2012 PART IA PAPER 4 SECTION A MICHAEL KELLY

CRIBS

1: (a) Show: $\tanh^{-1} x = 0.5 \ln[(1+x)/(1-x)]$ $(-1<x<+1)$

Put $y=\tanh^{-1}x$: $x=\tanh y =(e^y - e^{-y})/(e^y+e^{-y})$
With $s=e^y$ $(s-1/s))/(s+1/s)=x$ $s^2=(1+x)/(1-x)$
Take logs: $y= \tanh^{-1} x = 0.5 \ln[(1+x)/(1-x)]$

(b) Show that $\cosh^{-1} x = \ln [(x + \sqrt{(x^2-1)}]$ $(x \geq 1)$
Put $y= \cosh^{-1} x$ $x=\cosh y = (e^y+e^{-y})/2$
With $s=e^y$ $2x=(s+1/s)$ $s^2-2sx+1= 0$ $s=0.5[x \pm\sqrt{(x^2-1)}]$
$y = \cosh^{-1} x = \ln [(x + \sqrt{(x^2-1)}]$

2: $z^4 =r^4e^{i4\theta}$ $\sqrt{z} =r^{1/2}e^{i\theta/2}$ and $\ln(z) = \ln r + i\theta$

$P(i)=0 = P(-i)$ Divide by (z^2+1)
So $P(z)= (z^2+1)(z^2+5z+6)$
The zeroes are i, -i, -2, and -3

3: Vector form of plane: $\mathbf{r}.(A,B,C)=D$. Vector normal to plane is $\mathbf{n}=(A,B,C)$

Eqn of line through (l,m,n) and perpendicular to the plane is
$$\mathbf{r} = (l,m,n) + t(A,B,C)$$
Meets plane when $\mathbf{r}.(A,B,C)= lA+ mB+nC+ t (A,B,C).(A,B,C) = D$

Or: $t= [D - (lA+ mB+nC)]/[A^2+B^2+C^2]$

The point on the plane N where this line meets the plane is
$$(l,m,n) + \{[D - (lA+ mB+nC)]/[A^2+B^2+C^2]\}(A,B,C)$$

The length of PN $= \{[D - (lA+ mB+nC)]/[A^2+B^2+C^2]\} \sqrt{[A^2+B^2+C^2]}$

Particular question: $(l,m,n)=(2,-1,3)$ $(A,B,C,)= (2,-2,-1)$ and $D= 9$

$T= [9 - (4+2-3)]/9 = 2/3X\,3 =2$

~~Q5~~ Q4 a) i) $\det(\underline{\underline{Q}}\,\underline{\underline{Q}}^t) = \det(\underline{\underline{I}}) = 1$ $\qquad \det \underline{\underline{Q}} = \det \underline{\underline{Q}}^t$

$$\Rightarrow (\det(\underline{\underline{Q}}))^2 = 1 \quad \Rightarrow \quad \underline{\underline{\det \underline{\underline{Q}} = \pm 1}}$$

ii) If $\underline{\underline{Q}} = \begin{pmatrix} u_1 & u_2 & u_3 \\ | & | & | \end{pmatrix}$ $\qquad Q^tQ = \begin{pmatrix} \leftarrow \underline{u}_1 \rightarrow \\ \leftarrow \underline{u}_2 \rightarrow \\ \leftarrow \underline{u}_3 \rightarrow \end{pmatrix} \begin{pmatrix} \uparrow & \uparrow & \uparrow \\ \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \\ \downarrow & \downarrow & \downarrow \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$\Rightarrow \underline{u}_i \underline{u}_j = \delta_{ij}$ ie <u>columns are an orthonormed set</u>.

(Also true for rows).

b) (i) $\det(\underline{\underline{A}} - \lambda \underline{\underline{I}}) = \det \begin{pmatrix} 3/5-\lambda & 0 & -4/5 \\ 0 & 1-\lambda & 0 \\ -4/5 & 0 & -3/5-\lambda \end{pmatrix} = 0$

EXPAND BY 2nd row $\Rightarrow$ $(1-\lambda)\left[(3/5-\lambda)(-3/5-\lambda) - 16/25\right] = 0$

$(1-\lambda)\left[\lambda^2 - 9/25 - 16/25\right] = 0$

$(1-\lambda)(\lambda^2-1) = 0 \qquad \underline{\underline{\lambda = 1, 1, -1}}$

Repeated e-value so expect eigen-plane!

