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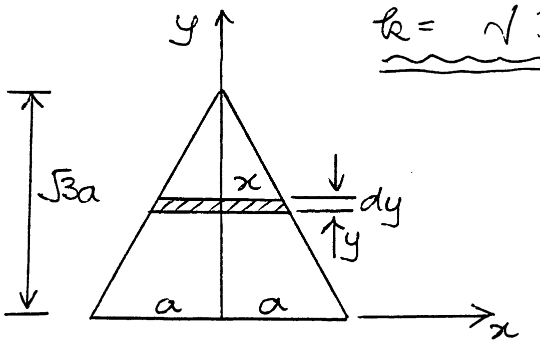
Part IB - 1996 - Paper 1 Mechanics - Solutions

1 (a) If a body mass m has moment of inertia I about a particular axis then the radius of gyration about that axis is defined as k where

$$\underline{k = \sqrt{I/m}}$$



Area of lamina $= \frac{1}{2} \cdot 2a \cdot \sqrt{3}a$

So if mass of lamina $= m$

surface density $\rho = \underline{\dfrac{m}{\sqrt{3}a^2}}$

$$I_{xx} = \int y^2 dm = \int_{y=0}^{\sqrt{3}a} y^2 \, 2x\rho \, dy$$

But $y = \sqrt{3}a - \sqrt{3}x$, so $x = a - \dfrac{y}{\sqrt{3}}$

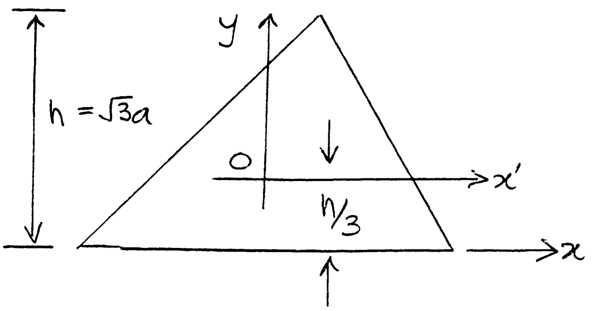
$$\therefore \quad I_{xx} = 2\rho \int ay^2 - \frac{y^3}{\sqrt{3}} dy$$

$$= \frac{2m}{\sqrt{3}a^2}\left[\frac{ay^3}{3} - \frac{y^4}{4\sqrt{3}}\right]_0^{\sqrt{3}a}$$

$$= \frac{2m}{\sqrt{3}a^2}\left\{\sqrt{3} - \frac{3\sqrt{3}}{4}\right\} a^4$$

$$= \frac{ma^2}{2} \qquad \therefore \quad \underline{k = \sqrt{I/m} = \frac{a}{\sqrt{2}}}$$

Alternatively - from Data Book (1989) ed p10



$$I_{x'x'} = \frac{mh^2}{18}$$

||el axis

$$\therefore I_{xx} = \frac{mh^2}{18} + m\left(\frac{h}{3}\right)^2$$

$$= \frac{mh^2}{6}$$

$$\therefore \quad k_{xx} = \frac{h}{\sqrt{6}} = \underline{\frac{a}{\sqrt{2}}}$$

(b) $\underline{OR} = \underline{OQ} + \underline{QR}$

$\underline{\mathring{OR}} = \underline{\mathring{OQ}} + \underline{\mathring{QR}}$ $\qquad$ $\underline{OQ}$ rotates at $\underline{\omega_2} = \omega_2\underline{k}$

$\qquad\qquad\qquad\qquad\qquad$ $\underline{QR}$ " " $(\underline{\omega_1}+\underline{\omega_2}) = (\omega_1\underline{i} + \omega_2\underline{k})$

$$\therefore\ \underline{\mathring{OR}} = \underline{\omega_2} \times \underline{OQ} + (\underline{\omega_1}+\underline{\omega_2}) \times \underline{QR}$$

Since both OQ and QR are of fixed length

$$\underline{\ddot{OR}} = \underline{\dot{\omega}_2} \times \underline{OQ} + \underline{\omega_2} \times \underline{\mathring{OQ}} + (\underline{\dot{\omega}_1} + \underline{\dot{\omega}_2}) \times \underline{QR} + (\underline{\omega_1}+\underline{\omega_2}) \times \underline{\mathring{QR}}$$

But $\underline{\dot{\omega}_2} = 0$ because $\underline{\omega_2}$ does not rotate

and $\underline{\dot{\omega}_1} = \underline{\omega_2} \times \underline{\omega_1}$ because $\underline{\omega_1}$ rotates at $\underline{\omega_2}$

$$\therefore\ \underline{\ddot{OR}} = \underline{\omega_2} \times \underline{\mathring{OQ}} + \underline{\dot{\omega}_1} \times \underline{QR} + (\underline{\omega_1}+\underline{\omega_2}) \times \underline{\mathring{QR}}$$

