ENGINEERING TRIPOS PART IA 2013

Paper 2 Structures and Materials

Solutions

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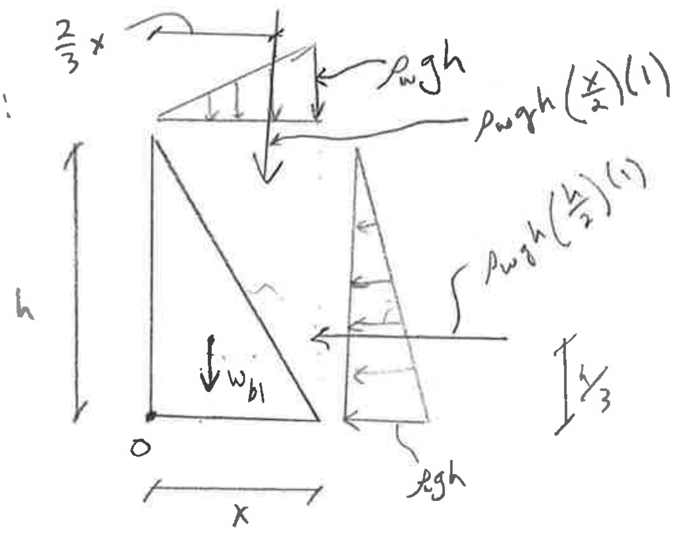
M. Dabney

① a) FBD:



$$W_{bl} = \rho_{bl}\, g\, V_{bl}$$

$$= 2\rho_w g \frac{hx(1)}{2}$$

$$= \rho_w g h x$$

$$\circlearrowleft + \quad \Sigma M_o \Rightarrow -W_{bl}\left(\frac{x}{3}\right) - \rho_w g h\left(\frac{x}{2}\right)\left(\frac{2}{3}x\right) + \rho_w g h\left(\frac{h}{2}\right)\frac{h}{3} = 0$$

$$(\rho_w g h x)\frac{x}{3} + \frac{\rho_w g h x^2}{3} - \frac{\rho_w g h^3}{6} = 0$$

$$h\frac{x^2}{3} + h\frac{x^2}{3} - \frac{h^3}{6} = 0$$

$$\frac{2x^2}{3} = \frac{h^2}{6_2} \rightarrow x^2 = \frac{h^2}{4} \rightarrow \underline{\underline{x = \frac{h}{2}}}$$

b) Horizontal force $= \rho_w g h\left(\frac{h}{2}\right)$

Vertical force $= W_{bl} + \rho_w g h\left(\frac{h}{2}\right) = \frac{3}{2}\rho_w g h^2$

$$\mu = \frac{\rho_w g\, h^2/2}{3\rho_w g\, h^2/2} = \frac{1}{3} \rightarrow \underline{\underline{\lambda = \mathrm{atan}\left(\frac{1}{3}\right)}}$$

(2) $$I = \frac{1(12)^3}{12}(2) + \frac{12(2)^3}{12} = 296 \text{ cm}^4$$

Check Shear

$$q = \frac{S A_c \bar{y}}{I} \rightarrow A_c \bar{y} = (6)(1)(3)(2) + (12 \times 1)0.5 = 42 \text{ cm}^3$$

$$q = \frac{1 \text{ kN}(2 \text{ bolts})}{0.5 \text{ m}} = 4 \text{ kN/m}$$

$$S = \frac{Iq}{A\bar{y}} = \frac{296 \text{ cm}^4 (4 \text{ kN/m})\left(\frac{1}{100}\frac{\text{m}}{\text{cm}}\right)}{42 \text{ cm}^3} = \underline{\underline{0.28 \text{ kN}}}$$

$$P_{allow} = 2(0.28) = \boxed{0.56 \text{ kN}}$$

← governs

Check Bending

$$\sigma = \frac{My}{I} \rightarrow M = \frac{I\sigma_y}{y} = \frac{296 \text{ cm}^4}{6 \text{ cm}}\left(\frac{1 \text{ m}}{100 \text{ cm}}\right)^3 355 \times 10^6 \text{ N/m}^2$$

