ENGINEERING TRIPOS PART IA

Tuesday 14 June 2011 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (**short**) Find the equations in Cartesian coordinates (i.e. in the (x,y) plane) of the locus of the complex number z given by

$$|z+1| + |z-1| = 4$$

and sketch the locus in Cartesian coordinates. [10]

2 (**short**) Find the following limits

(a) $$\lim_{x \to 0} \left[\frac{\sqrt{1+x^2} - 1 - x^2/2}{x^4} \right]$$ [4]

(b) $$\lim_{x \to 1} \left[\frac{\tan^{-1} x - \pi/4}{x - \sin(\pi x/2)} \right]$$

[6]

3 (**short**) For the 3×3 matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

find all the eigenvalues and one of the eigenvectors. Describe the mapping represented by the matrix **A**. [10]

4 (**long**)

(a) Find the shortest distance between the two lines given by

$$\mathbf{r} = (4, -2, 3) + t(2, 1, -1) \text{ and } \mathbf{r} = (-7, -2, 1) + s(3, 2, 1)$$

[10]

(b) A line is given by the parametric equation

$$\mathbf{r} = \mathbf{a} + s\mathbf{d}$$

Show that the perpendicular distance of this line from the origin can be written in two forms:

$$p = \frac{|\mathbf{a} \times \mathbf{d}|}{|\mathbf{d}|} \quad \text{or} \quad p = \left| \mathbf{a} - \frac{\mathbf{a} \cdot \mathbf{d}}{\mathbf{d} \cdot \mathbf{d}} \mathbf{d} \right|$$

[10]

(c) By considering $\mathbf{d} \times (\mathbf{a} \times \mathbf{d})$ and sketching the result along with the line $\mathbf{r} = \mathbf{a} + s\mathbf{d}$, explain why the two expressions for p in part (b) are equivalent. [10]

5 (**long**) A tank initially contains 80 litres of pure water. From $t = 0$ minutes, a salt solution containing 0.25 kg of salt per litre flows into the tank from the top at a rate of 8 litres per minute. The contents of the tank are kept homogeneous by constant stirring and, again from $t = 0$, liquid is allowed to flow out of the bottom of the tank at 12 litres per minute.

(a) Determine the volume of liquid and the concentration of salt in the tank at time t ($0 < t < 20$ minutes), and hence show that the amount of salt $x(t)$ in kilograms in the tank is given by

$$\frac{dx}{dt} + \frac{3x}{20 - t} = 2 \quad (0 < t < 20)$$

[10]

(b) (i) Solve the equation for $x(t)$

(ii) Evaluate x at time $t = 10$ minutes.

(iii) Find the maximum value of $x(t)$ in the interval $0 < t < 20$ minutes. [20]

(TURN OVER

SECTION B

6 **(long)**

(a) Show that

$$\int_{\tau=0}^{t} f(\tau)g(t-\tau)d\tau \quad = \quad \int_{\tau=0}^{t} f(t-\tau)g(\tau)d\tau \qquad [3]$$

(b) A linear system is governed by the differential equation

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = f(t)$$

Find and sketch the step response and the impulse response of this system. [8]

(c) Using a convolution integral (and no other method), find the response of this system to an input $f(t)$ given by

$$f(t) = \begin{cases} e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

where α is a positive constant ($\alpha \neq 1$). [12]

(d) Find the response of the system for the case when $\alpha = 1$. [7]

7 **(short)** Find the Fourier Series representation of the function $f(t)$ over the range $-\pi \leq t \leq \pi$, where

$$f(t) = \begin{cases} -1 & -T \leq t < 0 \\ 1 & 0 \leq t \leq T \\ 0 & \text{otherwise} \end{cases}$$

for the case when the constant T satisfies $0 < T < \pi$. [10]

8 **(short)** Two darts players A and B have six darts each and throw alternately until one of them wins by hitting the bull. The probability of A hitting the bull with any throw is $\frac{1}{8}$, while that of B hitting the bull with any throw is $\frac{1}{10}$. If A throws first, find

(a) The probability that all darts are used with neither player winning [4]

(b) The probability that A wins. [6]

9 **(short)** Using Laplace transforms (and no other method), find the solution of the differential equation

$$\frac{dx}{dt} + 3x = \sin t$$

subject to the initial condition $x(0) = 1$. [10]

10 **(long)**

(a) Sketch contours of the function $f(x,y)$ given by

$$f(x,y) = (x-1)\left(x^2 + y^2 - 4\right)$$ [8]

(b) Find the stationary points of f and classify them using your sketch. [14]

(c) Verify that your classification agrees with that obtained using auxiliary conditions on appropriate combinations of second derivatives, which are given in the Maths Databook. [8]

(TURN OVER

SECTION C

11 (**short**)

The C++ function shown in Fig. 1 is intended to calculate the square root of the number passed as its argument. In testing, it is found that the function does not always operate correctly but instead runs forever without returning a value.

