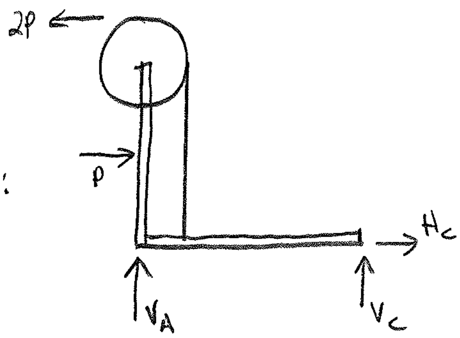
2012 - Part IA - P2 - Structures & Materials - Section A

M. DeJong

① FBD:

Global Equilibrium:



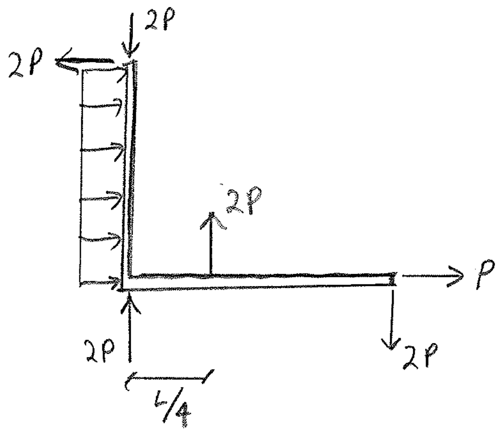
ΣF_H: $H_c = 2P - P = P$

ΣM_A: $V_c(L) + 2P\left(\frac{5}{4}L\right) - P\left(\frac{L}{2}\right) = 0$

$V_c = -2P$

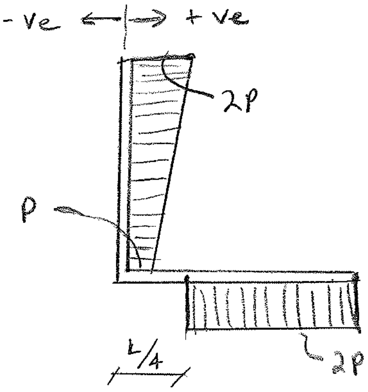
ΣF_V: $V_A = -V_c = 2P$

(b) FBD w/out pulley:

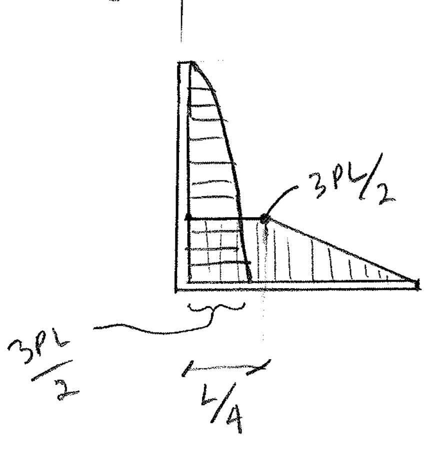


sign convention:

Shear Force Diagram:



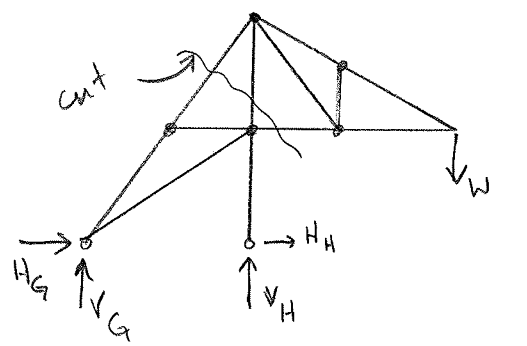
Bending Moment Diagram:



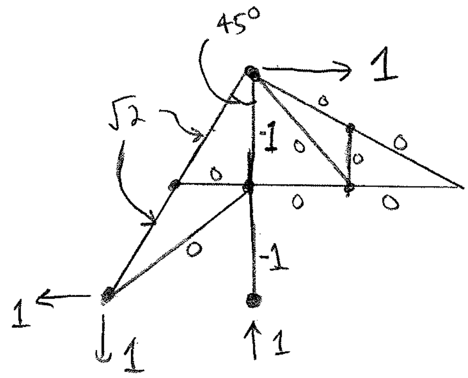
Bending Moment @ A = Area of trapezium

$$M_A = \frac{P+2P}{2}L = \frac{3PL}{2}$$

(2) Real:



Virtual:



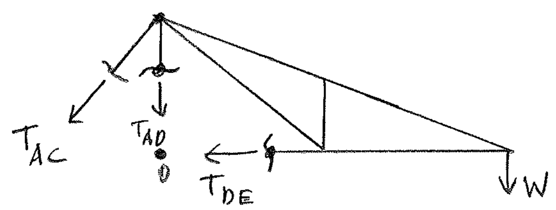
Global Equilibrium:

$H_H = 0$, $H_G = 0$

ΣM_G: $V_H (2L) = W(4L)$

$V_H = 2W$

FBD of "cut" portion:



ΣM_D: $T_{AC}\left(\frac{\sqrt{2}}{2}L\right) = W(2L)$

$\underline{\underline{T_{AC} = 2\sqrt{2}\,W}}$

ΣF_y: $-T_{AC}(\sin 45°) - T_{AD} - W = 0$

$\underline{\underline{T_{AD} = -3W}}$

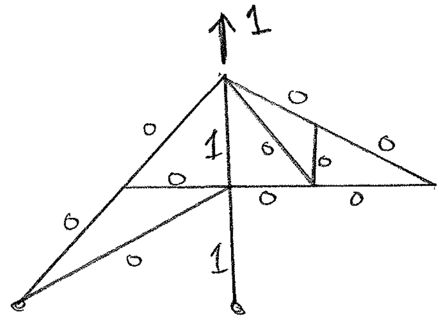
	T	L	e $\left(\times\frac{WL}{AE}\right)$	T*	T*e $\left(\times\frac{WL}{AE}\right)$
AC	$2\sqrt{2}W$	$\sqrt{2}L$	4	$\sqrt{2}$	$4\sqrt{2}$
CG	$2\sqrt{2}W$	$\sqrt{2}L$	4	$\sqrt{2}$	$4\sqrt{2}$
AD	$-3W$	L	-2	-1	2
DH	$-2W$	L	-3	-1	3

$$\underline{\underline{\delta_{AH} = \left(5 + 8\sqrt{2}\right)\frac{WL}{AE}}}$$

(2)(b) Vertical displacement @ A

<u>Real:</u> Same as before

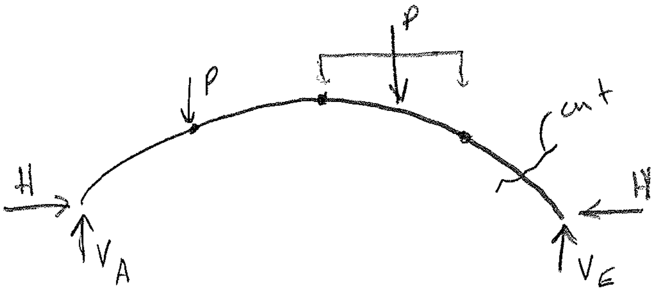
<u>Virtual:</u>



	$e\left(\times\frac{wL}{AE}\right)$	T^*	$T^*e\left(\times\frac{wL}{AE}\right)$
AD	-2	1	-2
DH	-3	1	-3

$$\delta_{AV} = -5\,\frac{wL}{AE}$$

③ (a) Global Equilibrium



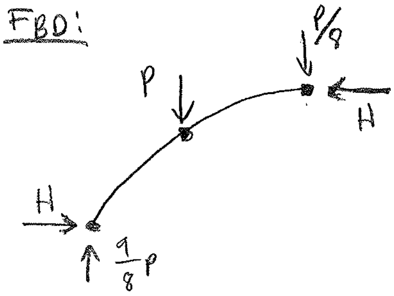
$\Sigma M_A: \; V_E(4L) - P\left(\frac{5}{2}L\right) - PL = 0$

$\underline{\underline{V_E = \frac{7}{8}P}}$

$\Sigma F_V: \; V_A + \frac{7}{8}P = P + P$

$\underline{\underline{V_A = \frac{9}{8}P}}$

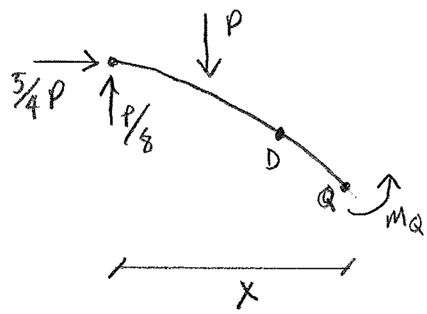
FBD:



$\Sigma M_A: \; H(L) - \frac{P}{8}(2L) - P(L) = 0$

$\underline{\underline{H = \frac{5}{4}P}}$

(b) FBD (cut portion)



$\Sigma M_Q = \; M_Q + P\left(x - \frac{L}{2}\right) - \frac{P}{8}x - \frac{5}{4}P(y) = 0$

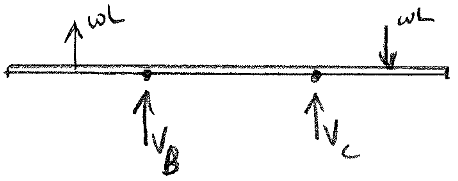
$M_Q = \frac{-7}{8}Px + \frac{PL}{2} + \frac{5}{4}P\left(\frac{x^2}{4L}\right)$

$M_Q = \frac{-7}{8}Px + \frac{PL}{2} + \frac{5}{16L}Px^2$

$\frac{dM_Q}{dx} = \frac{-7}{8}\cancel{P} + \frac{10}{16L}\cancel{P}x = 0 \quad \longrightarrow \quad x = \frac{7}{8}\frac{16}{10}L = \frac{7}{5}L$

$\underline{\underline{\text{Max magnitude @ } x = \frac{7}{5}L}}$

④ Global Equilibrium:



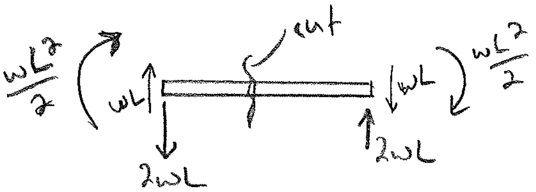
$$\Sigma M_B: \quad V_C L = wL\left(\frac{L}{2}\right) + wL\left(\frac{3L}{2}\right)$$

$$\underline{\underline{V_C = 2wL}}$$

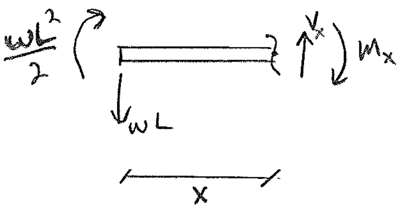
$$\Sigma F_y: \quad V_C + V_B = 0$$

$$\underline{\underline{V_B = -2wL}}$$

FBD of BC:



FBD of cut section:



ΣM_{cut}:

$$M_x + \frac{wL^2}{2} - wLx = 0$$

$$M_x = wLx - \frac{wL^2}{2} = -EI\frac{d^2v}{dx^2}$$

$$\frac{d^2v}{dx^2} = \frac{1}{EI}\left(-wLx + \frac{wL^2}{2}\right)$$

$$\frac{dv}{dx} = \frac{1}{EI}\left(-\frac{wLx^2}{2} + \frac{wL^2}{2}x\right) + C_1$$

$$v = \frac{1}{EI}\left(\frac{-wLx^3}{6} + \frac{wL^2x^2}{4}\right) + C_1x + C_2$$

BC's: $v=0$ @ $x=0$ $\longrightarrow$ $C_2 = 0$

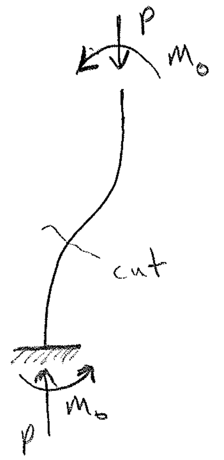
$v=0$ @ $x=L$ $\longrightarrow$

$$\frac{1}{EI}\left(-\frac{wL^{\cancel{4}}}{6} + \frac{wL^{\cancel{4}}}{4}\right) + C_1\cancel{L} = 0$$

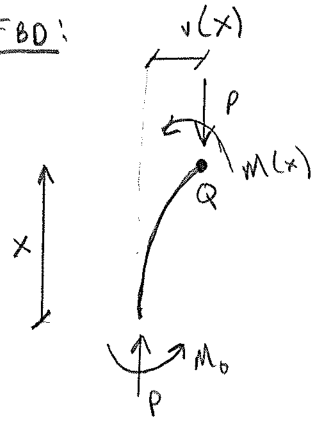
$$C_1 = \frac{1}{EI}\left(\frac{wL^3}{6} - \frac{wL^3}{4}\right) = \frac{-wL^3}{12\,EI}$$

$$\underline{\underline{v = \frac{1}{EI}\left(-\frac{wLx^3}{6} + \frac{wL^2x^2}{4} - \frac{wL^3}{12}\right)}}$$