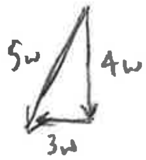
$$\underline{\underline{M_{allow} = 17,513 \text{ N-m}}}$$

$$M_{allow} = \frac{PL}{4} \rightarrow P_{allow} = \frac{4(17.5 \text{ kN-m})}{(8)} = \underline{\underline{8.76 \text{ kN}}}$$

(3) a)

$$\sum M_C \rightarrow \; 5w\,(2.5) = V_A\,(8)$$

$$V_A = 12.5 \text{ kN}$$

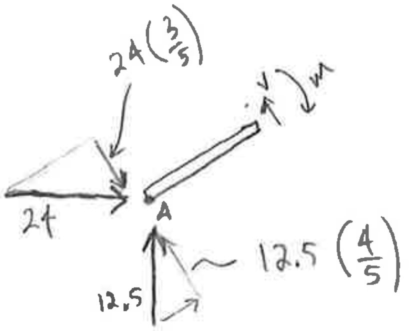


$\rightarrow$ $\sum F_H$: $H_A = 3w = 24$ kN

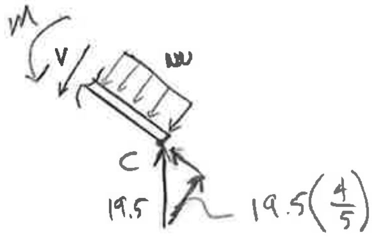
$\sum F_V$: $V_C = 4w - V_C$

$= 32 - 12.5 = 19.5$ kN

b)

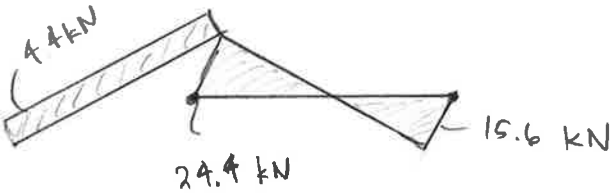


$\rightarrow$ Shear : $V = -12.5\left(\frac{4}{5}\right) + 24\left(\frac{3}{5}\right) = +4.4$ kN

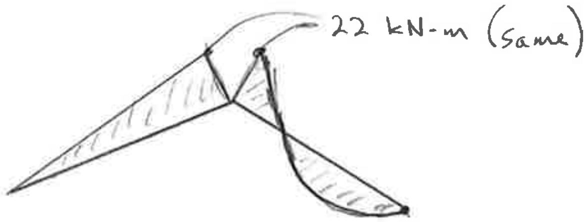


$$V = 19.5\left(\frac{4}{5}\right) - 8x = 15.6 - 8x$$

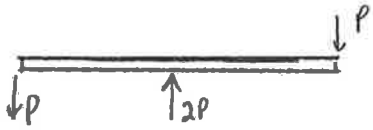
Shear diagram :

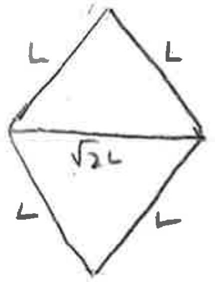


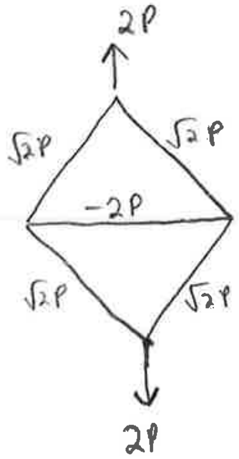
Bending diagram :



(4) a)







	e $\left(\times \frac{PL}{AE}\right)$	T	T × e $\left(\frac{P^2L}{AE}\right)$
4 edges	$\sqrt{2}$	$\sqrt{2}P$	2
DE	$-2\sqrt{2}$	$-2P$	$4\sqrt{2}$