$\underline{\underline{\lambda = 1}}$ (REPEATED) $\Rightarrow \begin{pmatrix} -2/5 & 0 & -4/5 \\ 0 & 0 & 0 \\ -4/5 & 0 & -8/5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\qquad x + 2z = 0$ & y = anything $\Rightarrow$

eigen plane – choose ~~tr~~ e-vectors $\Rightarrow$ $\underline{\underline{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}}$ & $\underline{\underline{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}}$
(note non-unique)

$\underline{\underline{\lambda = -1}} \Rightarrow \begin{pmatrix} 8/5 & 0 & -4/5 \\ 0 & 2 & 0 \\ -4/5 & 0 & 2/5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \underline{\underline{\begin{pmatrix} +1 \\ 0 \\ +2 \end{pmatrix}}}$

(ii) $\underline{\underline{A}}$ symmetric $\Rightarrow$ e-values are real, e-vectors for distinct e-vals are perpendicular & repeated e-vals (e-plane) can ~~be~~ choose e-vectors to be perpendicular.

$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0 \Rightarrow$ ~~tr~~ $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = 0$ ✓ $\qquad \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 0$ ✓

(iii) $\underline{\underline{A}}$ corresponds to reflection in a plane with normal $(1, 0, 2)^t$ (ie e-vector corresponding to $\lambda = -1$), other two e-values are both unit.

Q5

~~Q4~~ a) $\dfrac{d^2y}{dx^2} = k\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$ put $z = \dfrac{dy}{dx}$ $z=0 \;@\; x=0$ (zero slope)

$\Rightarrow \dfrac{dz}{dx} = k\sqrt{1+z^2}$

$$\int_0^z \frac{1}{\sqrt{1+z^2}}\,dz = \int_0^x k\,dx$$

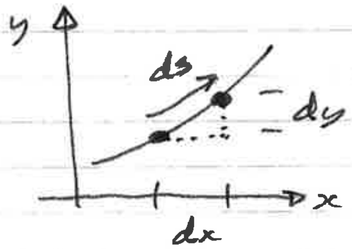
DATABOOK $\Rightarrow \left[\sinh^{-1}(z)\right]_0^z = \left[kx\right]_0^x$

$\Rightarrow \sinh^{-1}\left(\dfrac{dy}{dx}\right) = kx$

REARRANGE $\Rightarrow \dfrac{dy}{dx} = \sinh(kx)$

INTEGRATE $y = \dfrac{1}{k}\cosh(kx) + \text{const.}$

$y=0 \;@\; x=0 \Rightarrow \underline{\underline{y = \dfrac{1}{k}\left(\cosh(kx) - 1\right)}}$ $\underline{\underline{k \text{ chosen to fit through } (x_1, y_1)}}$

b) 

Pythagoras $ds^2 = dx^2 + dy^2$

$\Rightarrow \underline{\underline{\dfrac{ds}{dx} = \sqrt{1+\left(\dfrac{dy}{dx}\right)^2}}}$

$\dfrac{ds}{dx} = \sqrt{1+\sinh^2(kx)} = \cosh(kx)$

$\Rightarrow s = \dfrac{1}{k}\sinh(kx) + \text{const}$

$s=0 \;@\; x=0 \Rightarrow \underline{\underline{s = \dfrac{1}{k}\sinh(kx)}}$

$\underline{\underline{\text{Length} = \dfrac{2}{k}\sinh(kx_1)}}$

c) i) $y_1 \ll x_1 \Rightarrow |dy/dx| \ll 1 \Rightarrow \underline{\underline{\dfrac{d^2y}{dx^2} = k}}$

(ii) $\dfrac{dy}{dx} = kx + A \Rightarrow y = \dfrac{1}{2}kx^2 + Ax + B$

$y=0 \;@\; x=0$ & $dy/dx=0 \Rightarrow A=B=0 \Rightarrow \underline{\underline{y = \dfrac{1}{2}kx^2}}$

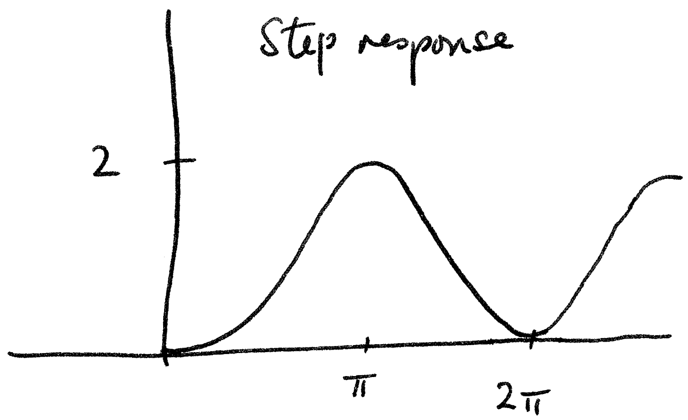
(iii) $y = \dfrac{1}{k}\left(\cosh(kx) - 1\right) = \dfrac{1}{k}\left(\underbrace{\left[1 + \dfrac{(kx)^2}{2!} + O(x^4)\right]}_{\text{DATABOOK.}} - 1\right) = \underline{\underline{\dfrac{1}{2}kx^2 + O(x^4)}}$