	$\underline{i}$	$\underline{j}$	$\underline{k}$
$\underline{\omega_2}$	0	0	ω_2
$\underline{OQ}$	q	0	0
$\underline{\mathring{OQ}} = \underline{\omega_2} \times \underline{OQ}$	0	$q\omega_2$	0
$\underline{\omega_1}+\underline{\omega_2}$	ω_1	0	ω_2
$\underline{QR}$	0	0	r
$\underline{\mathring{QR}} = (\underline{\omega_1}+\underline{\omega_2}) \times \underline{QR}$	0	$-r\omega_1$	0
$\underline{v_R} = \underline{\mathring{OR}}$	0	$(q\omega_2 - r\omega_1)$	0
$\underline{\omega_2}$	0	0	ω_2
$\underline{\mathring{OQ}}$	0	$q\omega_2$	0
$\underline{\omega_2} \times \underline{\mathring{OQ}}$	$-q\omega_2^2$	0	0
$\underline{\omega_1}$	ω_1	0	0
$\underline{\dot{\omega}_1} = \underline{\omega_2} \times \underline{\omega_1}$	0	$\omega_1\omega_2$	0
$\underline{QR}$	0	0	r
$\underline{\dot{\omega}_1} \times \underline{QR}$	$r\omega_1\omega_2$	0	0
$\underline{\omega_1}+\underline{\omega_2}$	ω_1	0	ω_2
$\underline{\mathring{QR}}$	0	$-r\omega_1$	0
$(\underline{\omega_1}+\underline{\omega_2}) \times \underline{\mathring{QR}}$	$r\omega_1\omega_2$	0	$-r\omega_1^2$
$\underline{a_R}$	$(-q\omega_2^2 + 2r\omega_1\omega_2)$	0	$-r\omega_1^2$

2 (a) Required speed is magnitude of the component of velocity of A in the direction of AB

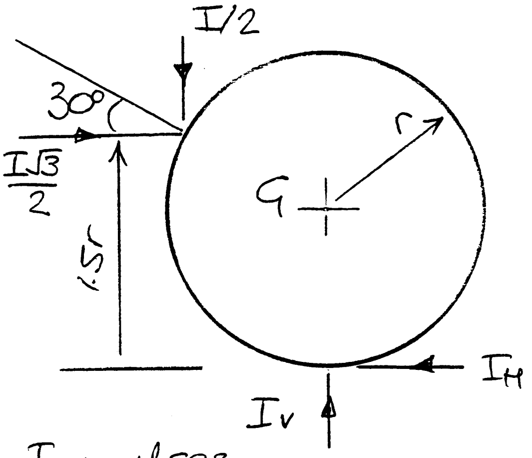
i.e. $|\underline{v}_A \cdot \underline{e}_{AB}|$ $\qquad$ $\underline{e}_{AB}$ is unit vector along $\underline{AB}$

But $\underline{v}_A = \underline{\omega} \times \underline{OA}$

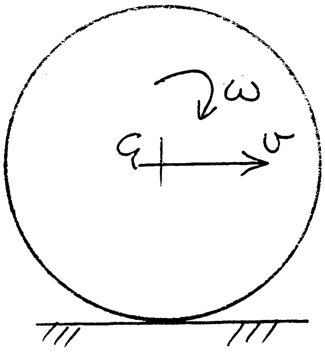
		$\underline{i}$	$\underline{j}$	$\underline{k}$	
$\underline{\omega}$		0	5	0	rad s^{-1}
$\underline{OA}$		1	3	1	m
$\underline{v}_A$	$\underline{\omega} \times \underline{OA}$	5	0	−5	m s^{-1}
$\underline{AB}$		1	1	−1	m
$\underline{e}_{AB}$		$\frac{1}{\sqrt{3}}$ (1	1	−1)	

$$\therefore \; |\underline{v}_A \cdot \underline{e}_{AB}| = \frac{5+5}{\sqrt{3}} = \frac{10}{\sqrt{3}} \text{ m s}^{-1}$$

(b)



Impulses



Suppose impulsive forces >> mg

Motion

I_V and I_H are components of the impulse acting at point of contact with ground. C of G moves with velocity v and sphere has angular vel^y ω.

To test for slip, evaluate v and ω on basis of limiting friction $\mu = 0.2$ and test for $v = r\omega$,

$$I_V = I/2 \quad \text{and} \quad I\frac{\sqrt{3}}{2} - I_H = mv$$

$$\therefore \text{ If } I_H = 0.2\, I/2 \qquad v = \frac{\sqrt{3} - 0.2}{2}\frac{I}{m} = 0.766\frac{I}{m}$$

Rotation about G, $I_G = \frac{2}{5} mr^2$ ← Data Book p12

$$\therefore \quad 0.2 \times \frac{I}{2} \times r = \frac{2}{5} m r^2 \omega$$

$$r\omega = 0.25 \frac{I}{m}$$

Clearly $v > r\omega$ so sphere does indeed slip.

