T.A

2011 Part 1A P4 Mathematical Methods

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Q1

$$|z+1| + |z-1| = 4$$

$$|z+1| = 4 - |z-1|$$

$$\sqrt{(x+1)^2+y^2} = 4 - \sqrt{(x-1)^2+y^2}$$

$$(x+1)^2 + \cancel{y^2} = 16 - 8\sqrt{(x-1)^2+y^2} + (x-1)^2 + \cancel{y^2}$$

$$4x - 16 = -8\sqrt{(x-1)^2+y^2}$$

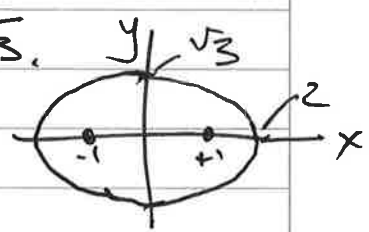
$$x - 4 = -2\sqrt{(x-1)^2+y^2}$$

$$x^2 - 8x + 16 = 4(x^2 - 2x + 1 + y^2)$$

$$12 = 3x^2 + 4y^2 = \text{ellipse}$$

Semi Major Axis $= 2$ Semi-minor axis $= \sqrt{3}$.

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$



OR: Ellipse, by inspection: locus of point, the sum of whose distance from two points, the foci, is constant.

Q2 (i)

$$\lim_{x\to 0} \frac{\sqrt{1+x^2} - 1 - \frac{x^2}{2}}{x^4}$$

$$\sqrt{1+x^2} = 1 + \tfrac{1}{2}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^4 \ldots$$

Numerator $= -\frac{1}{8}x^4 + \ldots$

Denominator $= x^4$ $\qquad$ limit $x \to 0 = -\frac{1}{8}$.

(ii)

$$\lim_{x\to 1}\left[\frac{\tan^{-1}x - \frac{\pi}{4}}{x - \sin\frac{\pi x}{2}}\right].$$

Put $x = 1+t$

$$f(x+a) = f(a) + x\left.\frac{df}{dx}\right|_{x=a} : \tan^{-1}(1+t) = \tan^{-1}1 + \frac{t}{1+1^2} + \ldots$$

$$= \frac{\pi}{4} + \frac{t}{2}$$

Denominator $= 1 + t - \sin\frac{\pi}{2}\cos\frac{\pi t}{2} - \cos\frac{\pi}{2}\sin\frac{\pi t}{2}$

$$= 1 + t - 1 - 0 = t$$

$$\text{limit } t \to 0 = \frac{1}{2}$$

Q3: $$\left|A-\lambda I\right| = \left|\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 0 \\ 0 & 1 & -\lambda \end{bmatrix}\right| \equiv \lambda^3 = 1$$

$\lambda = 1$, exp(i2π/3), exp(-i2π/3) .

Substitute back with $\lambda = 1$: eigenvector = $\frac{1}{\sqrt{3}}[1,1,1]$

$|A| = +1$

≡ Rotation by $\frac{2\pi}{3}$ about the axis [111].

Or = two successive rotations around x & y axes by 90°.

Q4. (a) $$\underset{\sim}{r} = \begin{pmatrix} 4 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

$\underset{\sim}{a}$

$$\underset{\sim}{r} = \begin{pmatrix} -7 \\ -2 \\ 4 \end{pmatrix} + s\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$$

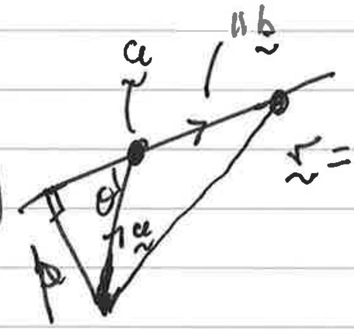
$\underset{\sim}{b}$

Common normal
$$\underset{\sim}{n} = \begin{vmatrix} i & j & k \\ 2 & 1 & -1 \\ 3 & 2 & 1 \end{vmatrix} = 3i - 5j + k$$

$$\hat{n} = \frac{1}{\sqrt{35}}\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}$$

