{p}{E}(1-\nu) + \alpha\Delta T \underbrace{=}_{\text{contrained}} 0$$

$$\therefore \quad p = \left(\frac{E}{1-\nu}\right)\alpha\Delta T \qquad [4]$$

(b) $$\varepsilon_{zz} = \frac{p}{E}(2\nu) + \alpha\Delta T = \frac{H-h}{h}$$

Substitute for p:

$$\left(\frac{2\nu}{1-\nu}\right)\alpha\Delta T + \alpha\Delta T = \frac{H}{h} - 1$$

$$\frac{2\nu}{1-\nu} + 1 = \frac{1}{\alpha\Delta T}\left(\frac{H}{h} - 1\right)$$

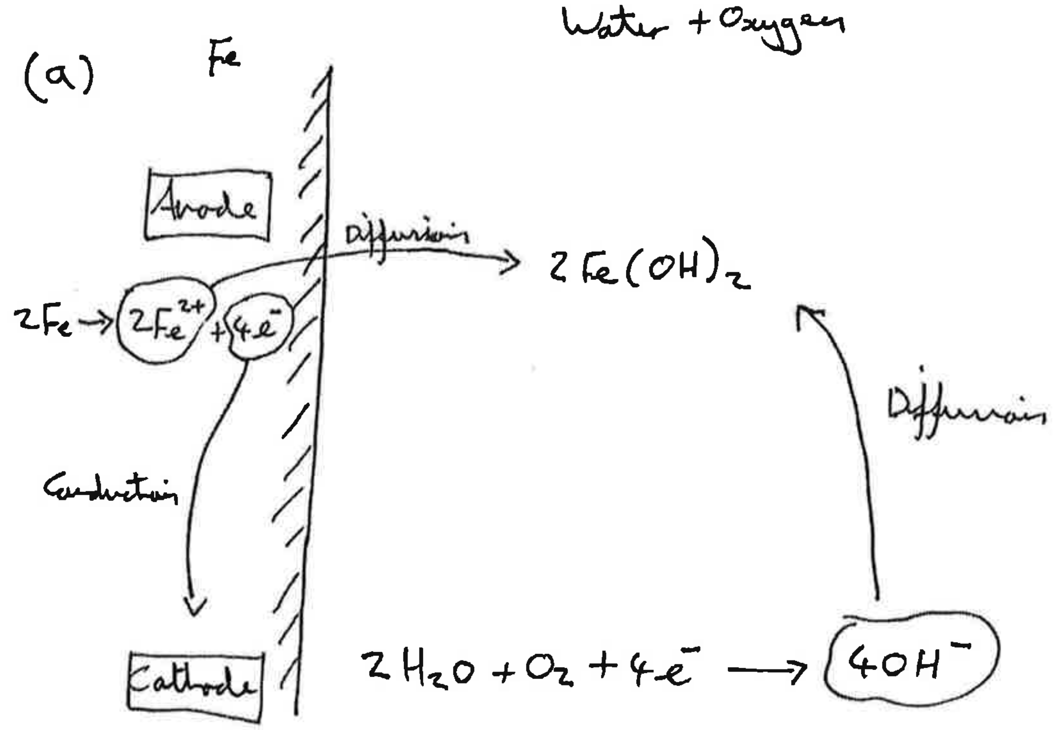
$$\frac{1+\nu}{1-\nu} = \frac{1.01 - 1}{(20\times10^{-6})(245)} = 2.04$$

$$\therefore \nu(1+2.04) = 2.04 - 1$$

$$\therefore \quad \nu = 0.342 \qquad [6]$$

Q8 (SHORT)

(a)



[6]

(b) (i)
- Driving force for net corrosion is the electro-chemical potential of the metal (relative to a standard hydrogen electrode)
- The more negative, the larger the driving force
- Gold has a <u>positive</u> electrochemical potential => <u>no driving force</u> for corrosion

(ii)
- Aluminium has a more negative potential than Fe – driving force for corrosion is higher.
- In air, Al_2O_3 (aluminium oxide) forms, which is an <u>effective barrier</u> (unlike Fe_2O_3)
- Obstructs diffusion, preventing net corrosion

[4]

Q9 (SHORT)

(a) • "Weakest link" idea: fracture of a brittle material depends on the largest flaw present in the sample

• The larger the volume, the higher the chance that there is a flaw above a critical size, therefore the lower the probability of survival subject to a given stress. [3]