(5) (a) Deformed shape:



(b) FBD:



$$\Sigma M_Q: \quad M(x) + M_o - P\big(v(x)\big) = 0$$

$$-EI \frac{d^2v}{dx^2} - P\,v(x) + M_o = 0$$

$$\frac{EI}{P} \frac{d^2v}{dx^2} + v(x) = \frac{M_o}{P}$$

General solution: $v = A\sin(\alpha x) + B\cos(\alpha x) + \dfrac{M_o}{P}$

where $\alpha^2 = \dfrac{P}{EI}$

@ $x=0$, $v=0$ $\rightarrow$ $0 = B + \dfrac{M_o}{P}$ $\rightarrow$ $\underline{\underline{B = \dfrac{-M_o}{P}}}$

@ $x=0$, $\dfrac{dv}{dx}=0$ $\rightarrow$ $\dfrac{dv}{dx} = \alpha A\cos(\alpha x) - \alpha B\sin(\alpha x)$

$0 = \alpha A$ $\rightarrow$ $\underline{\underline{A=0}}$

$$\therefore \quad \underline{\underline{v = \frac{-M_o}{P}\cos(\alpha x) + \frac{M_o}{P}}}$$

(c) @ $x=L$, $\dfrac{dv}{dx}=0$ $\rightarrow$ $0 = +\dfrac{M_o}{P}\sin(\alpha L)$ $\rightarrow$ $\sin(\alpha L) = 0$

$\alpha L = 0, \pi, 2\pi$

use $\alpha = \dfrac{\pi}{L}$

$$\alpha^2 = \frac{P}{EI}$$

$$\frac{\pi^2}{L^2} = \frac{P}{EI} \rightarrow \underline{\underline{P_E = \frac{\pi^2 EI}{L^2}}}$$

5(d) $L_e = \frac{L}{2}$ for fixed-fixed

$\therefore$ $P_E = 4\times$ larger for fixed-fixed

(e) (i) $P_E = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 (210 \times 10^9)(10^{-6})}{9} = 230.3$ kN

$\underline{W_{buckling} = 460.6 \text{ kN}}$ $\left(< W_{yield} = 550 \text{ kN} \right)$

(ii) $r = \sqrt{\frac{I}{A}} = \sqrt{\frac{10^{-6}}{10^{-3}}} = 0.032$ m

$\frac{L}{r} = \frac{3}{0.032} \approx 95$

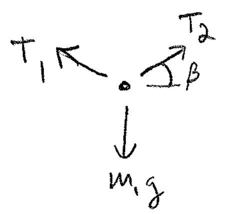
$\left(\frac{L}{r}\right)_{cr} = \sqrt{\frac{\pi^2 E}{\sigma_y}} = \sqrt{\frac{\pi^2 \; 210{,}000}{275}} \approx 87$

From graph, failure load is $\approx$ 50-60% of perfect strut

$\Rightarrow$ say 55%

$\underline{W_{failure} \approx 250 \text{ kN}}$

6 (a)



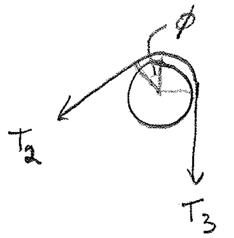
$$T_1 = T_2$$

$$2 T_1 \sin\beta = m_1 g$$

$$\underline{\underline{T_1 = \frac{m_1 g}{2\sin\beta}}}$$

$$\underline{\underline{T_{CD} = m_2 g}} = T_3$$

(b) Sliding around pipe:



From data book: $T_2 = T_1 e^{\mu\theta}$

$$\theta = \pi/2 + \beta$$

If m_2 goes down: (just sliding) $T_3 > T_2 \rightarrow T_3 = T_2 e^{\mu\theta}$

$$\frac{T_3}{T_2} = e^{\mu\theta}$$

$$\mu\theta = \ln\left(\frac{T_3}{T_2}\right)$$

$$\mu = \frac{1}{\theta} \ln\left(\frac{m_2 \cancel{g}}{m_1 \cancel{g}/2\sin\beta}\right)$$

