ENGINEERING TRIPOS PART IA

Tuesday 12 June 2012 9 to 12

Paper 4

MATHEMATICAL METHODS

*Answer **all** questions.*

*The **approximate** number of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

There are no attachments.

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculated allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

SECTION A

1 (**short**)

(a) By making an appropriate substitution show that

$$\tanh^{-1} x = 0.5 \ln[(1+x)/(1-x)] \quad (-1 < x < +1)$$ [5]

(b) By making an appropriate substitution show that

$$\cosh^{-1} x = \ln\left[x \pm \sqrt{x^2 - 1}\right] \quad (x \geq 1)$$ [5]

2 (**short**)

(a) Given the general complex number $z = re^{i\theta}$ where r is the distance from the origin and θ is the angle with the positive x-axis, write down the expressions for

z^4, $\sqrt{z}$ and $\ln(z)$

Draw an Argand diagram plotting these points when $z = e^{i\pi/3}$ [5]

(b) Show that $z_1 = -i$ is a zero of the polynomial

$$P(z) = z^4 + 5z^3 + 7z^2 + 5z + 6$$

and find all the other zeros. [5]

3 (**short**)

(a) Find the distance from the point P = (l,m,n) to the plane whose equation is given by

$$Ax + By + Cz = D$$ [5]

(b) What is the distance from (2, −1, 3) to the plane $2x - 2y - z = 9$? [5]

4 [**long**]

(a) An orthogonal matrix is one that satisfies the property

$$\mathbf{Q}^t\mathbf{Q} = \mathbf{Q}\mathbf{Q}^t = \mathbf{I}$$

Explain, with justification, what can be said about the following:

(i) the determinant of an orthogonal matrix; [4]

(ii) the relationship between the individual columns of the matrix $\mathbf{Q}$. [4]

(b) The symmetric matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{bmatrix} \frac{3}{5} & 0 & -\frac{4}{5} \\ 0 & 1 & 0 \\ -\frac{4}{5} & 0 & -\frac{3}{5} \end{bmatrix}$$

(i) Determine the eigenvalues and eigenvectors of the matrix $\mathbf{A}$. [12]

(ii) Provide, with justification, a simple check that the eigenvectors are correct. [6]

(iii) Provide an interpretation of the mapping specified by the matrix $\mathbf{A}$. [4]

5 [**long**]

Figure 5 shows a wire that is suspended symmetrically between the two points (x_1, y_1) and $(-x_1, y_1)$. The middle of the wire passes through the origin. You may assume that the differential equation governing the shape of the wire is

$$\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \text{[Eqn 1]}$$

where k is a constant.

(a) By using the substitution $z = dy/dx$, or otherwise, show that the shape of the wire is given by

$$y = \frac{1}{k}\left[\cosh(kx) - 1\right]$$

and explain how the value for k would be determined. [10]

(b) If s is the distance along the wire, measured from the origin, show that

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

and hence determine an expression for the total length of the wire. [8]

(c) For the case where the vertical displacement of the wire is small, $y_1 << x_1$, determine:

(i) the simplified form for Eqn. 1; [2]

(ii) the shape of the wire; [4]

(iii) that the solutions found in (a) and (c)(ii) are consistent. [6]

(cont.

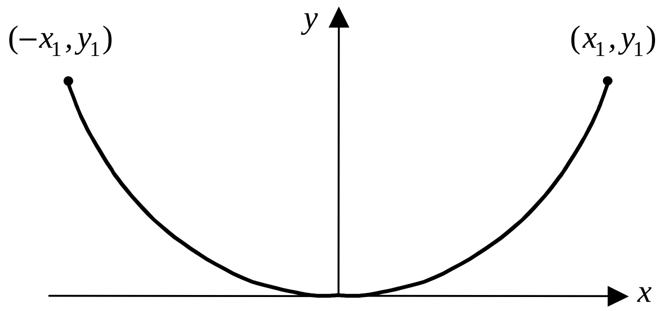

Fig. 5

SECTION B

6 (**short**) The impulse response of a system is given by

$$g(t) = \begin{cases} \sin t & t \geq 0 \\ 0 & t < 0 \end{cases}$$

(a) Find and sketch the step response. [6]

(b) What is the response of the system when the input is
$\delta(t) + \delta(t-\pi)$ [4]

7 (**long**)
A function f is given by

$$f(t) = \begin{cases} 0 & -\pi \leq t \leq -\frac{\pi}{2} \\ \cos t & -\frac{\pi}{2} \leq t \leq \frac{\pi}{2} \\ 0 & \frac{\pi}{2} \leq t \leq \pi \end{cases}$$

and is periodic with period 2π.

