ENGINEERING TRIPOS PART IA 2013

Paper 1   Mechanical Engineering

Solutions

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Mechanical Engineering Tripos, Part IA Section A, 2013

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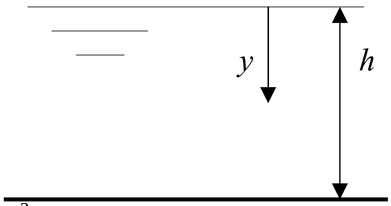
Question 1 (short)

Select coordinate system with y starting from the water surface vertically downwards, as shown in the sketch.

a) $p = \rho g h$; $F_{hyd} = \int_A p dA = \int_0^h p w dy = \rho g w(\frac{1}{2}h^2) = \boxed{\frac{1}{2}\rho g w h^2}$ [5]



b) Moment due to hydrostatic force about the hinge:

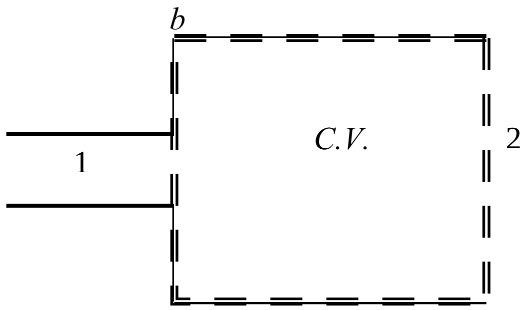
$$dm = (h-y)pwdy = \rho g w(hy - y^2)dy ;$$

$$M = \int_0^h dm = \int_0^h (h-y)pwdy = \rho g w(hy - y^2)dy = \rho g w[h\frac{h^2}{2} - \frac{h^3}{3}] = \frac{1}{6}\rho g w h^3$$

Equilibrium: $M = FH$; $\boxed{F = \frac{1}{6}\rho g w \frac{h^3}{H}}$ [5]

Examiner's comments: This is a straightforward question and well done by majority of the candidates. Many used the linear distribution of the pressure with water depth without integration, which is allowed. A common mistake was confusion with the centre of the force being referenced from the top or from the hinge, as was with the origin of the moment calculation.

Question 2 (short)



Continuity: $A_1V_1 = A_2V_2$; $\frac{A_1}{A_2} = \frac{V_2}{V_1}$

SFME: $p_1A_2 + \rho A_1V_1V_1 = p_2A_2 + \rho A_1V_1V_2$; $(p_2 - p_1)A_2 = \rho A_1V_1^2 - \rho A_1V_1^2\frac{V_2}{V_1}$

$A_2\frac{p_2 - p_1}{\rho A_1V_1^2} = 1 - \frac{A_1}{A_2}$; $\frac{A_2}{A_1}\frac{p_2 - p_1}{\rho V_1^2} = 1 - \frac{A_1}{A_2}$; $\boxed{\frac{p_2 - p_1}{\rho V_1^2} = \frac{A_1}{A_2}(1 - \frac{A_1}{A_2})}$ [10]

Examiner's comments: This again is a question well done by most candidates. Some did not realise that the mixing process between the expansion opening and section 2 where flow is uniform is irreversible and used Bernoulli's Principle between sections 1 and 2, which is *NOT* applicable. On applying SFME a number of candidates thought that the pressure force upstream only contributes to area of A_1 but not the whole A_2.

Question 3 (long)

a). Bernoulli between the top of the water and nozzle exit 1, at both points pressure is atmospheric:

$$p_3 + \cancel{\frac{1}{2}\rho V_3^2} + \rho g h_2 = p_2 + \frac{1}{2}\rho V_1^2 \; ; \; V_1 = \sqrt{2gh_2} = \sqrt{2 \cdot 9.81 \cdot 5} = \boxed{9.905\,\mathrm{ms}^{-1}}$$

$$\dot{m} = \rho A_1 V_1 = 1000. \cdot 0.002 \cdot 9.905 = \boxed{19.81\ \mathrm{kgs}^{-1}}$$ [6]