Alternatively - evaluate μ for no slip and compare to given value.

Linear motion: $mv = \frac{I\sqrt{3}}{2} - I_H$

rotation: $I_H . r = \frac{2}{5} m r^2 \omega$

no slip so rolling and $v = r\omega$

$$\therefore \quad \frac{I\sqrt{3}}{2} - I_H = \frac{5}{2} I_H$$

$$\therefore \quad I_H = \frac{I\sqrt{3}}{7}$$

But $I_V = I/2$ $\therefore$ reqd. $\mu = \frac{2\sqrt{3}}{7} = 0.495$

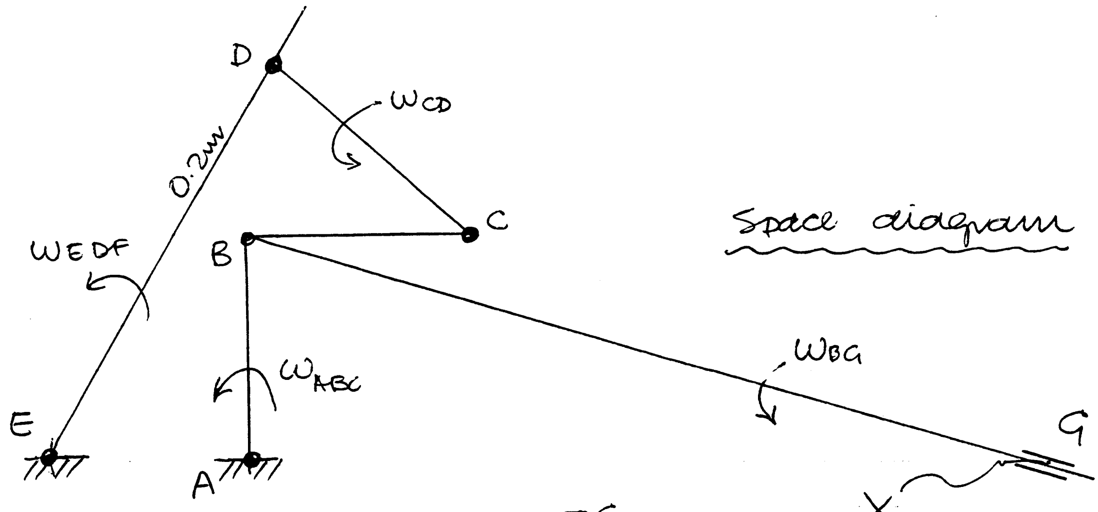
as this is > 0.2, there must be slip.

Examiner's comments on Q's 1 & 2 below:

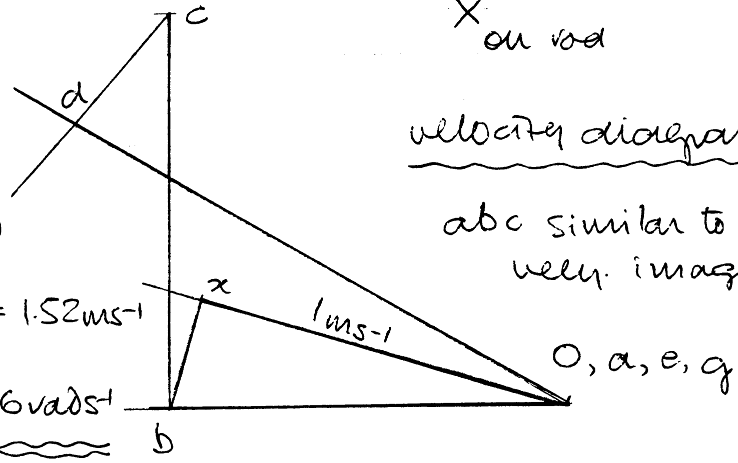
1. Apart from those candidates who misinterpreted the phrase equilateral triangle part (a) was reasonably well done - usually from first principles - less frequently from the general triangle case in the Data Book plus the parallel axis theorem. In part (b) the successful solutions started from first principles differentiating twice and noting the appropriate vector differentiations. Application of data book formula was less successful (though not always so) most errors arising from a failure to appreciate that the fact that the magnitude of the vector $\underline{\omega}_1$ is constant does not imply that $\underline{\dot{\omega}}_1$ is equal to zero.

2. In part (a) many candidates lost marks because having correctly established the velocity of point A they resolved this in the direction of <u>AB</u> by taking a dot product with <u>AB</u> rather than the *unit* vector along <u>AB</u>. In (b) some confusion between forces and impulses, often both appeared on the same diagram. Many miscalculated the angle to the horizontal made by *I* and others misread the Data Book value for the moment of inertia of a sphere (taking that of spherical shell - there is some scope for improved clarity in the Data Book here).

3.