Total : $\left[2(4) + 4\sqrt{2}\right]\frac{P^2L}{AE}$

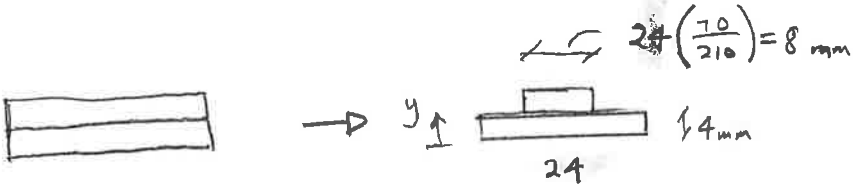
$\sum P\delta = \sum Te \rightarrow 2P\delta = 8 + 4\sqrt{2}\ \frac{P^2L}{AE}$

$$\delta_B = (4 + 2\sqrt{2})\frac{PL}{AE}$$

b) Disp. @ C =

$$\delta_C = 2\delta_B$$

(5) a)



$$\bar{y} = \frac{\Sigma A y}{\Sigma A} = \frac{24(4)(2) + 8(4)(6)}{24(4) + 8(4)} = \frac{384}{128} = 3 \text{ mm}$$

$$I_{xx,steel} = \frac{24(4)^3}{12} + 24(4)(1)^2 + \frac{8(4)^3}{12} + 8(4)(3)^2 = 555 \text{ mm}^4$$

$$EI = 210 \times 10^9 \left(555 \times 10^{-12}\right) = \underline{\underline{116.5 \text{ N}\cdot\text{m}^2}}$$

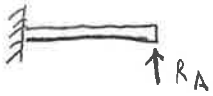
b) $K = 0.001(20) = 0.02$

$$\frac{d^2v}{dx^2} = -0.02 \rightarrow \frac{dv}{dx} = -0.02x + C_1 \rightarrow \left.\frac{dv}{dx}\right|_{x=0} = 0 \rightarrow C_1 = 0$$

$$v(x) = -\frac{0.02x^2}{2} + C_2 \rightarrow v(0) = 0 \rightarrow C_2 = 0$$

$$v(x) = -0.01x^2 \rightarrow \underline{\underline{v(1) = -0.01 \text{ m}}}$$

c) Superpose: Part (b) +

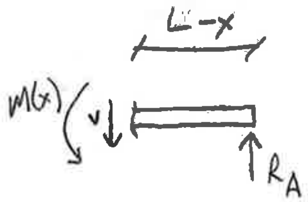


$= 0$

$$\therefore \delta_{end} = \frac{R_A L^3}{3EI} \rightarrow R_A = \frac{3EI\,\delta_{end}}{L^3} = \frac{3(116.5)(0.01)}{1^3}$$

$$\underline{\underline{R_A = 3.49 \text{ N}}}$$

(5) c) (ii) Cut end of beam:



$$\Sigma M \Rightarrow (L-x)R_A + M(x) = 0 \;\Rightarrow\; M(x) = -R_A(1-x)$$

$$\frac{d^2v}{dx^2} = \frac{R_A}{EI}(1-x)$$

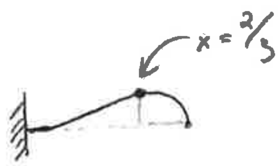
$$\frac{dv}{dx} = \frac{R_A}{EI}\left(x - \frac{x^2}{2}\right) + \cancel{C_1} \to 0 \qquad \frac{dv}{dx} = 0 \;@\; x=0$$

Add temperature:
$$\frac{dv}{dx} = 0.03\left(x - \frac{x^2}{2}\right) - 0.02x$$

$$\frac{dv}{dx} = 0.01x - 0.015x^2$$

$$\frac{dv}{dx} = 0 \;\Rightarrow\; x = 0,\ \frac{2}{3}$$

$$\therefore\; x = \frac{2}{3}$$



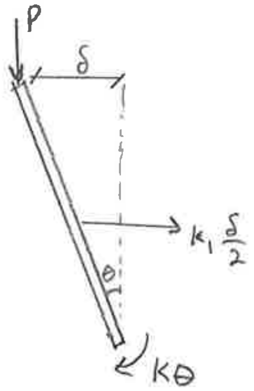
d) Curvature must be equal and opposite for both cases.