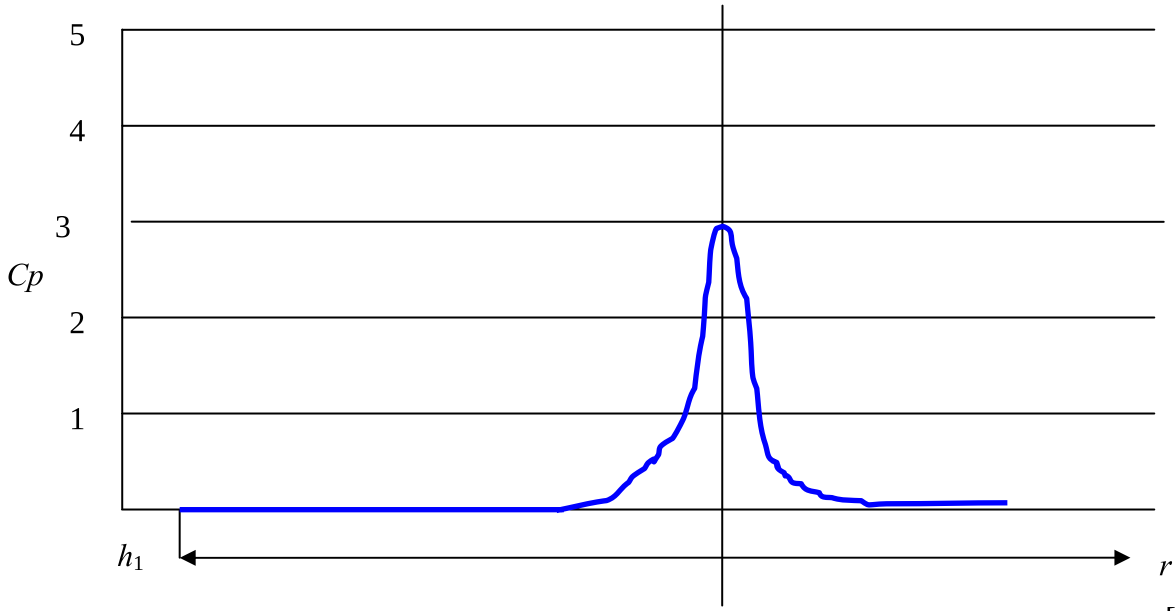
b). at 1, $p = p_{atm}$; $\boxed{Cp_1 = \dfrac{p - p_{atm}}{\frac{1}{2}\rho V_1^2} = 0}$

at 0, $V = 0$; $p = p_0 = p_{atm} + \rho g(h_1 + h_2)$; $p - p_{atm} = \rho g(h_1 + h_2)$

$$Cp_0 = \frac{p - p_{atm}}{\frac{1}{2}\rho V_1^2} = \frac{\rho g(h_1 + h_2)}{\rho g} = \frac{(h_1 + h_2)}{h_2}\boxed{= (\frac{h_1}{h_2} + 1); \quad \text{function of } \frac{h_1}{h_2} \text{ only}}$$

$$\boxed{Cp_0 = \frac{p - p_{atm}}{\frac{1}{2}\rho V_1^2} = \frac{h_2 + h_1}{h_2} = \frac{10 + 5}{5} = 3.0}$$ [10]

c). Along most part of the vertical jet streamlines are almost parallel, $p = p_{atm}$, $Cp = 0$; approaching 0, p increase to reach the stagnation pressure at 0, $Cp_0 = 3.0$ (part b. above). Then as water flows out on the surface of the plate, the pressure approaches to p_{atm} when r is large. ($p_{plate} = \rho g \delta h$; δh as a function of r is the local water film thickness. $\delta h \propto \frac{1}{r}$; $p_{plate} = \delta h / h_2$)



[8]

d).

$$\frac{dp}{dn} = \rho \frac{V_{edge}^2}{R_C} = \rho \frac{2\rho g(h_1 + h_2)}{R_C} = \frac{2 \cdot 9{,}81 \cdot (10+5) \cdot 10^3}{0.1} = \boxed{2.943 \times 10^6 \, \mathrm{pam}^{-1} = 2.943 \mathrm{Mpam}^{-1}}$$

[6]

Examiner's comments: Part a) was very well done, most candidates got it correct. In Parts b) and c) on pressure coefficients, a common mistake being unable to realise that along the vertical column, except for a region close to the floor plane the static pressure is atmospheric thus the pressure coefficient is zero – large amount of candidates thought that the static pressure **linearly** increase with the distance towards the stagnation value on the floor. For the last part, on the pressure gradient due to the stream curvature, those who did not get it right mainly were not able to find the flow speed at the edge of the stream where radius of streamline curvature was given, the speed can calculated using Bernoulli's Principle with pressure being atmospheric and total potential head being $\rho g(h_1 + h_2)$.