Space diagram



velocity diagram

ED = 0.2m

abc similar to ABC
vely. image.

from vely. diagram

ed $\simeq \frac{76}{50} \times 1$ ms-1 = 1.52 ms-1

$\therefore \omega_{EDF} = \frac{1.52}{0.2} = 7.6$ rads-1

Evaluation of accelerations requires knowledge of all angular velocities:

$\omega_{ABC} = \frac{ab}{AB} = \frac{1.06 \text{ ms}^{-1}}{0.1 \text{ m}} = \underline{10.6 \text{ rads}^{-1}}$

$\omega_{CD} = \frac{cd}{CD} = \frac{0.4 \text{ ms}^{-1}}{0.117 \text{ m}} = \underline{3.42 \text{ rads}^{-1}}$

$\omega_{BG} = \frac{xb}{GB} = \frac{0.3 \text{ ms}^{-1}}{0.366 \text{ m}} = \underline{0.82 \text{ rad s}^{-1}}$

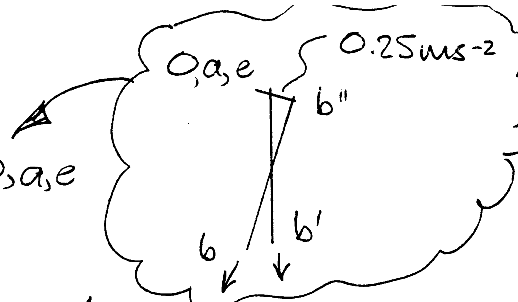
To draw the acceleration diagram we note that ABCDE forms a four bar chain and is such that point B has no acceleration ||el to BG other than the centripetal term because the ram is extending at a constant speed of 1 ms-1.

B accelerates towards G at $\omega_{BG}^2 . GB = 0.82^2 \times 0.366$
$= \underline{0.25 \text{ ms}^{-2}}$

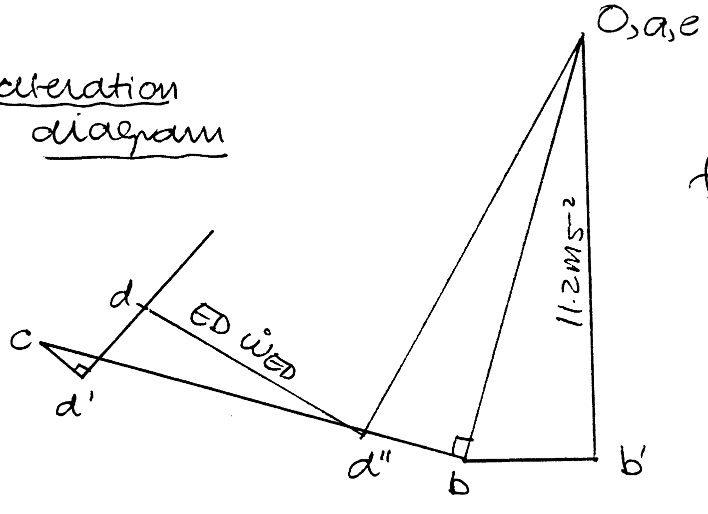
and B " " A at $\omega_{ABC}^2 . AB = \underline{11.2 \text{ ms}^{-2}}$

D " " C at $3.42^2 \times 0.117 = \underline{1.37 \text{ ms}^{-2}}$

D " " E at $7.6^2 \times 0.2 = \underline{11.6 \text{ ms}^{-2}}$



<u>acceleration diagram</u>



from diagram

$$\dot{\omega}_{ED} = \frac{6.8}{0.2} = 34 \text{ rad s}^{-2}$$

abc similar to ABC
(accln. image)

<u>Alternatively</u>, and a bit more laboriously we can start the acceleration diagram at G. Since the cylinder/ram is both extending and rotating (so that there is both an $\dot{r}$ and a $\dot{\theta}$) there will be a Coriolis accln. As the position of the piston, X, within the cylinder is not specified we choose to put it in the most convenient position i.e. coincident with G; this has the advantage of putting $r = 0$.

Coriolis accln at G $= 2\,\underline{\omega_{BG}} \times \underline{gx}$ ← from vely diagram

i.e. component of accln of X $\perp^r$ to GB $= 2 \times \frac{xb}{XB} \times gx$ ↙ to BG

$$= 2 \times \frac{0.3 \text{ ms}^{-1}}{0.366 \text{ m}} \times 1 = 1.65 \text{ ms}^{-2} \swarrow$$

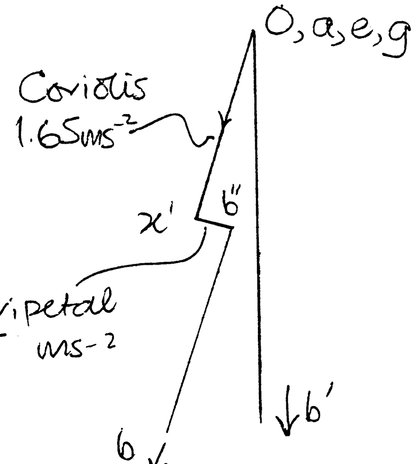
X has no component of acceleration // GB because ram moves at const. speed