(b) Uniform stress:

$$P_s(V) = \exp\left[-\left(\frac{\sigma}{\sigma_0}\right)^m \frac{V}{V_0}\right]$$

$$\therefore \frac{\ln(P_{s1})}{\ln(P_{s2})} = \left(\frac{\sigma_1}{\sigma_2}\right)^m \left(\frac{V_1}{V_2}\right) \qquad \text{(2 sets of data)}$$

$$\therefore m = \frac{\ln\left[\frac{V_2}{V_1}\frac{\ln(P_{s1})}{\ln(P_{s2})}\right]}{\ln\left(\frac{\sigma_1}{\sigma_2}\right)}$$

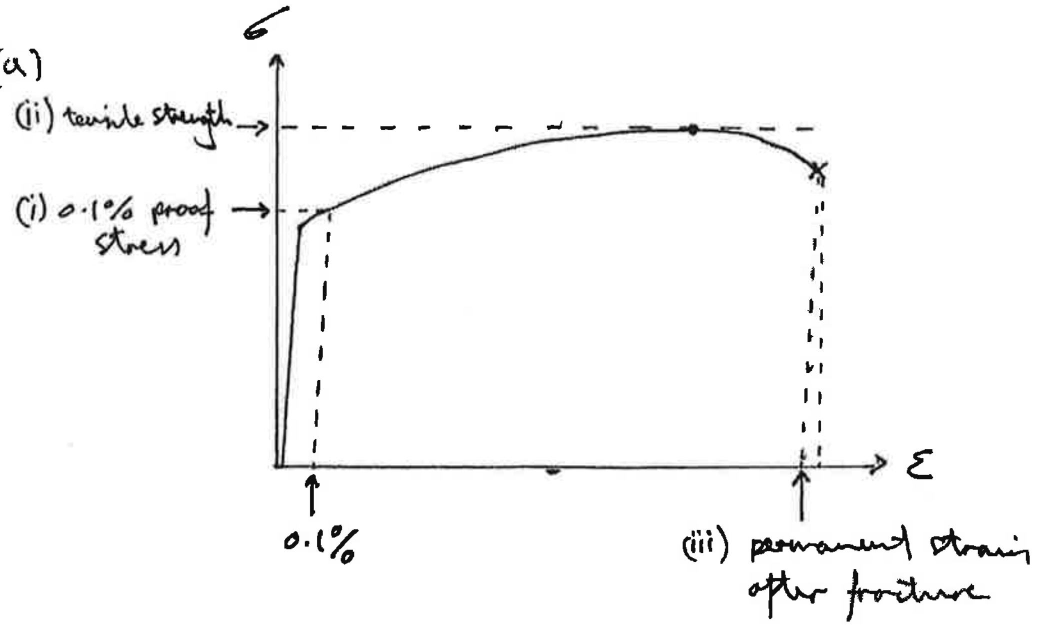
$$\therefore m = \frac{\ln\left[\frac{150}{120}\frac{\ln(0.66)}{\ln(0.81)}\right]}{\ln\left(\frac{90}{95}\right)} = \underline{10.0} \qquad [5]$$

(c) Survival probability increases:

• lower average stress
• lower volume of material subjected to tensile stress [2]

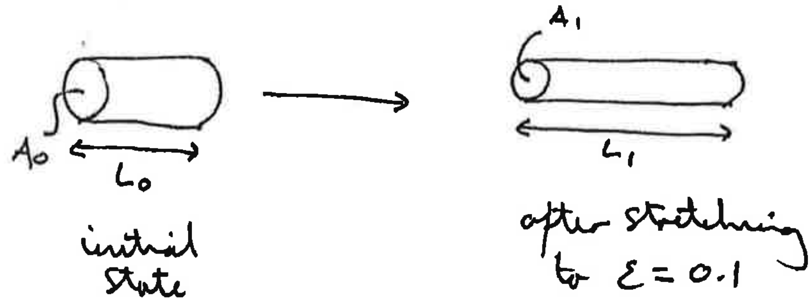
Q10 (SHORT)

(a)



[3]

(b)



- Nominal stress at unloading:

$$\sigma_n = \frac{F}{A_0} = 190 \text{ MPa} \quad \text{(from Fig. 8)}$$

- Cross sectional area at unloading:

$$V = A_1 L_1 = A_0 L_0 \quad \text{(volume conserved)}$$

$$\varepsilon_n = \frac{L_1 - L_0}{L_0} = 0.1 \quad \text{(nominal strain of 0.1)}$$

$$\therefore A_1 = \frac{A_0}{1+\varepsilon_n} = \frac{A_0}{1.1}$$

- Re-loading: yield strength of deformed specimen: [4]

$$\sigma_y = \frac{F}{A_1} = \sigma_n(1+\varepsilon_n) = 1.1\,\sigma_n = \underline{209 \text{ MPa}}$$

Q10 CONT.