Temp:


$$\theta(x) = C(1-x)$$

C ← Constant

⑥ a)



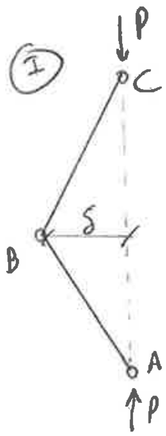
$$\delta \approx 2L\theta$$

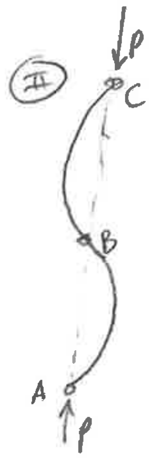
$$\Sigma M_A: \; P\delta - k_1 \frac{\delta}{2} L + k\theta = 0$$

$$P(2L\theta) - k_1(L\theta)L + k\theta = 0$$

$$P = \frac{k_1 L^2 + K}{2L}$$

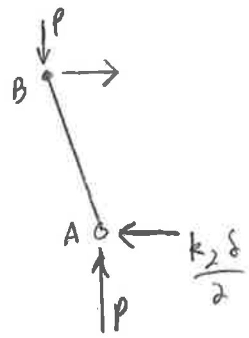
b) (i)





Mode (II): $P_{cr} = \frac{\pi^2 EI}{L^2}$

(ii) Mode (I):



$$\delta \approx L\theta$$

$$\Sigma M_B: \; PL\theta - \frac{k_2 L\theta}{2} L = 0$$

$$P_{cr} = \frac{k_2 L}{2}$$

Find k_{cr}: $\frac{\pi^2 EI}{L^2} = \frac{k_2 L}{2} \rightarrow k_{2,cr} = \frac{2\pi^2 EI}{L^3} \rightarrow$ If $k_2 < k_{2,cr} \rightarrow$ Mode (I)

If $k_2 > k_{2,cr} \rightarrow$ Mode (II)

(c) If $k_2 = 0 \rightarrow P_{cr} = \frac{\pi^2 EI}{4L^2}$

If $k_2 = \infty \rightarrow P_{cr} = \frac{\pi^2 EI}{L^2}$

$$\therefore \; \frac{\pi^2 EI}{4L^2} < P_{cr} < \frac{\pi^2 EI}{L^2}$$

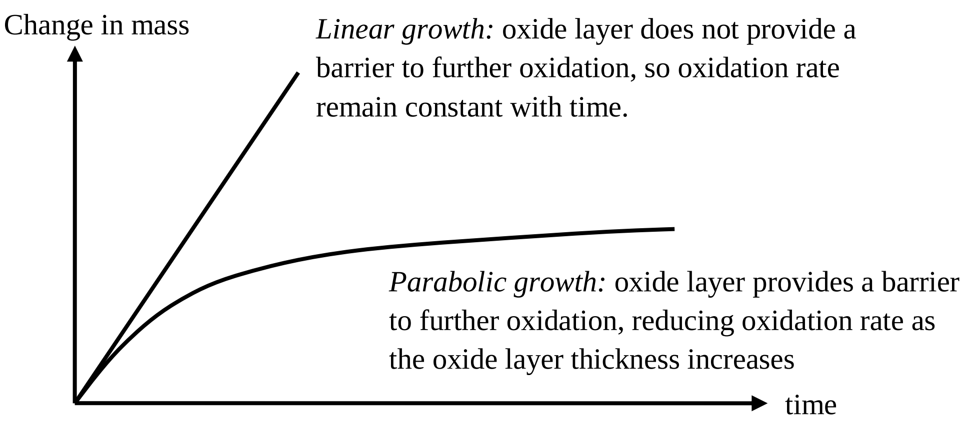
Part IA Paper 2 2013 Section B: Materials G J McShane

Q7 (Short)

(a) Formation of positive metal ions: $M \rightarrow M^{2+} + 2e^-$
Formation of negative oxygen ions: $O_2 + 4e^- \rightarrow 2O^{2-}$
Formation of the oxide: $M^{2+} + O^{2-} \rightarrow MO$