Question 4

(a)

Ideal gas law

$$\rho_1 = \frac{P_1}{R_A T_1} = \frac{100\,\text{kPa}}{0.287\,\frac{\text{kJ}}{\text{kg K}} \cdot 300\text{ K}} = 1.16\,\frac{\text{kg}}{\text{m}^3}$$

$$\dot{m}_1 = \rho_A Q_1 = 1.16\frac{\text{kg}}{\text{m}^3} \cdot 0.25\,\frac{\text{m}^3}{\text{s}} = \boxed{0.290\,\frac{\text{kg}}{\text{s}}}$$

$$w_x = \frac{\dot{W}_x}{\dot{m}} = \frac{-20\,\text{kW}}{0.29\,\frac{\text{kg}}{\text{s}}} = \boxed{-68.9\,\frac{\text{kJ}}{\text{kg}}}$$

(b)

For isentropic, adiabatic SSSF

$$\cancel{q} - w_x = (h_2 - h_1) + \cancel{\Delta KE} + \cancel{\Delta PE}$$

$$-w_x = c_p(T_2 - T_1) \Rightarrow T_2 = \frac{-w_x}{c_p} + T_1$$

$$T_2 = \frac{68.9\,\frac{\text{kJ}}{\text{kg}}}{1.006\,\frac{\text{kJ}}{\text{kg K}}} + 300\,\text{K} = \boxed{368.5\,\text{K}}$$

$$P_2 = P_1\left(\frac{T_2}{T_1}\right)^{\frac{\gamma}{\gamma-1}} = 100\text{ kPa}\cdot\left(\frac{368.5}{300}\right)^{\frac{7}{2}} = \boxed{205.5\text{ kPa}}$$

(c)

$$\frac{\rho_2}{\rho_1} = \frac{\frac{P_2}{RT_2}}{\frac{P_1}{RT_1}} = \frac{P_2\,T_1}{P_1\,T_2} = \frac{205.5\text{ kPa}\cdot 300\text{ K}}{100\text{ kPa}\cdot 368.5\text{ K}} \cdot 100\% = \boxed{167\%}$$

Alternatively, $\frac{\rho_2}{\rho_1} = \frac{v_1}{v_2} = \left(\frac{P_2}{P_1}\right)^{1/\gamma} \cdot 100\% = 167\%$

Examiners comments: The vast majority of students completed this problem correctly or had only minor mistakes. The most common error was using c_v rather than c_p for part (b). Although the process is not at constant pressure the steady state steady flow relationship allows for the calculation of useful work, w_x, used to increase the density of the fluid within the compressor.

Question 5

(a)

Conservation of mass

$\dot{m}_1 + \dot{m}_2 = 2\,\frac{\mathrm{kg}}{\mathrm{s}} + 3\,\frac{\mathrm{kg}}{\mathrm{s}} = \underline{\dot{m}_3 = 5\,\frac{\mathrm{kg}}{\mathrm{s}}}$

Conservation of energy with no heat transfer or external work

$\cancel{\dot{Q}} - \cancel{\dot{W}} = -\dot{m}_1(h_1 + V_1^2/2) - \dot{m}_2(h_2 + V_2^2/2) + \dot{m}_3(h_3 + V_3^2/2)$

Constant pressure process of an ideal gas with constant specific heat, $h = c_p T$. Substitute mass and heat capacity relationship into conservation of energy equation.