2012 Part IA Mathematical Methods Tom Hynes

6 (a) Step response $= \int$ impulse response

$\Rightarrow$ Step resp $= -\cos t + A$

System at rest before $t=0 \Rightarrow A=1 \Rightarrow$ step response $= 1-\cos t$



$$\text{(b) Response is } \quad y(t) \begin{cases} = 0 & t \leq 0 \\ = \sin t & 0 < t \leq \pi \\ = \sin t + \sin(t-\pi) & t > \pi \end{cases}$$

And $\sin(t-\pi) = \sin t \cos\pi - \cos t \sin\pi = -\sin t$

$\therefore$ Response is $y = \sin t \qquad 0 < t < \pi$

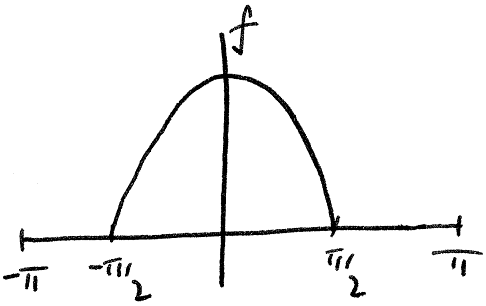
$= 0 \qquad$ otherwise

Examiner's Note

345 Candidates 344 attempted this question. Average 68%.

Well done, in general. Only common error was failure to realise that the answer to part (b) has different forms for different time intervals. Some candidates tried to do part (b) by convolution and sank without trace.

7. (a)



Series in databook with $T = 2\pi \Rightarrow \omega_0 = \frac{2\pi}{2\pi} = 1$

$$\therefore f(t) = \frac{1}{\pi} + \cos t + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\cos 2mt}{4m^2-1}$$

(b)
$$a_6 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos 6t\, dt = \frac{2}{\pi}\int_0^{\pi/2}\cos t\cos 6t\, dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{2}(\cos 7t + \cos 5t)dt = \frac{1}{\pi}\left[\frac{\sin 7t}{7} + \frac{\sin 5t}{5}\right]_0^{\pi/2}$$

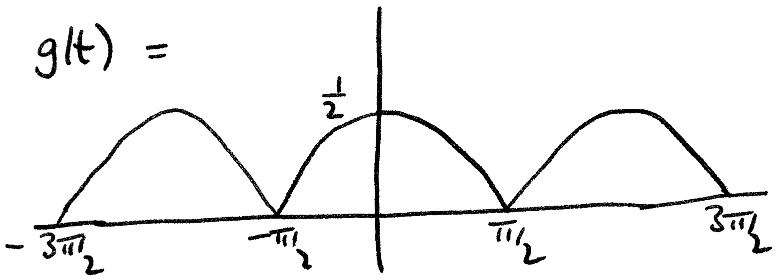
$$= \frac{1}{35\pi}\left[5\underbrace{\sin\frac{7\pi}{2}}_{=-1} + 7\underbrace{\sin\frac{5\pi}{2}}_{=+1}\right] = \frac{2}{35\pi}$$

Part (a) $\Rightarrow$ (with $m = 3$) coeff $= \frac{2}{\pi}\frac{(-1)^4}{4.3^2-1} = \frac{2}{35\pi}$

(c) (i) f is an even function $\Rightarrow$ no sines

(ii) f is continuous but discontinuous slope $\Rightarrow$ coeff $= O(\frac{1}{n^2})$

(d) $g(t) =$



<u>Either</u>

Symmetric about $\pi/2$

$\Rightarrow$ only terms with this symmetry

$\Rightarrow$ $\cos t$, $\cos 3t$, $\cos 5t$, etc not present

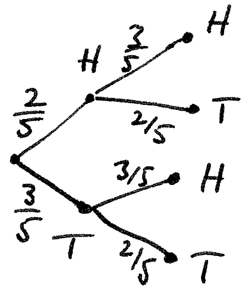
<u>Aliter</u> $g(t)$ <u>has period π $\Rightarrow$ only terms with period π.</u>

<u>Examiner's Note:</u> 343 attempts Average 64%

All parts quite well done except for part (d) – the sting in the tale.