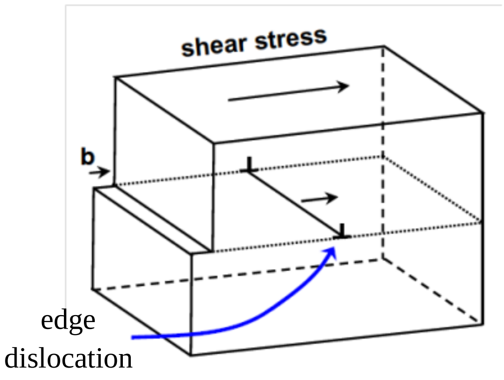
Free energy of oxidation: The net free energy change that takes place during the reaction. It can be interpreted as the thermodynamic driving force for the reaction. Positive free energy change: there is no driving force, the reaction will not spontaneously happen. The more negative the change: the greater the driving force for the reaction.

(b)

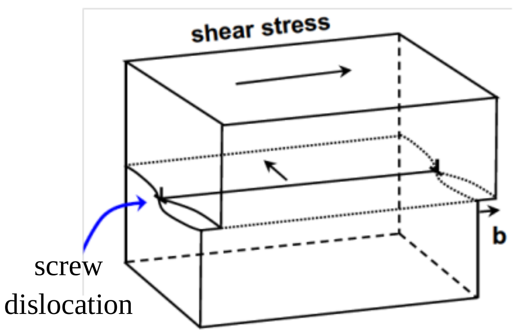


Q8 (short)

(a) (i)



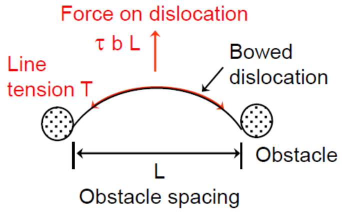
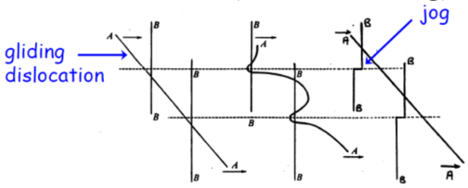
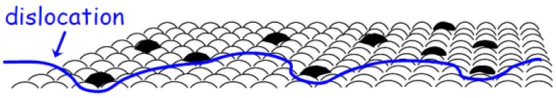
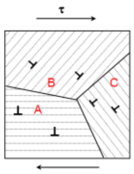
line direction and Burgers vector (**b**) are perpendicular



line direction and Burgers vector (**b**) are parallel

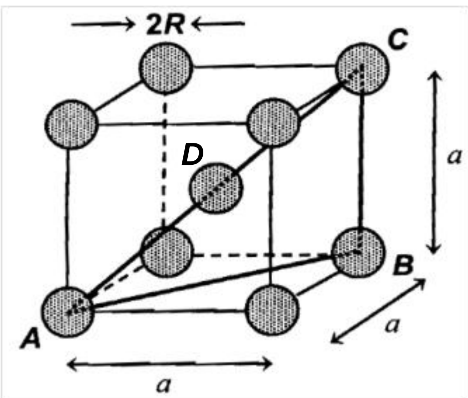
(ii) Dislocations glide through the material in response to an applied shear stress (as shown above). Passage of the dislocation results in a permanent shear deformation of the crystal lattice. The cumulative effect of a large number of gliding dislocations leads to macroscopic plastic deformation.

(b) Any two of the following:

Microstructural feature	Parameter that can be controlled
Precipitate (precipitation hardening)  	Precipitate size and spacing
Other dislocations (work hardening)  	Dislocation density, through plastic straining
Solute atoms (solid solution hardening)  	Type and weight fraction of solute atoms
Grain boundaries 	Grain size

Q9 (short)

(a)



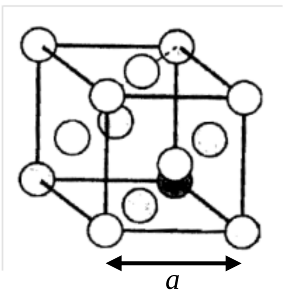
BCC unit cell: Atom D lies in the centre of a cube with one atom at each corner. Atomic radius is R. The atoms touch along ADC (atoms shown smaller, for clarity).