B accelerates → X at $\omega_{BG}^2 \times GB = \left(\frac{0.3}{0.366}\right)^2 \times GB = 0.25 \text{ ms}^{-2}$ ↘

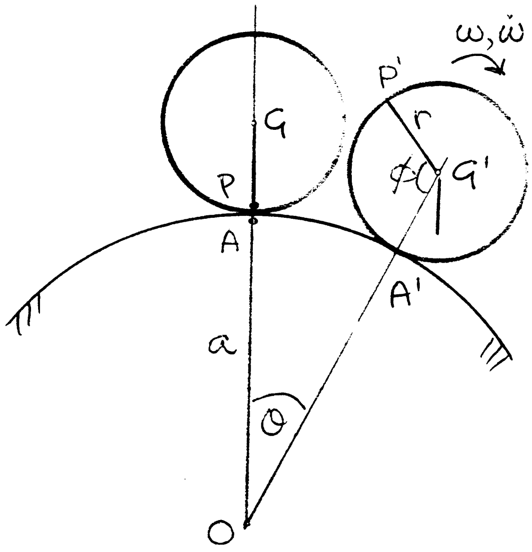
Thus b can be located on acceleration diagram

(drawn at an enlarged scale for clarity)

remainder of diagram as <u>before</u>



4.



No slip, so that

$$a\theta = r\phi$$

but hoop has rotated through an angle $(\theta + \phi)$

ie. through $\theta + \frac{a}{r}\theta$

$$= \underline{\left(1 + \frac{a}{r}\right)\theta}$$

P and G move to P' and G' respectively

$$\therefore \quad \omega = \left(1 + \frac{a}{r}\right)\dot{\theta}$$

$$\dot{\omega} = \left(1 + \frac{a}{r}\right)\ddot{\theta}$$

In moving from G to G' mechanical energy is conserved, so that:

$$\text{loss of pe} = mg(a+r)(1-\cos\theta)$$

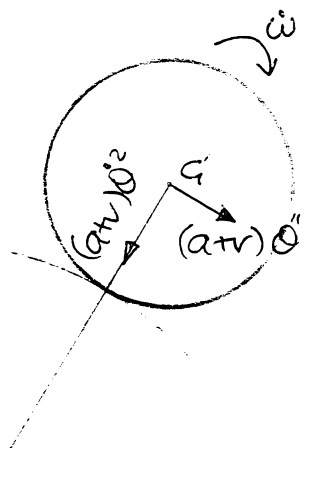
$$= \text{gain of ke} = \underbrace{\tfrac{1}{2}m(a+r)^2\dot{\theta}^2}_{\text{linear}} + \underbrace{\tfrac{1}{2}I_G\,\omega^2}_{\text{rotation}}$$

But I_G hoop $= mr^2$

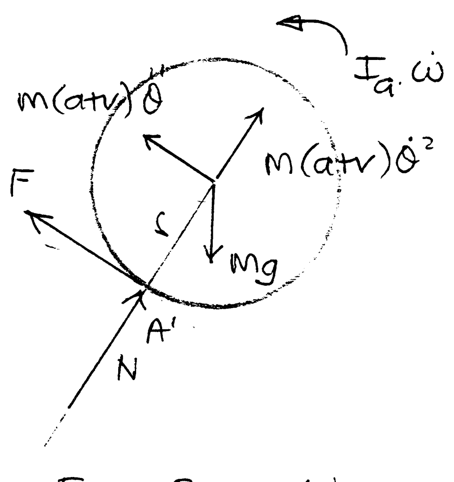
$$\therefore \quad mg(a+r)(1-\cos\theta) = \tfrac{1}{2}m(a+r)^2\dot{\theta}^2 + \tfrac{1}{2}.mr^2\frac{(a+r)^2}{r^2}\dot{\theta}^2$$

$$\text{ie.} \quad g(1-\cos\theta) = \tfrac{1}{2}(a+r)\dot{\theta}^2 + \tfrac{1}{2}(a+r)\dot{\theta}^2$$

$$\text{ie.} \quad \underline{\dot{\theta} = \sqrt{\frac{(1-\cos\theta)g}{a+r}}}$$



accelerations



Free Body Diagram

From F.B.D including D'Alembert forces

$$F = mg\sin\theta - m(a+r)\ddot{\theta}$$

$$N = mg\cos\theta - m(a+r)\dot{\theta}^2$$

Moments about point of contact A'

$$mgr\sin\theta = m(a+r)\ddot{\theta}\cdot r + mr^2\cdot\dot{\omega}$$

$$\text{ie. } g\sin\theta = (a+r)\ddot{\theta} + (a+r)\ddot{\theta}$$

$$\text{So } \ddot{\theta} = \frac{g\sin\theta}{2(a+r)}$$

$$\therefore \begin{cases} F = mg\sin\theta - \frac{1}{2}mg\sin\theta \\ N = mg\cos\theta - mg(1-\cos\theta) = mg(2\cos\theta - 1) \end{cases}$$