8. On one round

$$P(\text{win}) = P(2H) + P(2T)$$
$$= \frac{2}{5}\cdot\frac{3}{5} + \frac{3}{5}\cdot\frac{2}{5} = \frac{12}{25}$$



$$P(\text{loss}) = 1 - P(\text{win}) = \frac{13}{25}$$

For £1 stake

$$\text{Expected Return} = 2 \times \frac{12}{25} + 0 \times \frac{13}{25} \qquad \left(= \text{mean} = \sum_i x\, P(X=x)\right)$$
$$= \frac{24}{25} \Rightarrow \text{expected loss} = 4p$$

After 10 rounds, expected loss $= 40p$

Examiner's Note: 335 attempts, average 66%

Average of 66% came from half the candidates producing perfect solutions and the other half getting nowhere. To those who expected to lose more than £10, I would like to invite you to come and play!

Commonest errors:

(i) many candidates thought average of integers must be an integer

(ii) Some thought "expected" ⇒ most probable

N.B. mean(X) is usually denoted $E[X]$

i.e. mean = expectation.

Two candidates earned my undying admiration by evaluating all possible cases and concluded, after 3 pages, that

$$\text{expected loss} = \sum_{w=0}^{10} {}_{10}C_w \left(\frac{12}{25}\right)^w \left(\frac{13}{25}\right)^{10-w} (10-2w) = 40p$$

Awesome!

9(a) $\mathcal{L}\left\{\int_0^t f(\tau)\, g(t-\tau)\, d\tau\right\} = F(s)\,G(s)$ ← from databook

(b) Let $y = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \Rightarrow Y(s) = F(s)\,G(s)$

$$L(t) = \frac{1}{s^2} \qquad L(e^{-2t}) = \frac{1}{s+2}$$

$$\therefore \quad Y = \frac{1}{s^2(s+2)} = \frac{As+B}{s^2} + \frac{C}{s+2}$$

Cover up rule $s=-2 \Rightarrow C = \frac{1}{4}$

" " " $s = 0 \Rightarrow B = \frac{1}{2}$

Put $s=1 \Rightarrow \frac{1}{3} = A + B + \frac{C}{3} = A + \frac{1}{2} + \frac{1}{12}$

$$\Rightarrow A = \frac{4-6-1}{12} = -\frac{3}{12} = -\frac{1}{4}$$

$$\therefore \quad Y(s) = -\frac{1}{4s} + \frac{1}{2s^2} + \frac{1}{4(s+2)}$$

$$\Rightarrow \quad y(t) = \frac{t}{2} - \frac{1}{4} + \frac{1}{4}e^{-2t} \qquad \left(\begin{matrix} t \geq 0 \\ 0 \text{ otherwise} \end{matrix}\right)$$

(c) $L\left(\frac{d^n y}{dt^n}\right) = s^n Y - s^{n-1}y(0) - s^{n-2}\dot{y}(0) \ldots - y^{(n-1)}(0)$

So taking L.T. of both sides of a linear, constant coeff differential equation yields

Polynomial$(s)\,Y$ = polynomial from b.c.(s) + L(right hand side)

Solve for Y, then invert.

(d) From part (b)

$$Y(s) = \frac{1}{s^3+2s^2} \Rightarrow s^3 Y + 2s^2 Y = 1$$

Now $L\left(\frac{d^3y}{dt^3}\right) = s^3 Y - s^2 y(0) - s\dot{y}(0) - \ddot{y}(0)$

$L\left(\frac{d^2y}{dt^2}\right) = s^2 Y - sy(0) - \dot{y}(0)$

$$\Rightarrow L\left(\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2}\right) = (s^3+2s^2)Y - y(0)s^2 - s(\dot{y}(0) + 2y(0)) - \ddot{y}(0) - 2\dot{y}(0) = 0$$

Comparing this with $(s^3+2s^2)Y - 1 = 0 \Rightarrow$ satisfies eq^n

and $\Rightarrow$ $y(0) = 0$ $\quad \dot{y}(0) + 2y(0) = 0$ $\quad \ddot{y}(0) + 2\dot{y}(0) = 1$

$\Rightarrow y(0) = \dot{y}(0) = 0 \quad \ddot{y}(0) = 1$

Examiner's Note: 343 attempts, Average 78%

Candidates did very well on this question – well done. Some candidates tackled part (d) by substituting the solution from (b) into the differential equation. Some of these got into trouble having said

$$\frac{1}{2s^2} - \frac{1}{4s} + \frac{1}{4(s+2)} \longrightarrow \frac{t}{2} - H(t) + \frac{1}{4}e^{-2t} = y$$

then $\frac{dy}{dt} = \frac{1}{2} - \delta(t) - \frac{1}{2}e^{-2t}$ & didn't know what to do with the δ fn.