$$AB = \sqrt{2}a$$

$$AC = \sqrt{AB^2 + BC^2} = \sqrt{3a^2} = 4R$$

$$\therefore a = \frac{4R}{\sqrt{3}}$$

(b)



FCC, atoms per unit cell: $n = 8\left(\frac{1}{8}\right) + 6\left(\frac{1}{2}\right) = 4$

Size of the unit cell (Data Book): $a = 4.0496 \times 10^{-10}$ m

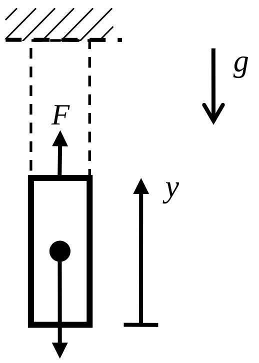
Molar mass of aluminium (Data Book): $M = 26.982$ kg kmol^{-1}

Mass per atom: $m = \frac{M}{N_A} = \frac{26.982}{6.022\times10^{26}} = 4.48\times10^{-26}$ kg atom^{-1} (N_A = Avogadro's number)

Theoretical density: $\rho = \frac{nm}{a^3} = 2699$ kg m^{-3}

Q10 (short)

(a) Consider a free body diagram for an element of column of height y :



Weight: $Mg = \rho A y g$

Tensile force at y (vertical equilibrium):

$$F = \rho A y g$$

Tensile stress at y :

$$\sigma = \frac{F}{A} = \rho g y$$

(b) Survival probability for non-uniform stress distribution (from Data Book):

$$P_s(V) = exp\left[-\int_V \left(\frac{\sigma}{\sigma_0}\right)^m \frac{dV}{V_0}\right]$$

(σ_0, V_0, m are material constants). Evaluating the integral for the stress distribution in the column:

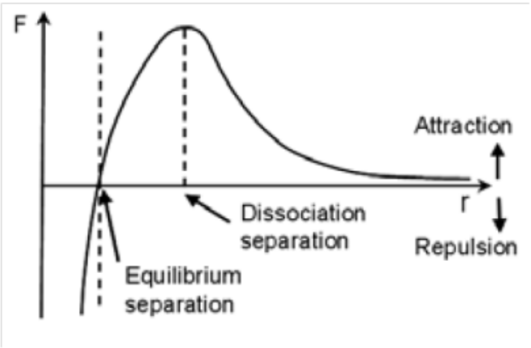
$$\int_V \left(\frac{\sigma}{\sigma_0}\right)^m \frac{dV}{V_0} = \int_0^L \left(\frac{\rho g y}{\sigma_0}\right)^m \frac{A dy}{V_0} = \left(\frac{\rho g}{\sigma_0}\right)^m \frac{A}{V_0}\left[\frac{y^{m+1}}{m+1}\right]_0^L = \left(\frac{\rho g}{\sigma_0}\right)^m \frac{AL^{m+1}}{V_0(m+1)}$$

Hence:

$$P_s(V) = exp\left[-\left(\frac{\rho g}{\sigma_0}\right)^m \frac{AL^{m+1}}{V_0(m+1)}\right]$$

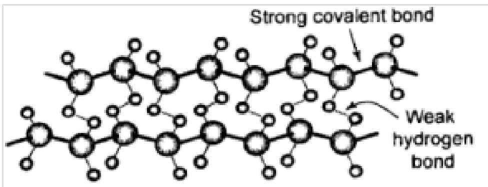
Q11 (long)

(a) (i) Primary bonds in metals behave as stiff elastic springs, with force-displacement response of the form:



The gradient of the force-displacement plot at the equilibrium separation gives the bond stiffness (which is approximately constant for small changes in separation). The bond stiffness is directly related the Young's modulus.

(ii) Thermoplastics consist of long molecular chains (formed by strong covalent bonds) bonded together via relatively low stiffness secondary bonds. The stretching of these secondary bonds determines the Young's modulus of the polymer.