$$\therefore \frac{F}{N} = \frac{\sin\theta}{2(2\cos\theta - 1)}$$

If on point of slipping when $\theta = 30°$

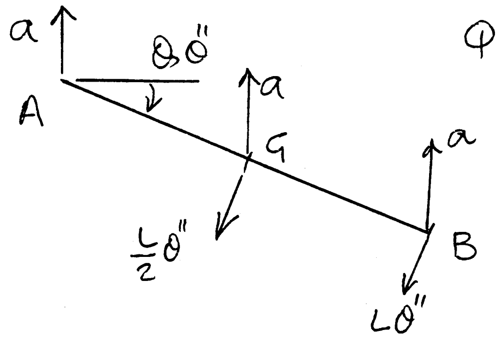
$$\text{So } \mu = \frac{1/2}{2(2.\sqrt{3}/2 - 1)} = \frac{1}{4(\sqrt{3}-1)} = 0.342$$

Examiner's comments on Q's 3 & 4.

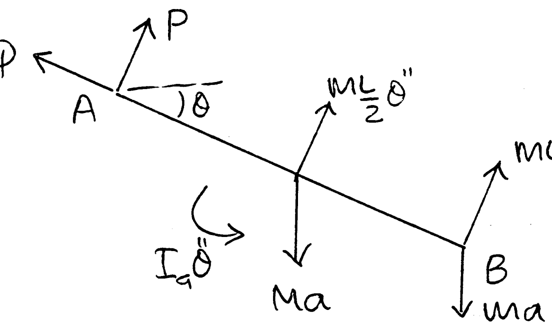
3. Most drew acceptable velocity diagrams and confirmed the required result. The acceleration diagram caused more problems often because the position of the ram in the cylinder is not specified - the neat solution being to put it at G. Some candidates realised that since ECDA is a four bar chain actually there is no necessity to get embroiled with Coriolis at this effective slider pivot at all. Solution drawn to larger scale.

4. The first sentence suggests that energy is the approved method and the initial question about the angle turned through by the hoop meant as a strong hint that this is not just $\frac{a}{r}\theta$; it is, of course $(\frac{a}{r} + 1)\theta$; a point appreciated by only about a dozen candidates despite this part of the problem being almost identical to an examples paper question. Some candidates got the angle wrong but then wrote down the correct relationship between the angular velocities. If you both make this error *and* ignore the moment of inertia of the hoop the resultant expression for θ is that displayed with a two on the top under the square root. Several candidates achieved this result. Other variants are possible by taking the moment of inertia of the hoop as $\frac{1}{2}mr^2$ etc. Many candidates solved the final part without any reference to dynamics at all - as if the hoop were glued on the cylinder.

5. D'Alembert's technique of introducing inertia forces and torques, equal to $-m\underline{a}$ and $-I_G\ddot{\theta}$ respectively enable problems in dynamics to be analysed by the techniques of statics.



Kinematics
$(\dot{\theta}=0)$



Free Body Diagram

Taking moments about end A

$$0 = ma\frac{L}{2}\cos\theta + maL\cos\theta - m\frac{L}{2}\ddot{\theta}\cdot\frac{L}{2} - mL\ddot{\theta}\cdot L - I_G\ddot{\theta}$$

But for uniform rod length L, mass m $I_G = \frac{mL^2}{12}$

$$\therefore\ 0 = \frac{3}{2}maL\cos\theta - \frac{5}{4}mL^2\ddot{\theta} - \frac{1}{12}mL^2\ddot{\theta}$$

i.e. $$\ddot{\theta} = \frac{12}{16}\cdot\frac{3}{2}\frac{a}{L}\cos\theta = \frac{9a\cos\theta}{8L}$$

Resolving ⊥r rod:

$$P + m\frac{L}{2}\ddot{\theta} + mL\ddot{\theta} - 2ma\cos\theta = 0$$

and ||el rod

$$\Phi - 2ma\sin\theta = 0$$

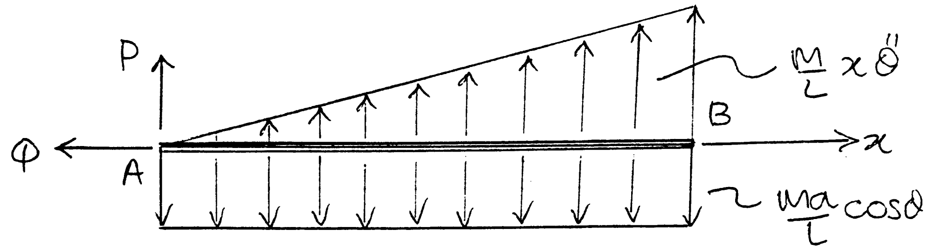
Now, if $\theta = 30°$ $\cos\theta = \sqrt{3}/2$ and $\sin\theta = 1/2$