Shortest distance = $\hat{\underset{\sim}{n}}\cdot(\underset{\sim}{a} - \underset{\sim}{b})$
$$= \frac{1}{\sqrt{35}}\begin{pmatrix} 3 \\ -5 \\ 1 \end{pmatrix}\cdot\begin{pmatrix} 11 \\ 0 \\ 2 \end{pmatrix} = \frac{35}{\sqrt{35}} = \sqrt{35}.$$

(b)



$\underset{\sim}{r} = \underset{\sim}{a} + s\underset{\sim}{d}$

$|\underset{\sim}{a} \times \hat{\underset{\sim}{d}}| = |\underset{\sim}{a}|\sin\theta = h$ = shortest distance

$$p = \frac{|\underset{\sim}{a} \times \underset{\sim}{d}|}{|\underset{\sim}{d}|}.$$

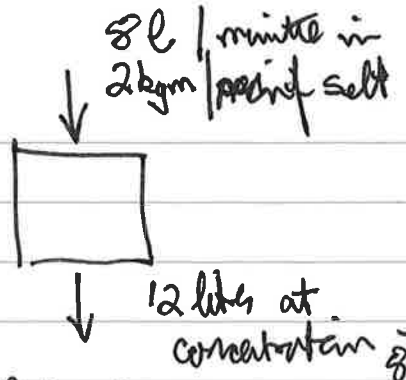
Also: Component of $\underset{\sim}{a} \parallel \underset{\sim}{d} = \frac{\underset{\sim}{a}\cdot\underset{\sim}{d}}{|\underset{\sim}{d}|^2}\underset{\sim}{d}$

the rest is perpendicular, so
$$p = \left|\underset{\sim}{a} - \frac{\underset{\sim}{a}\cdot\underset{\sim}{d}}{|d|^2}d\right|$$

(c) $\underset{\sim}{d} \times (\underset{\sim}{a} \times \underset{\sim}{d}) = \underset{\sim}{d}\cdot\underset{\sim}{d}\,\underset{\sim}{a} - (\underset{\sim}{d}\cdot\underset{\sim}{a})\,\underset{\sim}{d} \Rightarrow \frac{|\underset{\sim}{a}\times\underset{\sim}{d}|}{|d|^2} = \left|\underset{\sim}{a} - \frac{\underset{\sim}{a}\cdot\underset{\sim}{d}}{|d|^2}\underset{\sim}{d}\right|$

↳ Direction: from origin to shortest distance to line.

Q5:



After t: Volume $= 80-4t$

(a) Rate of salt increase = rate of salt in − rate of salt out

$$\frac{dx}{dt} = 2 - \frac{12x}{80-4t}$$

or $$\frac{dx}{dt} + \frac{3x}{20-t} = 2 \qquad 0<t<20.$$

(b) $\frac{dx}{dt} + f(t)x = g(t)$: Integrating factor $= e^{\int f(t)dt}$

$f(t) = \frac{3}{20-t}$ $\quad \int \frac{3\,dt}{20-t} = -3\ln(20-t) = (20-t)^{-3}$.

$$(20-t)^{-3}\frac{dx}{dt} + 3(20-t)^{-4}x = 2(20-t)^{-3}$$

$$\frac{d}{dt}\left((20-t)^{-3}x\right) = 2(20-t)^{-3}$$

$$(20-t)^{-3}x = (20-t)^{-2} + \text{constant}$$

$$\therefore\ x(t) = (20-t) + A(20-t)^3$$

$t=0,\ x=0 \Rightarrow A = -\frac{1}{400}$

$$x(t) = (20-t)\left[1 - \left(\frac{20-t}{20}\right)^2\right]$$

$t=10$: $\quad x(10) = (20-10)\left[1-\left(\frac{10}{20}\right)^2\right] = 7.5$ kgm.