$0 = -2\cancel{\mathrm{kg/s}}(T_1 c_p + v_1^2/2) - 3\cancel{\mathrm{kg/s}}(T_2 c_p + v_2^2/2) + 5\cancel{\mathrm{kg/s}}(T_3 c_p + v_3^2/2)$

$T_3 = \frac{2}{5}T_1 + \frac{3}{5}T_2 + \frac{1}{c_p}(\frac{1}{5}V_1^2 + \frac{3}{10}V_2^2 - \frac{1}{2}V_3^2)$

$$T_3 = \underbrace{\frac{2}{5}\cdot 600\ \mathrm{K} + \frac{3}{5}\cdot 300\ \mathrm{K}}_{420.00\ \mathrm{K}} + \underbrace{\frac{1}{1005\,\frac{\mathrm{J}}{\mathrm{kg\,K}}}\left(1/5\left(100\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 + 3/10\left(50\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2 - 1/2\left(125\,\frac{\mathrm{m}}{\mathrm{s}}\right)^2\right)}_{-5.04\ \mathrm{K}}$$

$\boxed{T_3 = 414.96\ \mathrm{K}}$

(b)

Apply second law relation given in the thermofluids databook

$$\cancel{\frac{\mathrm{d}S_{CV}}{\mathrm{d}t}} + \sum \dot{m}_{out}s_{out} - \sum \dot{m}_{in}s_{in} = \int \frac{\cancel{\mathrm{d}\dot{Q}}}{T} + \dot{S}_{irrev}$$

$\dot{S}_{irrev} = \dot{m}_3 s_3 - \dot{m}_2 s_2 - \dot{m}_1 s_1 = \dot{m}_1(s_3 - s_1) + \dot{m}_2(s_3 - s_2)$

Apply ideal gas entropy relationship

$\dot{S}_{\mathrm{irrev}} = \dot{m}_1\left(c_p \ln(T_3/T_1) - R\ln(P_3/P_1)\right) + \dot{m}_2\left(c_p \ln(T_3/T_2) - R\ln(P_3/P_2)\right)$

Note that $P_1 = P_2 = P_3 \therefore \ln(P_3/P_1) = \ln(P_3/P_2) = 0$

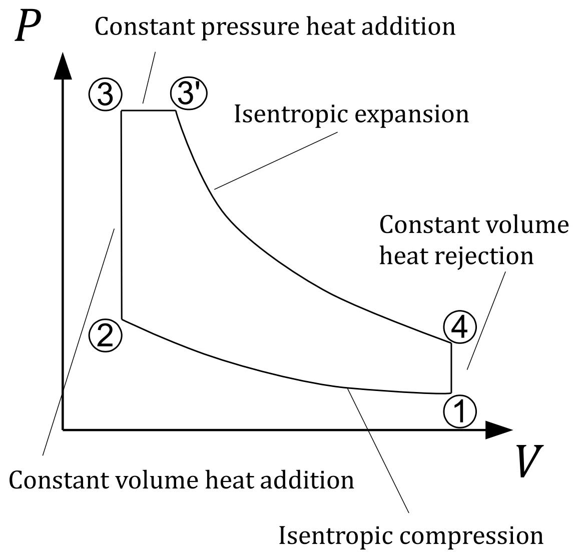
$$\dot{S}_{irrev} = \underbrace{\dot{m}_1\left(c_p \ln(T_3/T_1)\right)}_{-747.2\,\frac{\mathrm{J}}{\mathrm{K\,s}}} + \underbrace{\dot{m}_2\left(c_p \ln(T_3/T_2)\right)}_{978.1\,\frac{\mathrm{J}}{\mathrm{K\,s}}}$$

$\boxed{\dot{S}_{\mathrm{irrev}} = 236.9\,\frac{\mathrm{J}}{\mathrm{K\,s}}}$

Examiners comments: This problem proved challenging for many of the students. In part (a) the most most common error was neglecting the change of kinetic energy between the inlets and outlet of the fluid. This resulted in an overestimation of the outlet temperature by ~ 5 K. Part (b) was particularly challenging for many of the students and several did not make a valid attempt. Of those that used the second law steady flow equation and ideal entropy relation from the thermofluids databook, many did not separate the process into two changes of state, 1 to 3 and 2 to 3 with the corresponding mass flow for each process. Many calculated the change in entropy from 1 to 3 but assumed that the entire mass flow underwent that change.