$\therefore\ P/ma = \left(2 - \frac{3}{2}\cdot\frac{9}{8}\right)\cos\theta = \frac{5\sqrt{3}}{32}$ and $\Phi/ma = 1$

$\therefore$ Magnitude of resultant $= \sqrt{P^2+\Phi^2} = \frac{\sqrt{1099}}{32}ma$

$$= 1.036\,ma$$

Consider inertia loads acting ⊥ly to rod AB



shear force gradient $\frac{dF}{dx} = w(x)$

effective transverse loading

i.e. $\frac{dF}{dx} = \frac{Mx\ddot{\theta}}{L} - \frac{Ma}{L}\cos\theta$

$\therefore \quad F = \frac{Mx^2}{2L}\ddot{\theta} - \frac{Ma}{L} x \cos\theta + C_1$

But when $\theta = 30°$, $\ddot{\theta} = \frac{9\sqrt{3}a}{16L}$ and $F = P = \frac{5\sqrt{3}}{32} Ma$ at $x = 0$

$\therefore \quad C_1 = \frac{5\sqrt{3}}{32} Ma$

and $\frac{F}{Ma} = \frac{9\sqrt{3}}{32}x^2 - \frac{\sqrt{3}}{2}x + \frac{5\sqrt{3}}{32}$ writing $x \equiv x/L$

BM is a max when SF = 0, ie when

$9x^2 - 16x + 5 = 0$

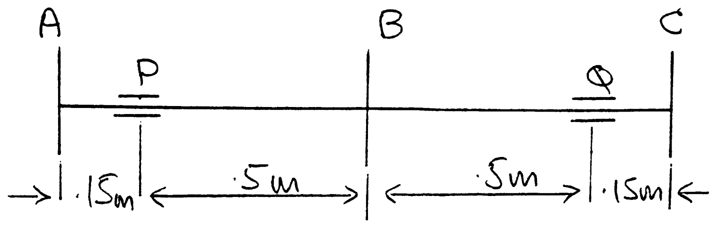
$x = \frac{16 \pm \sqrt{16^2 - 4.5.9}}{18} = \frac{16 \pm \sqrt{76}}{18}$

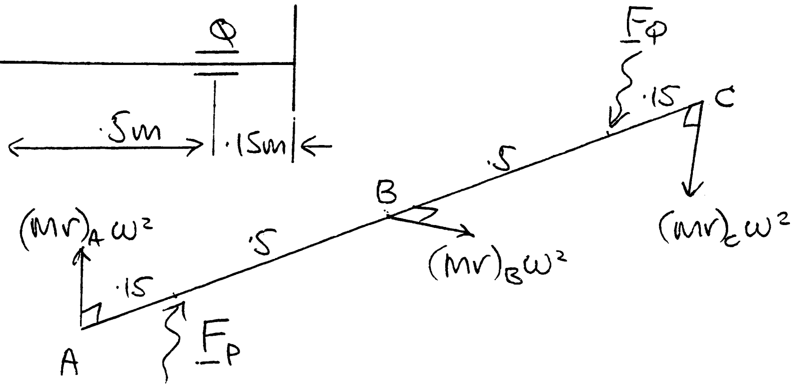
i.e. $\frac{x}{L} = 0.4046$ (since $0 < \frac{x}{L} < 1$)

Comments

A substantial number of candidates threw marks away by not even bothering to write anything in answer to the first part of the question. A high proportion of errors arose from not reading what the question says: horizontal table so no gravity forces, bar starts from rest so no angular velocity - an initial motion problem. Some candidates decided that it was obvious that the force at A needs no component perpendicular to the direction of the acceleration *a* whereas, of course, it does. Only a few sensible attempts at position of maximum bending moment in the beam.

6. In a <u>statically balanced</u> shaft the centre of mass of the rotating mass lies on the ℄ so that the shaft will sit in equilibrium at any angular position. When rotating the bearing forces sum to zero but may constitute a couple. In a <u>dynamically balanced</u> shaft the bearing forces for steady rotation are each equal to zero.