Maximum: $\frac{dx}{dt} = 0$ when $3x = 2(20-t)$

$$\tfrac{2}{3}(20-t) = (20-t)\left[1-\left(\frac{20-t}{20}\right)^2\right]$$

$$1-\left(\frac{20-t}{20}\right)^2 = \frac{2}{3}$$

$$t = 20\left(1-\frac{1}{\sqrt{3}}\right) = 8.4 \text{ min}.$$

Substitute: $\quad x(8.4) = \frac{40}{3\sqrt{3}} = 7.7$ kgm.

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Dr T.P. Hynes

Section B

6 a) Put $\tau = t - \tau' \Rightarrow t - \tau = \tau' \quad d\tau = -d\tau'$

$$\Rightarrow \int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau = \int_{\tau'=t}^{0} f(t-\tau')\,g(\tau')\,d(-\tau')$$

$$= \int_{\tau=0}^{t} f(t-\tau)\,g(\tau)\,d\tau \quad \text{when } \tau' \text{ renamed as } \tau.$$

Aliter:

$$\text{Laplace T}\left(\int_{\tau=0}^{t} f(\tau)\,g(t-\tau)\,d\tau\right) = F(s)\,G(s)$$

$$= G(s)\,F(s) = \text{Laplace T}\left(\int_{\tau=0}^{t} g(\tau)\,f(t-\tau)\,d\tau\right)$$

b) Step: Solve $\ddot{y} + 2\dot{y} + y = 1$ with $y(0) = \dot{y}(0) = 0$

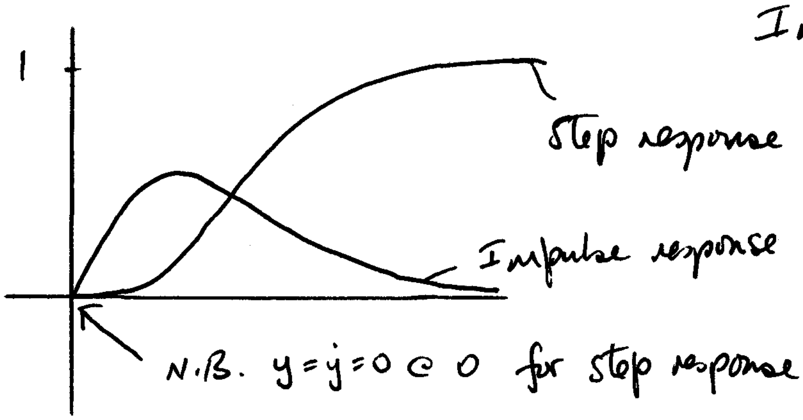
C.F. Put $y = e^{\lambda t} \Rightarrow \lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1$ (twice)

P.I. Put $y = c \Rightarrow c = 1$

$\therefore \; y = 1 + (At+B)e^{-t}$

B.C. $y(0) = 0 \Rightarrow 1 + B = 0 \quad \dot{y}(0) = 0 \Rightarrow A - B = 0$

$\therefore$ Step response $\underline{y = 1 - e^{-t} - te^{-t}}$



Impulse response $= \frac{d}{dt}$(step resp)

$= e^{-t} - e^{-t} + te^{-t}$

$= \underline{te^{-t}}$

c) Using part (a)

$$y(t) = \int_{\tau=0}^{t} e^{-\alpha(t-\tau)} \tau e^{-\tau} d\tau = e^{-\alpha t} \int_{\tau=0}^{t} \tau e^{(\alpha-1)\tau} d\tau$$

$$= e^{-\alpha t} \left\{ \left[\frac{\tau e^{(\alpha-1)\tau}}{\alpha-1} \right]_0^t - \int_0^t \frac{e^{(\alpha-1)\tau}}{\alpha-1} d\tau \right\}$$

$$= e^{-\alpha t} \left\{ \frac{t e^{(\alpha-1)t}}{\alpha-1} - \frac{e^{(\alpha-1)t} - 1}{(\alpha-1)^2} \right\}$$

$$= \underline{\frac{t e^{-t}}{\alpha-1} + \frac{e^{-\alpha t} - e^{-t}}{(\alpha-1)^2}} .$$