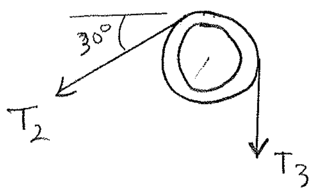
$$\therefore \quad \underline{\underline{\mu \geq \frac{1}{(\pi/2 + \beta)} \ln(2\sin\beta)}} \qquad \boxed{1}$$

If m_2 goes up: (just sliding)

$$\underline{\underline{\mu \geq \frac{1}{(\pi/2+\beta)} \ln\left(\frac{1}{2\sin\beta}\right)}} \qquad \boxed{2}$$

$\therefore$ Both $\boxed{1}$ & $\boxed{2}$ must be satisfied for equilibrium

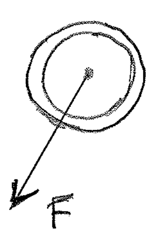
⑥ (c) For $\beta = 30°$: $\mu_{req'd} = 0 \rightarrow$ No friction.



$$T_2 = \frac{m_1 g}{2 \sin(30°)} = m_1 g = mg$$

$$T_3 = m_2 g = mg$$

||



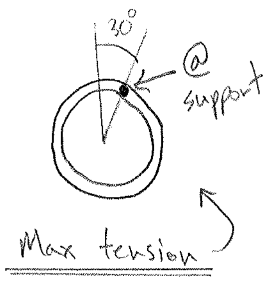
$$F = (mg \cos 30°)\,2 = \sqrt{3}\,mg$$

$$\therefore \; V_{max} = \sqrt{3}\,mg$$

$$M_{max} = \sqrt{3}\,mg \left(\frac{L}{2}\right) \quad \text{(at support)}$$

(ii) $\sigma_{max} = \dfrac{M y_{max}}{I}$ $\qquad I = \pi R^3 t$ (data book)

$$\sigma_{max} = \frac{\sqrt{3}\,mgL}{2}\;\frac{(R)}{\pi R^3 t} = \underline{\underline{\frac{\sqrt{3}}{2}\,\frac{mgL}{\pi R^2 t}}}$$

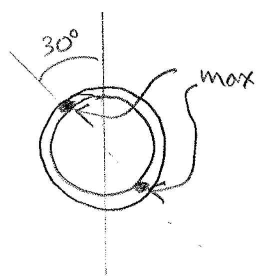


(iii) $q = \dfrac{S A_c \bar{y}}{I}$ $\rightarrow$ max @ neutral axis between support and cable

$$A_c = \pi R t$$

$$\bar{y} = R\,\frac{\sin \pi/2}{\pi/2} = \frac{2R}{\pi} \quad \text{(Mechanics data book)}$$

$$\tau\,(2t) = \frac{\sqrt{3}\,mg\,(\pi R t)\left(\frac{2R}{\pi}\right)}{\pi R^3 t} = \frac{2\sqrt{3}\,mg}{\pi R}$$



$$\therefore \; \tau_{max} = \underline{\underline{\frac{\sqrt{3}\,mg}{\pi R t}}}$$

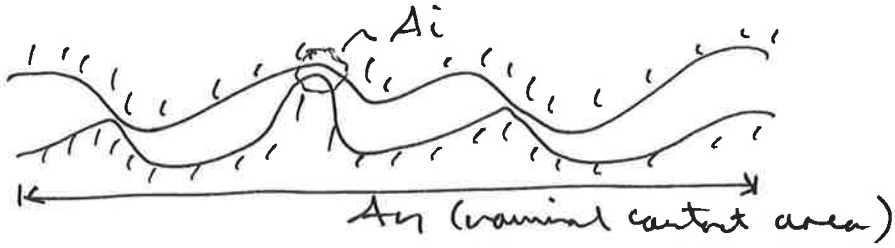
Engineering Tripos Part IA 2011-12

Solutions: Paper 2, Section B

Examiner: Dr GJ McShane

Q7

(a) Asperities touch:



Individual contact areas: A_i

True contact area: $A_t = \sum_i A_i \ll A_n$

Link to hardness: $A_i \approx \frac{P_i}{H}$, $A_t = \sum \frac{P_i}{H} = \frac{P}{H}$ [4]

($H \uparrow$, $A_t \downarrow$)

(b)(i) Frictional force at an asperity:

$F_i = A_i \tau_s$

Total frictional force: $F = \sum_i A_i \tau_s = A_t \tau_s$

Using: $A_t = \frac{P}{H} \Rightarrow F = \frac{P \tau_s}{H}$

$\therefore \mu = \frac{F}{P} = \frac{\tau_s}{H}$

(ii) If $\tau_s = k = \frac{1}{2}\sigma_y$, and $H \approx 3\sigma_y$

$\therefore \mu = \frac{(\sigma_y/2)}{(3\sigma_y)} = \frac{1}{6}$ (independent of metal)

(iii) $\tau_s = k$ doesn't work if:

- oil film lubricating interface
- oxide layer

[6]

Q8

(a) Cold drawing: molecules align with loading direction, unravel and slide relative to each other, until covalent bonds ("backbone" of molecule) dominate stiffness and strength. Stable neck propogation.

(i) Stiffness ↑, (ii) Strength ↑, (iii) Ductility ↓ [5]

(b) Strain ε is the same in fibre, matrix.

Total force: $F_x = (E_f \varepsilon) A_f + (E_m \varepsilon) A_m$

Stress (average): $\sigma_x = \dfrac{F_x}{A_f + A_m}$

$$= E_f \varepsilon \left(\frac{A_f}{A_f + A_m}\right) + E_m \varepsilon \left(\frac{A_m}{A_f + A_m}\right)$$

Volume fraction: $V_f = \dfrac{A_f \cancel{L}}{(A_f + A_m)\cancel{L}}$

$$V_m = \frac{A_m \cancel{L}}{(A_f + A_m)\cancel{L}} = 1 - V_f$$

Effective modulus: $E = \dfrac{\sigma_x}{\varepsilon} = E_f V_f + E_m (1 - V_f)$

[5]

Q9

(a) (i) K gives the amplitude of the singular stress field at a crack tip in a linear elastic body.

· Fracture : $K = K_{IC}$ (fracture toughness)

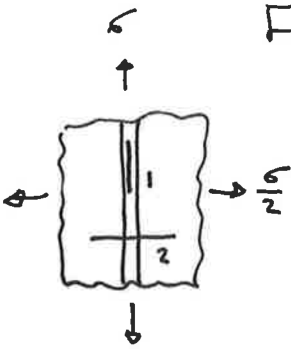
(ii) G gives the rate of release of elastic strain energy in the body per unit area of crack growth.

· Fracture : $G = G_{IC}$ (critical strain energy release rate)

[6]

(b) (i) Crack 1 propagates when: $K = \frac{\sigma}{2}\sqrt{\pi a_1} = \beta K_{IC}$

Crack 2 propagates when: $K = \sigma\sqrt{\pi a_2} = K_{IC}$



For crack 1 to propagate first:

$$\underbrace{\frac{2\beta K_{IC}}{\sqrt{\pi a_1}}}_{\sigma \text{ at fracture of crack 1}} < \underbrace{\frac{K_{IC}}{\sqrt{\pi a_2}}}_{\sigma \text{ at fracture of crack 2}} \qquad \therefore \quad \beta < \frac{1}{2}\sqrt{\frac{a_1}{a_2}}$$

[5]

(ii) For $a_1 = a_2$: required $\beta < \frac{1}{2}$ for crack 1 to be critical

$\therefore$ as $\beta = 0.75$, crack 2 will propagate first

$$\therefore \sigma = \frac{K_{IC}}{\sqrt{\pi a_2}} = \underline{276 \text{ MPa}}$$

· Validity : for low C steel, $K_{IC} = 60\text{MPa}\sqrt{m}$, $\sigma_y = 320\text{MPa}$ (both mid-range values), so will likely fracture before yield

But : process zone $r_p \sim \frac{1}{\pi}\left(\frac{K_{IC}}{\sigma_y}\right)^2 \approx 11\text{mm} \simeq \frac{2}{3}a$

$\therefore$ LEFM probably not reliable here

[5]

Q9 cont.