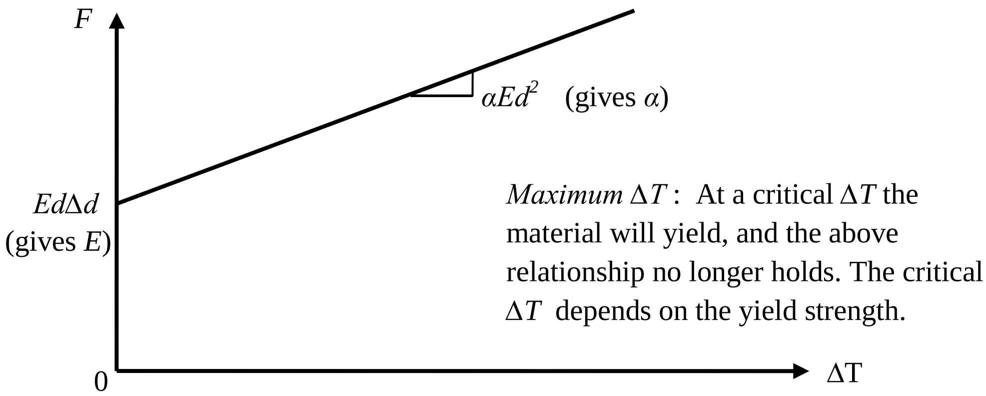


(b) Define compressive strains positive. The total strain is the sum of elastic and thermal components:

$$\varepsilon = \varepsilon^{\mathrm{el}} + \varepsilon^{\mathrm{th}} = \frac{\sigma}{E} - \alpha\Delta T = \frac{F}{Ed^2} - \alpha\Delta T \quad \text{and} \quad \varepsilon = \frac{\Delta d}{d} \quad \text{(held constant)}$$

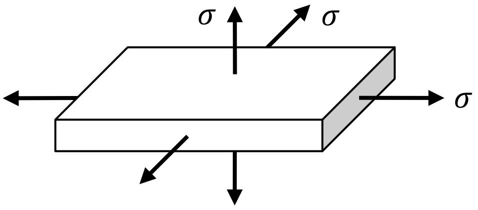
(*Note:* these are nominal stresses and strains. Many candidates overcomplicated the analysis by attempting to work out the evolution of true stress. Another common error at this stage was not accounting correctly for the signs of tensile and compressive strains - a clear sign convention is essential.) Eliminating ε and rearranging:

$$F = Ed^2\left(\frac{\Delta d}{d} + \alpha\Delta T\right)$$



(c) (i) The total strain in each direction is the sum of elastic and thermal components: $\varepsilon = \varepsilon^{\mathrm{el}} + \varepsilon^{\mathrm{th}}$. The thermal strain is all directions will be the same: $\varepsilon^{\mathrm{th}} = \alpha\Delta T$. The total strain in all directions will be the same, due to the constraint: $\varepsilon = 0$. The elastic strains will therefore also be the same in all three directions, and hence so will the stresses.

(ii) Taking tensile stresses and strains positive, and $\sigma = \sigma_x = \sigma_y = \sigma_z$, the elastic strain in any direction:



$$\varepsilon^{\mathrm{el}} = \frac{\sigma}{E} - \nu\left(\frac{\sigma}{E} + \frac{\sigma}{E}\right) = \frac{\sigma}{E}(1 - 2\nu) \qquad (1)$$

The thermal strain in any direction:

$$\varepsilon^{\mathrm{th}} = \alpha\Delta T \qquad (2)$$

The total strain in any direction:

$$\varepsilon = \varepsilon^{\mathrm{el}} + \varepsilon^{\mathrm{th}} = 0 \qquad (3)$$