(d) As $\alpha \to 1$, let $\alpha = 1+\varepsilon$ & $\varepsilon \to 0$

$$\Rightarrow y(t) = \frac{t e^{-t}}{\varepsilon} + \frac{e^{-t-\varepsilon t} - e^{-t}}{\varepsilon^2} = \frac{e^{-t}}{\varepsilon} \left\{ t + \frac{1 - \varepsilon t + \frac{\varepsilon^2 t^2}{2} + \dots - 1}{\varepsilon} \right\}$$

$$= \frac{t^2 e^{-t}}{2}$$

Aliter:

From convolution Integral $y(t) = \int_0^t e^{-(t-\tau)} \tau e^{-\tau} d\tau = e^{-t} \int_0^{\tau} \tau d\tau$

$$= \frac{t^2 e^{-t}}{2}$$

Aliter:

de l'Hopital $\dfrac{f(\alpha)}{g(\alpha)} = \dfrac{(\alpha-1) t e^{-t} + e^{-\alpha t} - e^{-t}}{(\alpha-1)^2}$

$$f'(\alpha) = t e^{-t} - t e^{-\alpha t} \qquad \text{(N.B. diff w.r.t. } \alpha\text{)}$$

$$g'(\alpha) = 2(\alpha-1)$$

$$f'(1) = g'(1) = 0$$

$$f''(\alpha) = t^2 e^{-\alpha t} \qquad g''(\alpha) = 2$$

$$\Rightarrow \quad \frac{f''(\alpha)}{g''(\alpha)} \to \frac{t^2}{2} e^{-t} \quad \text{as} \quad \alpha \to 1$$

Examiner's Note

322 attempts (out of 324 candidates) average 68%

Most candidates knew exactly what to do and did most of it correctly. Curve sketching was mediocre.

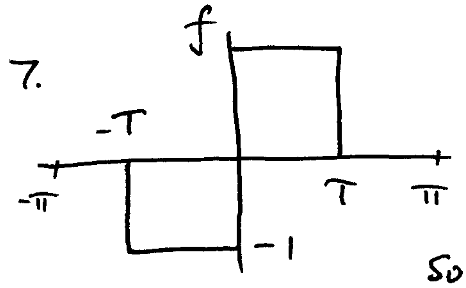
Commonest error: Expanding $e^{-\alpha t}$ in powers of α for part (d). α is tending to 1, so isn't small.

Brains not engaged: (i) A surprising number failed to tidy up the (correct) $g(t) = (1+t)e^{-t} - e^{-t}$

(ii) A significant number didn't realise that part (a) is really a hint for part (c)

Both (i) & (ii) suffered "death by convolution".

7.



f odd fn $\Rightarrow$ no cosines
no const (dc) term

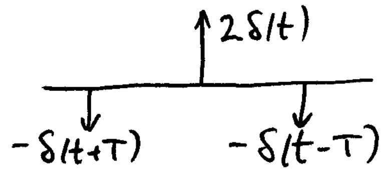
So that

$$f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

where
$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\sin nt\,dt$$
$$= \frac{2}{\pi}\int_0^T \sin nt\,dt = \frac{2}{\pi}\left[-\frac{\cos nt}{n}\right]_0^T$$
$$= \frac{2}{n\pi}(1-\cos nT)$$

$$\therefore f(t) = \sum_{n=1}^{\infty} \frac{2(1-\cos nT)}{n\pi}\sin nt$$

Aliter: $f'(t) =$



f' even with no const term $\Rightarrow$ $f'(t) = \sum_{n=1}^{\infty} a_n \cos nt$

with
$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[2\delta(t) - \delta(t-T) - \delta(t+T)\right]\cos nt\,dt$$
$$= \frac{2}{\pi}(1-\cos nT) \quad \text{since } \cos n(-T) = \cos nT$$

$$\Rightarrow f'(t) = \sum_{n=1}^{\infty} \frac{2}{\pi}(1-\cos nT)\cos nt$$

$$\Rightarrow f(t) = \sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos nT)\sin nt$$

Examiners Note:

318 attempts 71% average

odd/even function business handled brilliantly by the whole year.