(c) (i) Bridging contribution:

Take: $F = -\sigma_f \, t \, dx$

Integrate (superposition) given result:

$$K = \int_{a-b}^{a} \frac{-\sigma_f}{\sqrt{\pi a}} \frac{2a}{\sqrt{a^2 - x^2}} \, dx$$

$$= -\frac{2a\sigma_f}{\sqrt{\pi a}} \int_{a-b}^{a} \frac{1}{\sqrt{a^2-x^2}} \, dx$$

given in maths data book (or substitute $\frac{x}{a} = \sin u$)

$$= -\frac{2a\sigma_f}{\sqrt{\pi a}} \left[\sin^{-1}\left(\frac{x}{a}\right) \right]_{a-b}^{a}$$

$$= -\frac{2a\sigma_f}{\sqrt{\pi a}} \left[\frac{\pi}{2} - \sin^{-1}\left(\frac{a-b}{a}\right) \right]$$

[8]

(ii) Total stress intensity factor:

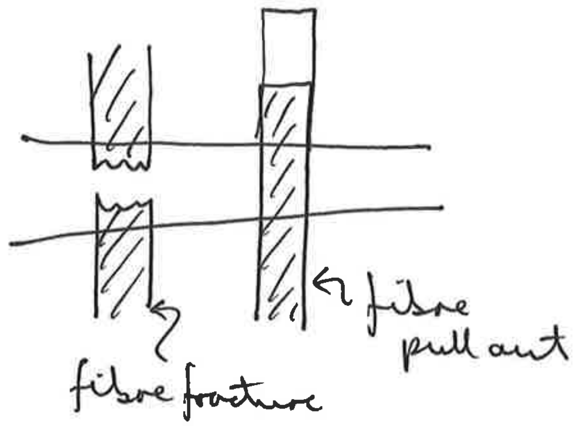
$$K = K_{(\sigma)} + K_{(\text{fibres})}$$

$\underline{b = 0}$: $K_{(\text{fibres})} = 0$ $\therefore$ $\underline{K = \sigma\sqrt{\pi a}}$

$\underline{b = a}$: $K_{(\text{fibres})} = -\sigma_f\sqrt{\pi a}$ $\therefore$ $\underline{K = (\sigma - \sigma_f)\sqrt{\pi a}}$

[4]

(iii)



dissipates energy
→ increases effective fracture toughness

[2]

$\boxed{Q10}$

(a) (i) Objective : $m = \rho L d^2$ ①

Constraint : $\delta = \dfrac{(\rho d L) L^3}{8 E I}$ ← data book

$I = \dfrac{d^4}{12}$ $\qquad \delta = \dfrac{3 \rho L^4}{2 E d^3}$ ②

Free variable : d

② → $d = \left(\dfrac{3 \rho L^4}{2 E \delta}\right)^{\frac{1}{3}}$

① → $m = \rho L \left(\dfrac{3}{2} \dfrac{\rho L^4}{E \delta}\right)^{\frac{2}{3}}$

$\therefore$ $M = \dfrac{E^{2/3}}{\rho}$ must be maximised [5]

(ii) 1. Ceramics – low fracture toughness

2. Composites – cost

3. Natural materials - size limitations [6]

(b) Objective : $C = C_m \rho L d^2$ ③

Deflection constraint : $\delta = \dfrac{3 \rho L^4}{2 E d^3}$ ④

③, ④ → $C = \left[\dfrac{C_m \rho}{E^{2/3}}\right] \left(\dfrac{3 \rho L^4}{2 \delta}\right)^{\frac{2}{3}} L$

Steel : $C = £38.95$

Al alloy: $C = £50.49$

$\boxed{\text{Q10 Cont.}}$

Stress constraint: $\sigma_y = \dfrac{M\left(\frac{d}{2}\right)}{I}$

$$M = \frac{p d L^2}{2}, \quad I = \frac{d^4}{12} \quad \therefore \quad \sigma_y = \frac{3pL^2}{d^2} \quad (5)$$

$$(3),(5) \rightarrow C = \left[\frac{C_m \rho}{\sigma_y}\right] 3 p L^3$$

Steel: $C = £4.88$

Al alloy: $C = £3.65$

- For both materials, <u>deflection</u> is critical constraint
- Best material (cheapest) = <u>steel</u>

[9]

(c) $\sigma_z = \dfrac{w}{b^2}$, $\sigma_x = \sigma_y = ?$

$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\left(\frac{\sigma_x}{E} + \frac{\sigma_z}{E}\right) = 0 \qquad \text{(constraint)}$$

$$\therefore \sigma_x = \frac{\nu \sigma_z}{1-\nu} = \left(\frac{\nu}{1-\nu}\right)\frac{w}{b^2}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\left(\frac{2\sigma_x}{E}\right) = \frac{w}{Eb^2}\left(1 - \frac{2\nu^2}{1-\nu}\right)$$

$$= \frac{w}{Eb^2}\left(\frac{1-\nu-2\nu^2}{1-\nu}\right)$$

Deflection: $\delta = \varepsilon_z h = \underline{\dfrac{wh}{Eb^2}\left(\dfrac{1-\nu-2\nu^2}{1-\nu^2}\right)}$

[10]

Q11

(a) (i) Metals : metallic bonding - atoms release electrons, becoming positively charged ions seperated by mobile free electrons.