(c). • The "ideal strength" corresponds to the applied stress required to break all atomic bonds (dissociation).

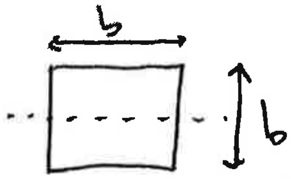
• The glide of dislocations in a metal requires only local breaking and reforming of atomic bonds.

• The material therefore deforms plastically (yields) at an applied stress that is significantly lower than the ideal strength.

[3]

Q11 (LONG)

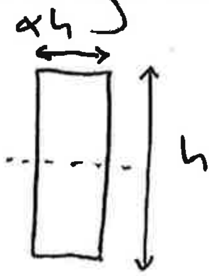
(a) (i) Square reference cross section :



Area : $A = b^2$

$I_{REF} = \frac{b^4}{12} = \frac{A^2}{12}$

Rectangular cross section :

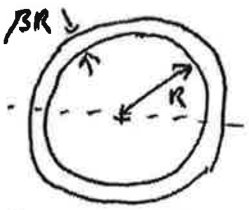


Area : $A = \alpha h^2$ $\quad \therefore I_{REF} = \frac{\alpha^2 h^4}{12}$

$I = \frac{\alpha h^4}{12}$ (structures data book)

$\therefore \Phi_e = \frac{I}{I_{REF}} = \frac{1}{\alpha}$

Tubular cross section :



(thin walled, $\beta \ll 1$)

Area : $A = 2\pi R(\beta R)$ $\quad \therefore I_{REF} = \frac{4\pi^2\beta^2R^4}{12}$

$I = \pi R^3(\beta R)$ (structures data book)

$\therefore \Phi_e = \frac{I}{I_{REF}} = \frac{3}{\pi\beta}$

[8]

(ii)

Material	Φ_e(MAX)	α(MIN) $= \frac{1}{\Phi_e(\text{MAX})}$	α(MIN) $= \frac{3}{\pi\,\Phi_e(\text{MAX})}$
Steel	64	0.0156	0.0149
Al Alloy	49	0.0204	0.0195
CFRP	36	0.0278	0.0265

$\boxed{\text{Q11 CONT.}}$

- Maximum shape factor is related to limitations on section thickness:
  - buckling failure can occur for very thin sections
  - manufacturing limitations
- Maximum shape factor is material dependent:
  - Young's modulus influences buckling failure
  - material type and manufacturing route are closely linked

[6]

(iii) Performance index, thin walled tube:

- Objective: mass $\quad m = \rho\,(2\pi/3\,R^2)\,L = \rho\left(\dfrac{6R^2}{\Phi_e}\right)L \quad ①$

- Constraint: deflection

$$\delta_{MAX} = \frac{FL^3}{3EI} = \frac{FL^3}{3E\,(\pi/3\,R^4)} = \frac{\Phi_e FL^3}{9ER^4} \quad ②$$

- Eliminate free variable, R:

$$② \rightarrow \quad R^2 = \left(\frac{\Phi_e FL^3}{9E\,\delta_{MAX}}\right)^{\frac{1}{2}}$$

$$① \rightarrow \quad m = \frac{6\rho L}{\Phi_e}\left(\frac{\Phi_e FL^3}{9E\,\delta_{MAX}}\right)^{\frac{1}{2}} = \frac{\rho}{\sqrt{E\,\Phi_e}}\left(\frac{2F^{\frac{1}{2}}L^{\frac{5}{2}}}{\delta_{MAX}^{\frac{1}{2}}}\right)$$

$\therefore$ Maximise index $\boxed{\dfrac{\sqrt{E\,\Phi_e}}{\rho}}$

[8]

Q11 CONT.

(iv) Material selection :

Material	$\sqrt{\frac{E\,\Phi_e(MAX)}{\rho}}$	
Steel	482	(0.0153 if E in GPa)
Al Alloy	686	(0.0217 " ")
CFRP	$\boxed{1265}$	(0.040 " ")

CFRP is the best choice

[4]