Commonest error: The δ-fn approach needs to take a great deal of care about $t = 0$, when claiming $f'(t)$ is even. Many people said $f'(t)$ even

$$\Rightarrow \quad a_n = \frac{2}{\pi}\int_0^{\pi} \{2\delta(t) - \delta(t-T)\}\cos nt \, dt$$

$$= \frac{2}{\pi}(2 - \cos nT)$$

This is incorrect because the $2\delta(t)$ is right on the limit of integration i.e. only "half in". Shame.

Brains not engaged: A significant number of candidates didn't really read the question which said that $f(t)$ is defined on $-\pi < t < \pi$

$$\Rightarrow \text{range is } 2\pi \Rightarrow \text{series is } f(t) = \sum_{n=1}^{\infty} b_n \sin nt$$

period $= 2\pi$

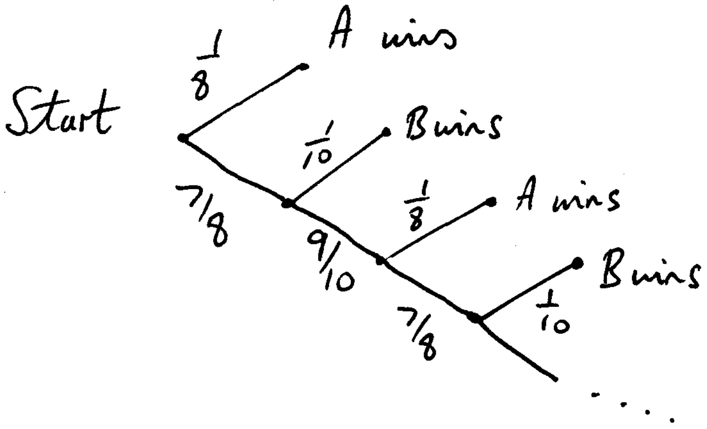
Many treated the function to be defined on $-T < t < T$

$$\Rightarrow \text{range is } 2T \Rightarrow \text{series is } f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{\pi n t}{T}$$

period $2T$

either that or they were blindly following the formulae in the databook without thinking about what T meant.

8)



(a) Prob no-one wins = Prob every dart misses bull

$$= \left(\frac{7}{8}\right)^6 \left(\frac{9}{10}\right)^6 \simeq .24$$

(b) $$\text{Prob (A wins)} = \frac{1}{8} + \frac{7}{8}\cdot\frac{9}{10}\cdot\frac{1}{8} + \left(\frac{7}{8}\cdot\frac{9}{10}\right)^2\cdot\frac{1}{8} + \cdots \left(\frac{7}{8}\frac{9}{10}\right)^5\frac{1}{8}$$

$$= \frac{1}{8}\,\frac{\left(1 - \left(\frac{63}{80}\right)^6\right)}{1 - 63/80}$$ G.P. formula from databook

$$\simeq .45$$

Examiner's Note

324 attempts (out of 324) Average 85%

Candidates murdered this question.

Commonest Error: There were hardly any.

Brains not engaged: Calculator problems.

Even when candidates wrote down $\left(\frac{7}{8}\right)^6\left(\frac{9}{10}\right)^6$ for part (a) ~~quite~~ a few had a 'senior moment' & got answers way adrift.

Numbers bigger than 1 or negative did not amuse the examiner.

9. Taking Laplace Transforms

$$sX - 1 + 3X = \frac{1}{s^2+1}$$

$$\Rightarrow \quad X = \left(1 + \frac{1}{s^2+1}\right)\frac{1}{s+3} = \frac{s^2+2}{(s+3)(s^2+1)}$$

$$= \frac{A}{s+3} + \frac{Bs+C}{s^2+1}$$

Cover up rule $(s=-3) \Rightarrow 11 = 10A \Rightarrow A = 11/10$

putting $s = 0 \Rightarrow \frac{2}{3} = \frac{A}{3} + C \Rightarrow C = \frac{3}{10}$

Equating coeff of s^2 in

$$s^2 + 2 = A(s^2+1) + (Bs+C)(s+3)$$

$$\Rightarrow \quad 1 = A + B \Rightarrow B = -\frac{1}{10}$$

$$\therefore \quad X = \frac{11}{10}\frac{1}{s+3} - \frac{1}{10}\frac{s}{s^2+1} + \frac{3}{10}\frac{1}{s^2+1}$$

$$\Rightarrow \quad x(t) = \frac{11}{10}e^{-3t} - \frac{1}{10}\cos t + \frac{3}{10}\sin t$$

Examiner's Note:

322 attempts 87% average

Another win for the students.