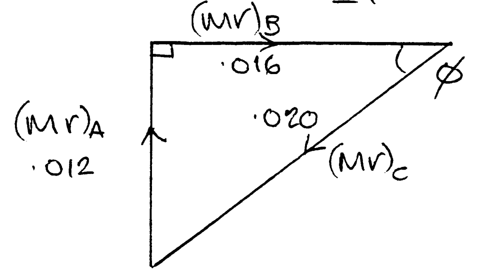


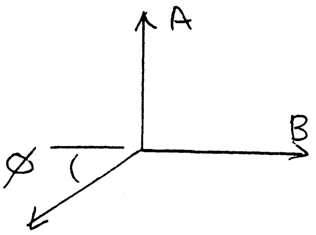


$(mr)_A = 0.012$ kgm
$(mr)_B = 0.016$ kgm
$(mr)_C = 0.020$ kgm

<u>Static balance</u>: look along shaft from A to B

$F_P + F_Q = 0$





by inspection 3:4:5 △.
$\phi = \tan^{-1} 0.75 = 36.9°$

Thus relative to pulley A, out of balance at B must be at <u>90°</u> and that at C at $(270-\phi) = $ <u>233°</u>

<u>Dynamic balance</u>:

Consider taking moments about one of the bearings, say P. Since all moment arms lie along the axis of the shaft moment vectors are all rotated by 90° from corresponding force vectors (note that because of overhang moment arm of A is negative).

Contribution from mass at A is $0.012 \times 0.015\,\omega^2 = 0.0018\,\omega^2$ Nm
" " B $0.016 \times 0.5\;\omega^2 = 0.008\,\omega^2$ Nm
" " C $0.020 \times 1.15\,\omega^2 = 0.023\,\omega^2$ Nm

Closing vector of moment polygon must represent

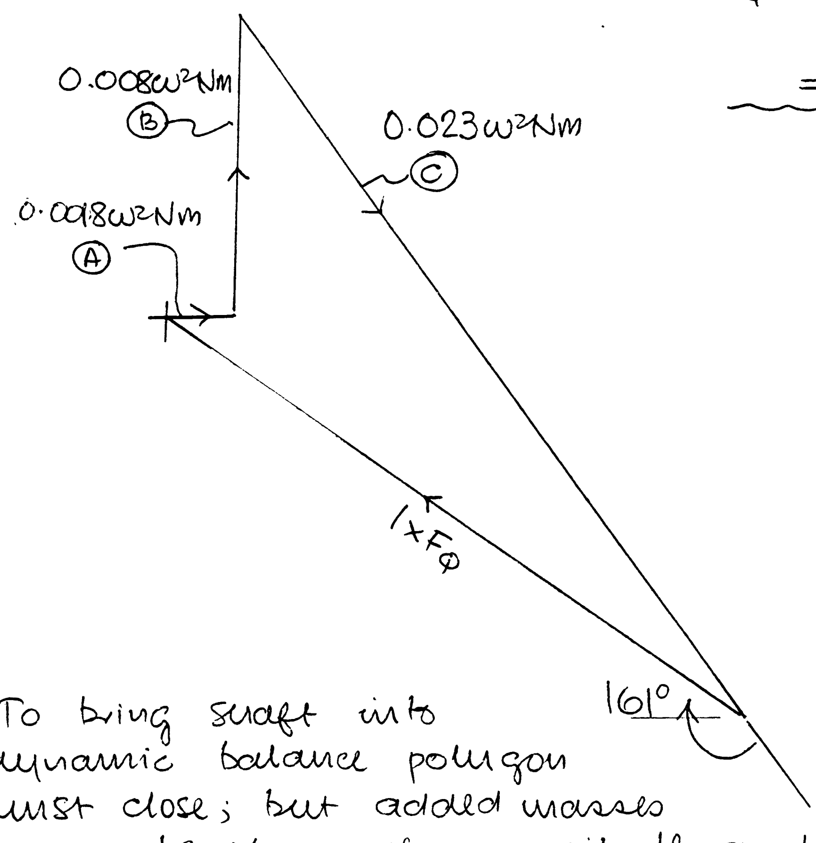
$$|\underline{PQ} \times \underline{F_\phi}| \quad \text{i.e.} \quad 1 \times F_\phi$$

Sketching to scale

gives $1 \times F_\phi = \frac{94}{40} \times 0.008\,\omega^2$ Nm

$$\therefore F_\phi = \frac{94}{40} \times \frac{0.008}{1} \times \left(\frac{420 \times 2\pi}{60}\right)^2 \text{ N}$$

$$= 36.4 \text{ N}$$



To bring shaft into dynamic balance polygon must close; but added masses must be of equal magnitude and 180° out of phase so as not to destroy static balance: Thus

$$(m \times 0.2 \times 1.15)\,\omega^2 + (m\,(\times 0.2 \times 0.15)\,\omega^2 = \frac{94}{40} \times 0.008\,\omega^2$$

specified radius

i.e. $\underline{\underline{m = 0.072 \text{ kg}}}$

Diagram indicates that mass at C should be added at a radius making angle 161° clockwise to out of balance at C, ie $233° + 161° = 394°$ or $34°$ to direction to that at A. Mass at A is thus at $34° + 180° = \underline{214°}$.

The least popular question but well done by those who attempted it and had obviously boned up on balancing. Similar to an examples paper question. Out of balance masses form a 3:4:5 triangle - recognised by most though not all who laboured away with the cosine rule - so calculations are pretty well simplified. About equal numbers of solutions by calculation and sketching to scale.

JAW - June 96.