Commonest error was not to realise that a third order differential equation needs three b.c.'s.

10

(a) Rate of change with distance in direction $\underline{n}$

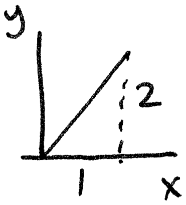
$$= \frac{df}{ds} = \underline{n}.\nabla f \quad \text{provided } \underline{n} \text{ unit vector.}$$

In this case

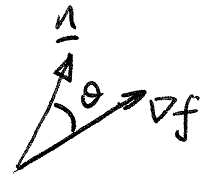
$$\nabla f = \left(2e^{x} - y - 1,\ -x - 1 + \frac{2}{(y+1)^3}\right)$$

At $(0,0)$ $\quad \nabla f = (1,1)$

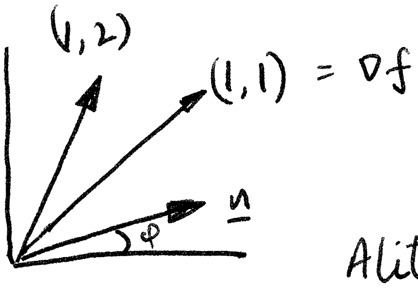
$$\underline{n} = \frac{(1,2)}{\sqrt{5}} \Rightarrow \frac{df}{ds} = \frac{(1,2)}{\sqrt{5}}.(1,1) = \frac{3}{\sqrt{5}}$$



(b) $\displaystyle \frac{df}{ds} = \underline{n}.\nabla f = |\underline{n}||\nabla f|\cos\theta$



$\Rightarrow$ need same $\cos\theta$



By symmetry this is in dirⁿ $(2,1)$

i.e. $y = \frac{1}{2}x$

Aliter $\quad \underline{n} = \dfrac{(a,b)}{\sqrt{a^2+b^2}} \Rightarrow \dfrac{a+b}{\sqrt{a^2+b^2}} = \dfrac{3}{\sqrt{5}}$

$$\Rightarrow 5(a^2+b^2+2ab) = 9(a^2+b^2) \Rightarrow 4a^2 - 10ab - 4b^2 = 0$$

$$\Rightarrow 2\left(\frac{a}{b}\right)^2 - 5\left(\frac{a}{b}\right) + 2 = 0 \Rightarrow \left(2\frac{a}{b} - 1\right)\left(\frac{a}{b} - 2\right) = 0$$

$$\Rightarrow \frac{a}{b} = \frac{1}{2} \text{ or } 2$$

Examiner's Note $\qquad$ Attempts 344, Average 65%

Most candidates knew what to do, although a disappointly large number didn't normalise $\underline{n}$. Some normalised ∇f!

Part (b) needs quite a bit of thought & stumped a lot of candidates.

2012 Part IA Paper 4 Mathematical Methods

Section C **James Matheson**

11 (a) There are 2^{48} or roughly 2.5×10^{14} possible MAC addresses. Even if only one byte of memory is required per table entry, this would require 2.5 TB of memory; clearly impractical! A hash table would require much less memory, typically only enough per entry (100) plus some spare to make collisions less likely, say 256 in all, i.e. a factor of 10^{12} less.

(b) We clearly need to avoid using the most significant 24 bits since these will be identical for all devices made by the same manufacturer, leading to collisions in the hash table. This leaves the least significant 24 bits of which the least significant N are most likely to be random, assuming that the manufacturer allocates addresses sequentially. If we want a hash table with 256 entries as suggested in (a) above, this implies using the least significant 8 bits. Any N less than 7 will results in unresolvable collisions and anything much larger than 9 or 10 will be a poor trade-off between wasting memory and reducing collisions, though how much this matters depends on the size of each entry in the hash table.

12 (a)

```
struct component {
float cost;
float total;
};
```

(b)There are 1000 item totals, 12 monthly totals and one year total. So, as the totals accumulate, the year total will be greatest, followed by the monthly totals, and the per-item totals will generally be the smallest. As soon as any particular total gets 2^{23} times larger than the smallest quantity being added (a penny), i.e. larger than 2^{23} pence or about 80,000 pounds, individual pence will no longer be added. And once a total exceeds 2^{23} times an item value, adding the item will have no effect on the total. This is because the mantissa of the sale value has to be shifted 23 bits to the right to get equal-exponent addition. This problem is going to affect the year total first, then the monthly totals and only at most a few of item totals, which is consistent with the accountant's findings.