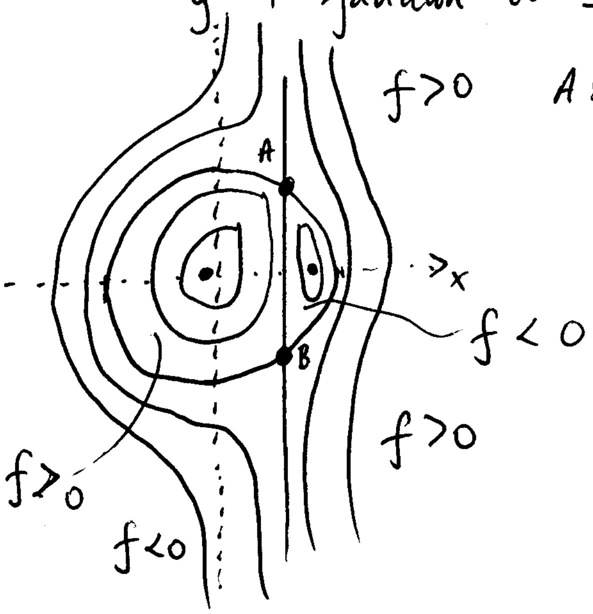
Brains not engaged:

Those students who took a moment to reflect on whether the answer looked right usually found algebraic errors, as did those who checked $x(0) = 1$

10.

(a) $f = 0 \Rightarrow x = 1$ or $x^2 + y^2 = 4$

+ function is symmetric in y



A & B contours cross $\Rightarrow$ saddle pts

$x = 1 \Rightarrow y = \pm\sqrt{3}$

Maximum (since $f > 0$) on $y = 0$ $x < 1$

Minimum (since $f < 0$) on $y = 0$ $x > 1$

(b) At stationary point $\nabla f = 0$

$$f = x^3 - x^2 + xy^2 - y^2 - 4x + 4$$

$$\frac{\partial f}{\partial x} = 3x^2 - 2x + y^2 - 4 \; ; \quad \frac{\partial f}{\partial y} = 2xy - 2y$$

$\frac{\partial f}{\partial y} = 0 \Rightarrow$ Either $x = 1$ or $y = 0$

(i) $y = 0$, $\frac{\partial f}{\partial x} = 0 \Rightarrow 3x^2 - 2x - 4 = 0$

$$\Rightarrow x = \frac{2 \pm \sqrt{4 + 48}}{6} \Rightarrow x = \frac{1+\sqrt{13}}{3} \text{ or } \frac{1-\sqrt{13}}{3}$$

$x = \frac{1+\sqrt{13}}{3}$ $y = 0$ is minimum (from sketch)

$x = \frac{1-\sqrt{13}}{3}$ $y = 0$ is maximum (.. ..)

(ii) $x = 1$ $\quad \frac{\partial f}{\partial x} = 0 \Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$

$x = 1 \quad y = \pm\sqrt{3}$ saddle points

(c) $\frac{\partial^2 f}{\partial x^2} = 6x - 2 \qquad \frac{\partial^2 f}{\partial y^2} = 2x - 2$

$\frac{\partial^2 f}{\partial x \partial y} = 2y$

x	y	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y^2}$	$\Delta = \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	
1	$\sqrt{3}$	4 (>0)	0	$4 \times 0 - 12$ (<0)	$\Rightarrow$ Saddle
1	$-\sqrt{3}$	4 (>0)	0	-12 (<0)	$\Rightarrow$ Saddle
$\frac{1+\sqrt{13}}{3}$	0	$2\sqrt{13}$ (>0)	$\frac{2\sqrt{13}-4}{3}$ (>0)	+ve × +ve (>0)	$\Rightarrow$ MIN
$\frac{1-\sqrt{13}}{3}$	0	$-2\sqrt{13}$ (<0)	$\frac{-2\sqrt{13}-4}{3}$ (<0)	−ve × −ve (>0)	$\Rightarrow$ MAX