(a) Explain briefly how computers represent floating point numbers and hence why this problem occurs. [4]

(b) Identify which line in the program allows the function to run forever and suggest how it might be altered to prevent this problem. [3]

(c) What in general terms would happen if the value passed to this function were negative? (Detailed numerical analysis is not required) [3]

(cont.

```
float square_root (float x)
{
  float t = 1;                // estimate of square root
  float t2 = 1;               // estimate squared
  float m = 10;               // factor to adjust estimate by
  bool inv;                   // using 1/x

  if (x < 1) {                // consider 1/x instead of x
    x = 1/x;
    inv = true;
  }
  else
    inv = false;

  while (t2 != x) {           // estimate not yet the answer
    while (t2 < x) {          // estimate still too small
      t = t * m;              //   so adjust up
      t2 = t * t;
     }
    if (t2 != x)              // estimate overshot
      t = t / m;              //   so adjust back
    t2 = t * t;
    m = (m + 1) / 2;          // reduce factor by which to adjust estimate
  }

  if (inv)                    // have been using 1/x
    t = 1 / t;                //   so invert answer

  return (t);                 // return the answer
}
```

Fig. 1

12 (**short**)

The C++ declarations shown in Fig. 2 are used to contain the data on a fleet of up to 10,000 cars in a program to manage a car rental business.

(a) By considering how this data would have to be represented without using structures, explain why their use in C++ aids software maintainability by providing data abstraction. [3]

(b) Write a short C++ function, `needs_service`, to determine whether returned cars need servicing before they can be hired out again. The function should take the car's `refno` as an argument and return `true` if it needs servicing and `false` otherwise. You may assume that there is data in the array for this car. [5]

(c) What is the algorithmic complexity of `needs_service`? [2]

```
struct car {
  int refno;               // unique number identifying car
  int mileage_start;
  int mileage_back;
  int service_interval;    // distance between services in miles
  int last_service;        // mileage
  float rental_charge;
};

car ourcars[10000];        // data on up to 10,000 cars
int num_cars;              // how many cars there actually are
```

Fig. 2

END OF PAPER

Engineering Tripos Part IA 2011

Paper 4: Mathematical Methods

Short Answers

1. $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$

2. (a) $-1/8$ (b) $1/2$

3. Eigenvalues $1, -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$ Eigenvector $(1,1,1)$ or $\left(\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}\right)$

4. (a) $\sqrt{35}$

5. (a) Volume $= 80 - 4t$ litres, Concentration $\dfrac{x}{80-4t}$ kg/litre

   (b) $20 - t - \dfrac{(20-t)^3}{20^2}$ (c) $\dfrac{15}{2}$

   (d) $\dfrac{40}{3\sqrt{3}}$

6. (b) Step Response $1-(t+1)e^{-t}$ Impulse Response $t e^{-t}$

   (c) $\dfrac{e^{-\alpha t} - e^{-t}}{(\alpha-1)^2} + \dfrac{te^{-\alpha t}}{\alpha - 1}$ (d) $\dfrac{t^2}{2}e^{-t}$

7. $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n\pi}(1-\cos nT)\sin nt$

8. (a) .24 (b) .45

9. $\dfrac{11}{10}e^{-3t} - \dfrac{1}{10}\cos t + \dfrac{3}{10}\sin t$

10. Maximum $\left(\dfrac{1-\sqrt{13}}{3}, 0\right)$, Minimum $\left(\dfrac{1+\sqrt{13}}{3}, 0\right)$, Saddle Points $\left(1, \pm\sqrt{3}\right)$

11. (b) Change "`while (t2 != x)`" to "`while (t2 !x && m > 1)`"

12. (c) $O(n)$