(a) Using one of the series in the Maths Databook, find the corresponding Fourier Series for $f(t)$. [4]

(b) Derive using integration the coefficient of cos $6t$, and show that the answer you derive agrees with that found when answering part (a) [10]

(c) Explain:

(i) why there are no sine terms in the Fourier Series for $f(t)$; [3]

(ii) what properties of the function $f(t)$ determine the rate at which the Fourier Series converges. [6]

(d) By considering $g(t) = f(t) - ½ \cos t$, or otherwise, explain carefully why the Fourier Series for $f(t)$ is missing the frequencies that it is. [7]

8 (**short**) A round of a betting game consists of tossing two coins. A player wins if the two tosses result in the same side showing for both coins and loses if they are different. When players win, they receive their original stake back and an equal matching sum. The bookmaker provides two biased coins, one that is biased to show heads with probability 2/5 and the other biased to show heads with probability 3/5. If a player stakes £1 on each round, what would the expected loss be after 10 rounds of the game? [10]

9 (**long**)

(a) What is the Laplace Transform of the convolution of *f* and *g*,

$$\int_{\tau=0}^{t} f(\tau)\, g(t-\tau)\, d\tau$$ [5]

(b) Using Laplace Transforms, find $y(t)$ which is the convolution of f and g where

$$f(t) = t \quad \text{and} \quad g(t) = e^{-2t}$$ [12]

(c) Explain how Laplace Transforms can be used to solve the differential equation

$$\frac{d^3y}{dt^3} + 2\frac{d^2y}{dt^2} = 0$$ [5]

(d) Using the method that you have described in section (c), show that the $y(t)$ that was found in part (b) is the solution of this differential equation and find the appropriate boundary conditions that are satisfied by y. [8]

10 (**short**)

(a) Find the rate of change with distance of the function

$$f(x, y) = 2e^{x} - (x+1)(y+1) - \frac{1}{(y+1)^2}$$

at $(x,y) = (0,0)$ in the direction of the line $y = 2x$. [5]

(b) Find a second line through (0,0), along which the rate of change is the same as that calculated for part (a). [5]

SECTION C

11 (**short**) In a wireless network, each connected device has a unique 48 bit address called its MAC address. The most significant 24 bits of a MAC address identify the manufacturer of the device, the least significant 24 bits are a number allocated by the manufacturer uniquely identifying the particular interface.

A wireless access point for a home network needs to maintain a table of data in which it can record, and look up by MAC address, information about each network device connected to it. It is designed to work with up to 100 devices.

(a) Explain why it would not be sensible simply to use the integer value of the MAC address as an array index for this data and hence why a hash table would be more suitable. [5]

(b) Discuss the suitability of using the most significant N bits or the least significant N bits of the MAC address as hash functions. Comment on the factors affecting the choice of a value for N. [5]

12 (**short**) The code shown in Fig. 12 is used to record the sale of components. All items cost a whole number of pence and all monetary values are floating point numbers representing pounds.

(a) The definition for the data type `component` is missing. Suggest a suitable definition for this, given the code shown. [5]

(b) As a year-end check, the value of `year_total` is compared to the sum of the monthly totals and the sum of the per-item totals. To the accountant's dismay, `year_total` is smaller than the sum of the monthly totals, which in turn is smaller than the sum of the item totals. Given that a typical monthly total is around 250,000 and that the type `float` is IEEE single precision, explain what has gone wrong. [5]

```
component items[1000];    // information about the items being sold
float year_total;         // total sales value in pounds for the year
float month_total[12];    // total sales value in pounds by month
int this_month;           // the current month number (Jan = 0, Dec = 11)
float value;

value = items[item_num].cost * quantity;
year_total = year_total + value;
month_total[this_month] = month_total[this_month] + value;
items[item_num].total = items[item_num].total + value;
```

Fig. 12

END OF PAPER

Answers

2(a) $r^4 e^{4i\theta}$; $\sqrt{r}e^{i\theta/2+in\pi}$ for n = 0,1; $\ln r + i\theta + 2n\pi i$ for any integer n

(b) i, –2, –3

3(a) $\left| \dfrac{Al + Bm + Cn \;-\; D}{\sqrt{A^2 + B^2 + C^2}} \right|$ (b) 2

4 λ = 1, 1, –1 e-vectors $\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ (eigenvectors are not unique)

5 (b) $2\,\dfrac{\sinh kx_1}{k}$ (c) (i) $\dfrac{d^2y}{dx^2} = k$ (ii) $y = k\dfrac{x^2}{2}$

6 $\sin t$ for $0 \le t \le \pi$, 0 otherwise

7 (a) $f(t) = \dfrac{1}{\pi} + \dfrac{1}{2}\cos t + \dfrac{2}{\pi}\sum_{m=1}^{\infty}(-1)^{m+1}\dfrac{\cos(2mt)}{4m^2-1}$ (b) $\dfrac{2}{35\pi}$

8 £ 0.4

9 (a) $F(s)G(s)$ (b) $\dfrac{t}{2} - \dfrac{1}{4} + \dfrac{1}{4}e^{-2t}$ (d) $y = 0,\ \dot{y} = 0,\ \ddot{y} = 1$ at $t = 0$

10 (a) $\dfrac{3}{\sqrt{5}}$ (b) $y = \dfrac{1}{2}x$