Examiner's Note

322 attempts 69% average

Commonest errors: (i) Confusing axis lines with contours

(ii) using 2 conditions from $\frac{\partial f}{\partial x} = 0$ or $\frac{\partial f}{\partial y} = 0$ rather than one from each.

Otherwise very good solutions.

Brain not engaged:

$3x^2 - 4 - 2x = 0$

Taking $a = 3$ $b = -4$ $c = -2$ in quadratic formula.

J. H. 5th July '11

2011 Part IA Paper 4 Mathematical Methods J. Matheson

Section C

Qu 11)

a) Floating point numbers are expressed as an exponent and a mantissa, both stored in a fixed number of bits. The fixed number of bits for the exponent (IEEE single precision uses 8 bits) limits the range of numbers which can be represented and for the mantissa (IEEE single precision uses 23 bits representing 2-1 through 2^{-23}) limits the precision. In this case it is the latter which is the problem as `m` gets gradually closer to 1 but without sufficient precision that `t2` ever exactly equals `x` (tests for equality with floating point numbers are always dangerous). It will work for some values of `x`, but for others, `m` will become equal to 1 to the available precision before equality is achieved and the outer `while` loop will then loop for ever.

b) Changing the outer loop from

```
while (t2 != x) {
```

to

```
while (t2 != x && m  > 1) {
```

will fix the problem.

c) If the function is passed a negative number almost everything goes wrong! First "`if (x < 1)`" is always true so we end up using the algorithm intended for |x| < 1. Next "`while (t2 < x)`" is never true so we simply apply the correction for overshooting and divide `t` by `m` and continue to do this for each successive value of `m`. So provided the outer loop terminates, which the correction in (b) will ensure, the value returned (which is `1/t`) will be the product of the successive values of `m` (10, 5.5, 3.25, 2.125, etc) completely independently of `x`.

Examiner's Comments:

A generally well-answered question with candidates realising that the problem would have to do with the limited precision provided by the mantissa. Some were less good at spotting what line of code needed changing and some seemed to confuse "`if`" and "`while`". The answer given to (c) is fuller than required by the wording of the question but no one noticed that `t` would be independent of `x`. An alternative in (b) is

```
while (x - t2 > acc){
```

where `acc` is the achievable accuracy (and simplifies the answer to (c)!).

Qu 12)

a) Without structures we would need to represent the data as separate arrays for each of the fields within the structure, e.g.

```
int refno[10000];
int mileage_start[10000];
etc ...
```

and make sure by careful programming (throughout the program) that the `i`'th element of each of these arrays contained the data for the same car. With a structure, there is only the one array so this is guaranteed to be the case.

Without structures we would have to pass each of fields separately to a function needing the attributes of a car rather than being able to pass the single data item of type car.

b)

```
bool needs_service (int  refno)
{
   int n;

   n = 0;
   while (ourcars[n].refno != refno)
      n = n + 1;
   if (ourcars[n].mileage_back >= ourcars[n].last_service
                                  + ourcars[n].service_interval) {
       return (true);
   else
       return (false);
}
```

c) O(n)

Examiner's Comments:

Some very good answers but some very minimal, producing a bi-modal mark distribution. A too common mistake in (a) was that using structures made things in some way easier for the computer (rather than the programmer) or greatly reduced the memory used.

In (b) in a real program the while loop should contain protection against the `refno` not being in the array, i.e.

```
while (n < num_cars && ourcars[n].refno != refno)
```

but the question says we don't need to bother about this risk. And C++ would in fact allow

```
return (ourcars[n].mileage_back >= ourcars[n].last_service +
                                    ourcars[n].service_interval);
```

in place of the longer version given above.