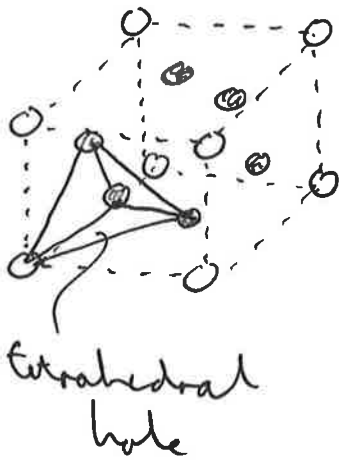
(ii) Ceramics : ionic bonding - electrons transfer permanently between atoms, which become attracting oppositely charged ions

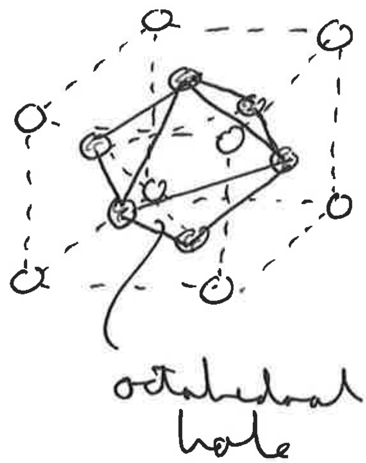
(iii) Polymers :
molecular chains - covalently bonded, atoms share electrons
between chains - van der Waals bonds, "secondary" bonds.

- Alumina : highest bond stiffness (ionic), hence largest E.
- Aluminium : intermediate bond stiffness (metallic), hence intermediate E.
- PE : modulus governed by secondary bonds, which have low stiffness

[6]

(b)

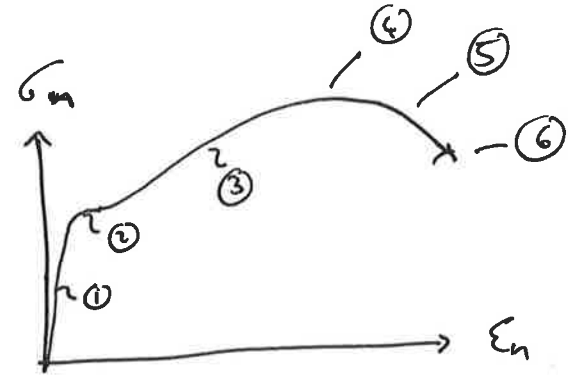




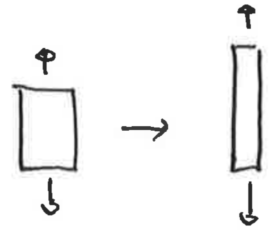
[4]

Q12

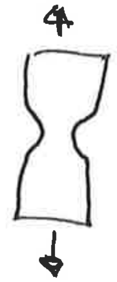
(a)



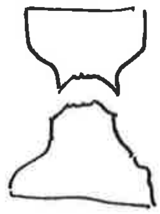
1. Elastic, uniform deformation
2. Yield – deformation still uniform
3. Stress increase:
   - deformation uniform



   - area reduction and work hardening both affect rise in σ_n
4. Onset of necking – deformation localises



5. Localised deformation causes fall in σ_n
6. Fracture of the specimen



[5]

(b) $\sigma_n = \frac{F}{A_0}$, $\varepsilon_n = \frac{L}{L_0} - 1$

$\sigma_t = \frac{F}{A}$ • Assume conservation of volume: $A_0 L_0 = AL$

$$\therefore \sigma_t = \frac{FL}{A_0 L_0} = \sigma_n (1 + \varepsilon_n)$$

Tensile strength: $\sigma_n = 670$ MPa, $\varepsilon_n = 0.18$

$$\therefore \underline{\sigma_t = 791 \text{ MPa}}$$

[5]