Question 6

(a)



Pressure versus volume diagram of modified Otto cycle. (Labelling processes between points was not required)

(b)

(i) ① → ② Isentropic compression with ideal gas

$$v_1 = \frac{RT_1}{P_1} = \frac{0.287\,\frac{\text{kJ}}{\text{kg K}} \cdot 300\text{ K}}{100\text{ kPa}} = 0.861\,\frac{\text{m}^3}{\text{kg}}$$

$$v_2 = v_1/r_v = \boxed{9.57 \cdot 10^{-2}\,\frac{\text{m}^3}{\text{kg}}}$$

Isentropic relation

$$T_2 = T_1\,(v_1/v_2)^{\gamma-1} = 300\text{ K}{\cdot}9^{0.4}$$

$$\boxed{T_2 = 722.5\text{ K}}$$

(ii) ② → ③ Constant volume heat addition

$$q_{2-3} - \cancel{w_{2-3}} = \Delta u_{2-3} = c_v(T_3 - T_2)$$

$$T_3 = \frac{q_{2-3}}{c_v} + T_2 = \frac{800\,\frac{\text{kJ}}{\text{kg}}}{0.718\,\frac{\text{kJ}}{\text{kg K}}} + 722.5\text{ K} = 1836.7\text{ K}$$

Constant volume process
$v_3 = v_2 = 9.57 \cdot 10^{-2}\text{ m}^3/\text{kg}$

Ideal gas relation

$$P_3 = \frac{RT_3}{v_3} = \frac{0.287\,\frac{\text{kJ}}{\text{kg K}} \cdot 1836.7K}{9.57 \cdot 10^{-2}\,\frac{\text{m}^3}{\text{kg}}}$$

$\boxed{P_3 = 5513.9 \text{ kPa}}$

(iii) ③ → ③′ Constant pressure heat addition

$q_{3-3'} = \Delta h_{3-3'} = c_p(T_{3'} - T_3)$

$$T_{3'} = \frac{q_{3-3'}}{c_p} + T_3 = \frac{400 \frac{\text{kJ}}{\text{kg}}}{1.005 \frac{\text{kJ}}{\text{kg K}}} + 1836.7 \text{ K}$$

$\boxed{T_{3'} = 2234.7 \text{ K}}$

(iv) ④ → ① Constant volume heat rejection

$q_{4-1} = c_v(T_1 - T_4)$

$$T_4 = T_{3'}(v_{3'}/v_4)^{\gamma-1} \text{ where } v_{3'} = \frac{RT_{3'}}{P_{3'}} = \frac{RT_{3'}}{P_3} \text{ and } v_4 = v_1$$

$$T_4 = T_{3'}^{\gamma}\left(\frac{R}{P_3 v_1}\right)^{\gamma-1} = (2234.7 \text{ K})^{1.4} \cdot \left(\frac{0.287 \frac{\text{kJ}}{\text{kg K}}}{5513.9 \text{ kPa} \cdot 0.861 \frac{\text{m}^3}{\text{kg}}}\right)^{0.4} = 1003.4 \text{ K}$$

$q_{4-1} = 0.718 \frac{\text{kJ}}{\text{kg K}} \cdot (300 \text{ K}–1003.4 \text{ K})$

$\boxed{q_{4-1} = -505.2 \frac{\text{kJ}}{\text{kg}}}$

(c)

$w_{out} = q_{in} + q_{out} = 1200 \frac{\text{kJ}}{\text{kg}} - 505.2 \frac{\text{kJ}}{\text{kg}}$

$\boxed{w_{out} = 694.8 \frac{\text{kJ}}{\text{kg}}}$

$$w_{out,max} = \eta_{Carnot} q_{in} = \left(1 - \frac{T_1}{T_{3'}}\right) q_{in}$$

$$w_{out,max} = \left(1 - \frac{300 \text{ K}}{2234.7 \text{ K}}\right) 1200 \frac{\text{kJ}}{\text{kg}}$$

$\boxed{w_{out,max} = 1038.9 \frac{\text{kJ}}{\text{kg}}}$

The work lost due to the modified Otto cycle differing from the Carnot cycle is

$w_{out,max} - w_{out} = 1038.9 \frac{\text{kJ}}{\text{kg}} - 694.8 \frac{\text{kJ}}{\text{kg}} = 344.1 \frac{\text{kJ}}{\text{kg}}$

The modified Otto cycle has lower work output than the Carnot cycle because the entire heat transfer in and out of the system does not occur at the highest and lowest cycle temperatures, respectively, i.e. q_{in} is not entirely at $T_{3'}$ and q_{out} is not entirely at T_1. Since losses due to friction, combustion and non-adiabatic pumping were neglected, these phenomena did not contribute to the lower work output of the modified Otto cycle.