Hence, substituting *(1)* and *(2)* into *(3)* and rearranging:

$$\frac{\sigma}{E}(1 - 2\nu) + \alpha\Delta T = 0$$

$$\therefore \sigma = -\frac{E\alpha\Delta T}{(1 - 2\nu)}$$

(*Note:* care is also needed with signs at this stage. With the tensile strain positive sign convention, a negative temperature change ΔT, i.e. thermal contraction, will lead to a positive tensile stress in the constrained adhesive.) Substituting in the given values: E = 2.5 GPa, ν = 0.40, $\alpha = 10^{-5}$ °C^{-1}, $\Delta T = -25$ °C:

$$\sigma = -\frac{(2.5 \times 10^{9})(10^{-5})(-25)}{(1 - 0.8)} = 3.13 \text{ MPa}$$

Q12 (long)

(a) Maximum bending moment: $M = \frac{wL^2}{2}$

Bending stress in the cross-section for a distributed load w:

$$\sigma = \frac{My}{I} = \frac{wL^2}{2}.\frac{d}{2}.\frac{12}{bd^3\left(1-\lambda^4\right)} = \frac{3wL^2}{bd^2\left(1-\lambda^4\right)} = 7940w$$

The mean stress in the beam is therefore:

$$\sigma_{\mathrm{m}} = 7940 \times 40 \times 10^3 = 317.6 \text{ MPa}$$

Similarly, bending stress range relates to the distributed load range Δw:

$$\Delta\sigma = \frac{3\Delta wL^2}{bd^2\left(1-\lambda^4\right)} = 7940\Delta w = 7940 \times 5 \times 10^3 = 39.7 \text{ MPa}$$

To apply Basquin's law using the given constants, the stress range at mean stress σ_{m} needs to be transformed to the zero mean stress equivalent using Goodman's rule:

$$\Delta\sigma_0 = \frac{\Delta\sigma}{1-\frac{\sigma_{\mathrm{m}}}{\sigma_{\mathrm{ts}}}} = \frac{39.7}{1-\frac{317.6}{400}} = 192.7 \text{ MPa}$$

Basquin's law, using the constants provided, gives the number of cycles to failure:

$$N_f = \left(\frac{C_1}{\Delta\sigma_0}\right)^{1/\propto} = \left(\frac{650}{192.7}\right)^{1/0.07} = 3.50 \times 10^7 \text{ cycles}$$

(b) (i) For a reference solid square cross-section of area $A = B^2$:

$$I_{\mathrm{REF}} = \frac{BB^3}{12} = \frac{A^2}{12}$$

The area of the shaped cross-section:

$$A = bd - \lambda^2 bd = bd(1-\lambda^2)$$

Substituting for A into the expression for I_{REF}:

$$I_{\mathrm{REF}} = \frac{b^2d^2\left(1-\lambda^2\right)^2}{12}$$

Using the given expression for I, the shape factor:

$$\phi = \frac{I}{I_{\mathrm{REF}}} = \frac{12}{b^2d^2\left(1-\lambda^2\right)^2}.\frac{bd^3\left(1-\lambda^4\right)}{12} = \frac{d(1+\lambda^2)(1-\lambda^2)}{b\left(1-\lambda^2\right)^2} = \frac{d(1+\lambda^2)}{b(1-\lambda^2)}$$

(ii) Objective equation, beam mass: $m = \rho A L$

Constraint equation, tip deflection (using Structures Data Book beam formulae):

$$\delta = \frac{wL^4}{8EI} = \frac{wL^4}{8E\phi I_{\mathrm{REF}}} = \frac{12wL^4}{8E\phi A^2}$$

Eliminate the free variable from the objective using the constraint, and group material and shape terms:

$$A = \sqrt{\frac{3wL^4}{2E\phi\delta}} \quad , \quad \therefore\ m = \rho L \sqrt{\frac{3wL^4}{2E\phi\delta}} = \frac{\rho}{\sqrt{E\phi}} \sqrt{\frac{3w}{2E\phi\delta}} L^3$$

Therefore, the performance metric to maximise: $M = \sqrt{E\phi}/\rho$

(iii) Using the selection charts in the Materials Data Book, wood has a slightly higher $\sqrt{E}/\rho$ than Al alloy (~100 versus ~98), if oriented longitudinally. But wood will have a much lower maximum shape factor ϕ due to a low E and manufacturing constraints. Hence, Al alloy is likely to give the lightest solution, accounting for shaping.