(b) Strength limited design :

$$\sigma_{MAX} = \frac{(FL)\,y_{MAX}}{I}$$

$\therefore$ need to shape so as to maximise $\frac{I}{y_{MAX}} = Z_e$

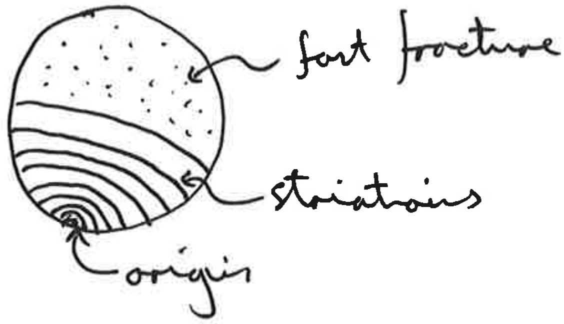
$$\therefore \quad \Phi_f = \frac{(I/y_{MAX})}{(I/y_{MAX})_{REF}} = \frac{Z_e}{Z_{e\,REF}}$$

Components in tension :

- stiffness and maximum stress depends only on <u>cross-sectional area, A</u> (not shape)

[4]

Q12 (LONG)

(a) 

[3]

(b) • Find the constants in Basquin's Law for the lab tests:

$$(N_f)^{\alpha} \Delta\sigma = C_1 \qquad \therefore \frac{\Delta\sigma_1}{\Delta\sigma_2} = \left(\frac{N_{f2}}{N_{f1}}\right)^{\alpha}$$ 2 sets of data

$$\therefore \alpha = \frac{\ln\left(\frac{\Delta\sigma_1}{\Delta\sigma_2}\right)}{\ln\left(\frac{N_{f2}}{N_{f1}}\right)} = \frac{\ln\left(\frac{35}{4}\right)}{\ln\left(\frac{5\times10^{10}}{7\times10^{6}}\right)} = \underline{0.244}$$

$$\therefore C_1 = (N_{f1})^{\alpha}\Delta\sigma_1 = \underline{1649 \text{ MPa}}$$

• Apply Goodman's rule to convert the component's stress amplitude to the equivalent at zero mean stress:

$$\Delta\sigma_0 = \frac{\Delta\sigma}{\left(1 - \frac{\sigma_m}{\sigma_{ts}}\right)} = \frac{20}{1 - \frac{150}{550}} = \underline{27.5 \text{ MPa}}$$

• Use Basquin's Law to predict the fatigue life of the component:

$$N_f = \left(\frac{C_1}{\Delta\sigma_0}\right)^{\frac{1}{\alpha}} = \left(\frac{1649}{27.5}\right)^{\frac{1}{0.244}} = \underline{18.8\times10^{6} \text{ cycles}}$$

[12]

Q12 CONT.

(c) Crack growth from a_1 to a_2:

$$\frac{da}{dN} = A\left(\Delta\sigma\, 1.12\sqrt{\pi a}\right)^4$$

$$\int_{a_1}^{a_2} \frac{da}{a^2} = A\left(\Delta\sigma\, 1.12\sqrt{\pi}\right)^4 \int_{N_1}^{N_2} dN$$

$$\left[-\frac{1}{a}\right]_{a_1}^{a_2} = (5\times10^{-12})(20\times1.12\sqrt{\pi})^4 \left[N\right]_{N_1}^{N_2}$$

$$\frac{1}{a_1} - \frac{1}{a_2} = (12.42\times10^{-6})\,\Delta N$$

For $a_1 = 2$ mm, $a_2 = 4$ mm:

$$\Delta N = \frac{\frac{1}{2}-\frac{1}{4}}{12.42\times10^{-6}} = \underline{20.1\times10^3 \text{ cycles}}$$

For $a_1 = 4$ mm, $a_2 = 6$ mm:

$$\Delta N = \frac{\frac{1}{4}-\frac{1}{6}}{12.42\times10^{-6}} = \underline{6.71\times10^3 \text{ cycles}}$$

- Crack growth rate accelerates, because for a given $\Delta\sigma$, ΔK increases as the crack length a increases.

[10]

(d) Initial crack depth corresponding to $\frac{da}{dN} = 0.001$ mm/cycle:

$$0.001 = A\left(1.12\Delta\sigma\sqrt{\pi a_{CRIT}}\right)^4$$

$$\therefore a_{CRIT} = \sqrt{\frac{0.001}{5\times10^{-12}}}\,\frac{1}{\pi(1.12\times20)^2} = \underline{8.97 \text{ mm}}$$

Q12 CONT.

Proof test stress, to cause fast fracture if $a \geq 8.97$ mm :

$$\sigma = \frac{K_{IC}}{1.12\sqrt{\pi a_{CRIT}}} \qquad (\text{i.e. } K = K_{IC})$$

$$\therefore \quad \sigma = \frac{50 \times 10^6}{1.12\sqrt{\pi \times 8.97 \times 10^{-3}}} = \underline{266 \text{ MPa}}$$

[5]

n.b. switching back to units of Pa and m (MPa and mm only required for Paris Law)