Examiners comments: A majority of the students made a valid attempt at this problem with several common errors throughout the process. Most students were able to correctly draw the P-v diagram and the subsequent results for those that were successful in part (a) were significantly better than those that were not, highlighting the benefits of problem visualization. Many confused the compression ratio, r_v, for pressure ratio, r_p, and solved part (b) accordingly. The primary challenge of part (b) was to recognize the difference in constant volume heat addition and constant pressure heat addition, and use c_v and c_p accordingly. Slightly more than half of the students that completed part (b) recognized this distinction. Most of the students that completed part (b, iii) did it correctly, but a common mistake was assuming that the difference in enthalpy from state 3' to 1 was all rejected as heat or that all was output as useful work (this would violate the second law). The vast majority of students who completed section (c) were able to determine the work output from a Carnot cycle operating between the same T_H and T_C, but attributed the difference between the work output of the Carnot and modified Otto cycle to losses due to friction and non-adiabatic pumping. These phenomena were neglected in this problem as no friction losses or heat transfer in the compression and expansion cycles were included in the problem statement.

Q7 (short)

a) Since the cables are assumed massless, the tension is constant along each cable. Since the movable pulley is also assumed massless, the sum of the forces acting on it must vanish, so

$$T_2 = 2T_1. \tag{1}$$

As mass C descends after it is released, the movable pulley rises, the magnitude of their accelerations are the same, a_2. Mass B remains stationary throughout the dynamics, so as the movable pulley rises distance x, mass A must rise $2x$ due to the inextensibility of the cable, so

$$a_1 = 2a_2,$$

and therefore

$$\frac{a_1}{a_2} = \frac{T_2}{T_1}.$$

b) Mass B remains stationary, so the forces on it must cancel out, giving

$$T_1 = 2mg,$$
$$T_2 = 4mg.$$

Newton's second law on mass A is

$$T_1 - mg = ma_1,$$

so

$$2mg - mg = ma_1,$$
$$a_1 = g,$$
$$a_2 = \frac{1}{2}g.$$

And Newton's second law on mass C then gives

$$Mg - 4mg = \frac{1}{2}Mg,$$
$$M = 8m$$

Q8 (short)

a) The moment of inertia of a circular element or radius r and width dr of the disc is $\rho r^2 2\pi r dr$, where $\rho = m/\pi a^2$ is the density. Next we integrate over the body radially outwards, the total moment of inertia is given by,

$$\begin{aligned} I_0 &= \frac{2m}{a^2}\int_0^a r^3\,dr \\ &= \frac{2m}{a^2}\left[\frac{r^4}{4}\right]_0^a \\ &= \frac{2m}{a^2}\left(\frac{a^4}{4} - 0\right) \\ &= \frac{ma^2}{2} \end{aligned}$$

b) Using the parallel axis rule, we have

$$\begin{aligned} I &= I_0 + m\left(\frac{a}{2}\right)^2 \\ &= \frac{ma^2}{2} + \frac{ma^2}{4} \\ &= \frac{3}{4}ma^2 \end{aligned}$$

c) The moment of Newton's second law is: torque = moment of inertia $\times$ angular acceleration, i.e.

$$\begin{aligned} mg\frac{a}{2} &= \frac{3}{4}ma^2\ddot{\theta} \\ \ddot{\theta} &= \frac{2}{3}\frac{g}{a} \end{aligned}$$

Q9 (long)

a) Given the circular motion, equate the force of gravity with the centripetal acceleration,

$$\frac{GMm}{R_P^2} = m\frac{v^2}{R_P}.$$

Using $v = R_P\omega$, we get

$$R_P^3 = \frac{GM}{\omega^2}$$
$$R_P = \sqrt[3]{\frac{GM}{\omega^2}}$$

On the surface of the Earth (whose radius is R_E), $GM/R_E^2 = g$, so the radius of the geostationary orbit can also be written as

$$R_P = \sqrt[3]{\frac{gR_E^2}{\omega^2}}$$

b) Applying the same procedure to the circular motion to both lower and upper orbits, we have

$$\frac{v_1^2}{R_P} = \frac{GM}{R_P^2}$$
$$\frac{v_2^2}{R_A} = \frac{GM}{R_A^2}$$

Therefore

$$v_1^2 R_P = GM = v_2^2 R_A$$

so

$$\frac{v_1}{v_2} = \sqrt{\frac{R_A}{R_P}}$$

c) After the first burn and before the second burn, i.e. while the spaceship is traveling on the elliptical orbit, there are no external forces other than gravity, so the total energy (including gravitational potential) is conserved,

$$\frac{1}{2}mv_P^2 - \frac{GMm}{R_P} = \frac{1}{2}mv_A^2 - \frac{GMm}{R_A}$$

Since gravity is a central force, the angular momentum is also conserved,

$$mv_P R_P = mv_A R_A$$

d) Writing $R_A/R_P = x$, we have

$$\frac{v_P}{v_A} = x,$$
$$\frac{v_1}{v_2} = \sqrt{x}.$$

Substituting the first into the expression of the energy conservation, we get

$$v_P^2 - \frac{1}{x^2}v_P^2 = 2GM\left(\frac{1}{R_P} - \frac{1}{R_A}\right)$$
$$v_P^2\left(1 - \frac{1}{x^2}\right) = \frac{2GM}{R_P}\left(1 - \frac{1}{x}\right)$$

Using $v_1^2 = GM/R_P$ from part b), we have

$$\frac{v_P^2}{v_1^2} = 2\frac{\left(1-\frac{1}{x}\right)}{\left(1-\frac{1}{x}\right)\left(1+\frac{1}{x}\right)}$$
$$\frac{v_P}{v_1} = \sqrt{\frac{2}{\left(1+\frac{1}{x}\right)}} = \sqrt{\frac{2x}{x+1}}$$

So the relative change in velocity is

$$\frac{v_P - v_1}{v_1} = \sqrt{\frac{2x}{x+1}} - 1.$$

Substituting for v_A and v_2, we get

$$\frac{xv_A}{\sqrt{x}v_2} = \sqrt{\frac{2x}{x+1}}$$
$$\frac{v_A}{v_2} = \sqrt{\frac{2}{x+1}}$$

So the relative change in velocity during the second burn is,

$$\frac{v_2 - v_A}{v_2} = 1 - \sqrt{\frac{2}{x+1}}.$$

Q10 (a) $\dot{\underline{r}} = \boxed{\hat{e}_r + \sqrt{3}\,\hat{e}_\theta}$

$\ddot{\underline{r}} = \boxed{-3\,\hat{e}_r + 3\sqrt{3}\,\hat{e}_\theta}$

(b) $\hat{e}_t = \frac{1}{2}\left(\hat{e}_r + \sqrt{3}\,\hat{e}_\theta\right)$

$\ddot{\underline{r}} = \ddot{s}\,\hat{e}_t + \frac{\dot{s}^2}{\rho}\,\hat{e}_n$

$\ddot{s}$ = acceleration along the path

$= \ddot{\underline{r}} \cdot \hat{e}_t = \frac{1}{2}(-3+9) = 3\ m/s^2$

$\dot{s} = |\dot{\underline{r}}| = 2\,m/s$

$|\ddot{\underline{r}}| = 6\ m/s^2$

$\therefore \frac{\dot{s}^2}{\rho} = \sqrt{36-9} = 3\sqrt{3}\ m/s^2$

ρ = radius of curvature $= \boxed{\frac{4}{3\sqrt{3}}\ m}$

Q11

(a)
$$ky + \lambda\dot{y} = 2\lambda(\dot{x} - \dot{y})$$
$$ky + 3\lambda\dot{y} = 2\lambda\dot{x} \quad \text{—— (1)}$$
$$\dot{y} + \frac{k}{3\lambda}y = \frac{2}{3}\dot{x}$$

(b)
$$Y(j\omega)\,[k + 3\lambda j\omega] = 2\lambda j\omega X(j\omega) \quad \text{— from (1)}$$
$$X(j\omega) = \left[\frac{k + 3\lambda j\omega}{2\lambda j\omega}\right] Y(j\omega)$$

Plugging in values of k, λ, ω:
$$X(j\omega) = \frac{(10 + 6\pi j)\,Y(j\omega)}{4\pi j}$$
$$|X(j\omega)| = \frac{\sqrt{100 + 36\pi^2}}{4\pi} \times 10\text{ mm} = \boxed{16.98\text{ mm}}$$

(c)
$$F(j\omega) = (k + \lambda j\omega)\,Y(j\omega)$$
$$|F| = \sqrt{100 + 4\pi^2}\,(0.01)$$
$$|F| = \boxed{0.118\text{ N}}$$

Q12

(a)

$$m\ddot{x}_2 = -k(x_2 - x_1) \qquad \text{— (1)}$$

$$4m\ddot{x}_1 = -2k(x_1 - y) - k(x_1 - x_2) \qquad \text{— (2)}$$

$$\begin{bmatrix} 4m & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 3k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2ky \\ 0 \end{bmatrix} \qquad \text{— (3)}$$

(b) To find resonant frequencies

$$|K - M\omega^2| = 0$$

$$\left| \begin{bmatrix} 3k - 4m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} \right| = 0$$

$$(3k - 4m\omega^2)(k - m\omega^2) - k^2 = 0$$

$$\Delta = 4m^2\omega^4 - 7m\omega^2 k + 2k^2 = 0$$

$$\omega^2 = \frac{7mk \pm \sqrt{49m^2k^2 - 32m^2k^2}}{8m^2}$$

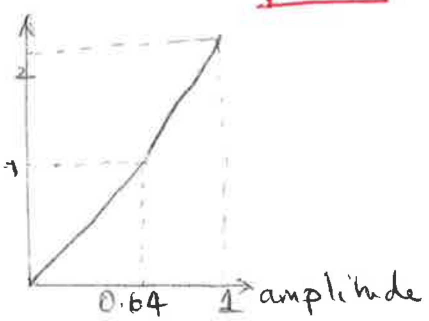
$$\omega^2 = \boxed{\frac{7 \pm \sqrt{17}}{8} \frac{k}{m}}$$

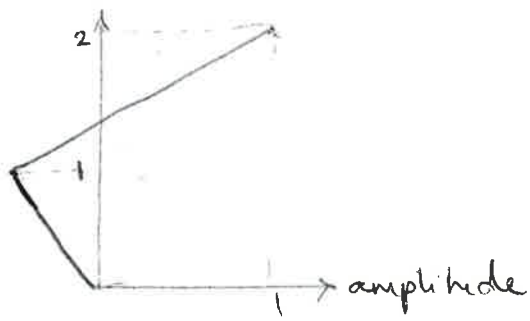
To sketch mode shapes use (1):

$$\frac{x_1}{x_2} = \frac{k - m\omega_n^2}{k} = 1 - \frac{m}{k}\omega_n^2$$

For $\omega_1^2 = \frac{7 - \sqrt{17}}{8}$, $\boxed{\frac{x_1}{x_2}} = 1 - \left(\frac{7}{8} - \frac{\sqrt{17}}{8}\right) = \boxed{0.64}$

For $\omega_2^2 = \frac{7 + \sqrt{17}}{8}$, $\boxed{\frac{x_1}{x_2}} = 1 - \left(\frac{7}{8} + \frac{\sqrt{17}}{8}\right) = \boxed{-0.39}$





(c) From (3)

$$\begin{bmatrix} 3k - 4m\omega^2 & -k \\ -k & k - m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 2kY \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \frac{1}{\Delta} \begin{bmatrix} k - m\omega^2 & k \\ k & 3k - 4m\omega^2 \end{bmatrix} \begin{bmatrix} 2kY \\ 0 \end{bmatrix}$$

$$\Delta = 4m^2\omega^4 - 7m\omega^2 k + 2k^2$$

$$X_1 = \boxed{\frac{(k - m\omega^2)2kY}{\Delta}}$$

$$X_2 = \boxed{\frac{2k^2Y}{\Delta}}$$